Higher Education
Australian Mathematical Sciences Institute

Maximal p-negative type for different norms on $\mathbb{R}^3$

Tingyu Mao
Supervisor: Associate Professor Ian Doust
UNSW Australia

Australian Government
Department of Education

5 February 2015

Abstract

Metric spaces of p-negative type zero have interesting properties. It is known that a metric space with no positive p-negative type can be isometric to no subspace of any L_p-space for which $0 < p \leq 2$. Lennard, Tonge and Weston gave an indirect proof that $\ell_q^{(3)}$ has p-negative type zero where $2 < q \leq \infty$ while later Doust, Sánchez and Weston gave a direct proof collecting two sets of points in the case when $p = \infty$. In this paper we develop that direct proof by the original definition of p-negative type. Furthermore, we take a brief look at things in the metric space $\ell_q^{(3)}$ where $2 < q < \infty$ with some findings and conjectures.

1 Introduction: negative type and generalized roundness

Given a metric space (X, d), a natural question is whether (X, d) can be identified as a subset of a standard space such as Euclidean space, or an L_p spaces. Since the standard spaces have many nice geometric properties and a well-developed theory, if one can embed (X, d) in such a space, one has many more tools with which to attack problems.

Many numeric quantities have been introduced to help answer the question of when a space (X, d) embeds in another space. This paper concerns a concept known as p-negative type and a closely related concept known as generalized roundness. The theory of spaces with 'negative type' has a long history, going back over 100 years to papers by Cayley [2], Menger [7, 8, 9] and Schoenberg [12, 13, 14]. Generalized roundness was introduced by Enflo [4] in 1969 in order to settle an old embedding problem of Smirnov.

Definition 1.1. Let $p \geq 0$ and let (X, d) be a metric space. Then:

(a) (X, d) has **p-negative type** if and only if for all integers $n \geq 2$, all finite subsets $\{z_1, \ldots, z_n\} \subseteq X$, and all choices of real numbers $\zeta_1, \ldots, \zeta_n$ with $\zeta_1 + \cdots + \zeta_n = 0$, we have:

$$\sum_{i,j=1}^{n} d(z_i, z_j)^p \zeta_i \zeta_j \leq 0. \tag{1.1}$$

(b) p is a **generalized roundness exponent** of (X, d) if and only if for all integers $n > 1$ and all choices of points $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, we have:

$$\sum_{i,j=1}^{n} \{d(x_i, x_j)^p + d(y_i, y_j)^p\} \leq 2 \sum_{i,j=1}^{n} d(x_i, y_j)^p. \tag{1.2}$$

(c) The **generalized roundness** of (X, d) is defined to be the supremum of the set of all generalized roundness exponents of (X, d).

It wasn't until 1997 that Lennard, Tonge and Weston [6, Theorem 2.4] showed that for any $p \geq 0$, conditions (a) and (b) above are equivalent. Thus, the generalized roundness and supremal p-negative type of any given metric space coincide.

An important result from Bretagnolle, Dacunha-Castelle and Krivine [1] in 1966 is that when $0 < p \leq 2$, a quasi-normed space X is linearly isometric to a subspace of some L_p space if and only if X has p-negative type. Rather later, Misiewicz [10] and Koldobsky [5] showed that $\ell_q^{(n)}$ doesn't have p-negative type for any $p > 0$ if $n \geq 3$ and $2 < q \leq \infty$. Here $\ell_q^{(n)}$ means the set $\mathbb{R}^n$ with the q norm:

$$\|(x_1, \ldots, x_n)\|_q = \left(\sum_{k=1}^{n} |x_k|^q \right)^{1/q}, \qquad 1 \leq q < \infty$$
$$\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

In particular $\ell_\infty^{(3)}$ has no p-negative type for any $p > 0$.

Unfortunately the methods that Misiewicz and Koldobsky use are quite indirect. It was known that if $\ell_q^{(n)}$ embedded isometrically in some L_p space with $0 < p \leq 2$ then the function $\mathbf{x} \mapsto \exp(-\|\mathbf{x}\|_q^p)$ would need to be positive definite, and what Misiewicz and Koldobsky show is that $\mathbf{x} \mapsto \exp(\phi(\|\mathbf{x}\|_\infty))$ is never positive definite unless $\phi \equiv 1$. In particular, their proofs give no real insight as to how one would find a subset $\{z_1, \ldots, z_n\} \subseteq \ell_\infty^{(3)}$ and reals $\zeta_1, \ldots, \zeta_n$ summing to 0 for which the inequality in (a) above fails.

Recently, Ian Doust, Stephen Sanchez and Anthony Weston [3], using a version of the equivalent formulation from (b), showed how one can choose appropriate sets of points proving $\ell_\infty^{(3)}$ has no positive p-negative type. Although their proof showed where the points need to lie, it did not explicitly determine how many points would be needed to construct a situation where inequality (1.2) failed.

The aims of this project then were

1. to examine the proof in [3] and to determine, given $p > 0$, how many points are needed to give an example in which inequality (1.2) is violated.
2. to see whether the proof in [3] can be adapted to the case of $\ell_q^{(3)}$ for $1 < q < \infty$.

2 Relations and reformulations

One of the significant challenges in this area is accurately calculating, or at least estimating, sums of the type that appear in (1.2). A useful technique is to consider the sums as Riemann sums for a suitable integral and use the value of the integral as an approximation for the sum. This idea was used in [3] (and before that in [11] for example). The following result appears in [3].

Theorem 2.1. *Let (X, d) be a metric space and suppose that $p \geq 0$. Then the following are equivalent:*

1. *p is a generalized roundness exponent of (X, d).*
2. *For all integers $N \geq 1$, all finite sets $\{x_1, \ldots, x_N\} \subseteq X$, and all collections of non-negative real numbers $m_1, \ldots, m_N, n_1, \ldots, n_N$ that satisfy $m_1 + \cdots + m_N = n_1 + \cdots + n_N$, we have:*

$$\sum_{i,j=1}^{N} \{m_i m_j + n_i n_j\} d(x_i, x_j)^p \leq 2 \sum_{i,j=1}^{N} m_i n_j d(x_i, x_j)^p.$$

3. For all regular Borel probability measures of compact support μ and ν on X, we have:

$$\iint_{X\times X} d(x,x')^p d\mu(x)d\mu(x') + \iint_{X\times X} d(y,y')^p d\nu(y)d\nu(y') \leq 2\iint_{X\times X} d(x,y)^p d\mu(x)d\nu(y) \tag{2.1}$$

For short, to prove the equivalence of the double sum edition and integral edition, we need that N be large enough, so much of this project was dedicated to explicitly determining just how large N needed to be for the purposes of the proof.

In the direct proof that $\ell_\infty^{(3)}$ has p-negative type zero in [3] they showed that for sufficiently large L, inequality (2.1) ws violated when μ and ν were one-dimensional Lebesge measure supported on the sets

$$S_\mu = \{(t,\pm1,0) : -L \leq t \leq L\} \quad \text{and} \quad S_\mu = \{(t,0,\pm1) : -L \leq t \leq L\}.$$

Since the integrands in (2.1) are continuous, it follows that if one takes sufficiently many equally spaced points $\{x_i\} \subseteq S_\mu$ and $\{y_j\} \subseteq S_\nu$ then inequality (1.2) would fail.

My first aim then was to estimate the errors in the Riemann sum approximations to give explicit collections of points $\{x_i\}$ and $\{y_j\}$ for which inequality (1.2) fails.

3 Finding the number of points needed

For the whole paper, we assume the real-valued function $f(x) \geq 0$ when $x \in D$, since it's more complicated to think of a function with negative value when adding a power p to it. And it also makes sense given that the functions we'll take into account are well-behaved. Non-negativity as well as the continuity are guaranteed for the distance function.

We begin with a simple estimate of the error in the Riemann sum approximation to the area of a degree one polynomial.

Lemma 3.1. *For a polynomial of degree 1 of the form $f(x) = ax+b$, the error ϵ of the estimation of the integral by Riemann sum is bounded by $\frac{L^2}{2N}$ when we divide the interval $[0,L]$ into N parts evenly.*

Proof. $\int_0^L (ax^2+b)dx = \frac{1}{2}aL^2 + bL$. Let $x_k = \frac{k}{N}L$, $k = 1,2,\dots,N$. Then

$$\sum_{k=1}^{N} \frac{L}{N} f(x_k) = \sum_{k=1}^{N} \frac{L}{N}(a\frac{kL}{N} + b) = \frac{a}{2}L^2 + bL + \frac{aL^2}{2N}.$$

Therefore,

$$\epsilon = |\sum_{k=1}^{N} \frac{L}{N} f(x_k) - \int_0^L (ax+b)dx| = \frac{aL^2}{2N}.$$

□

Theorem 3.2. *Let f be a continuous increasing (or decreasing) and piecewise differentiable function on $[0,L]$. Then*

$$|\int_0^L f^p(x)dx - \sum_{k=1}^{n} \frac{L}{n} f^p(x_k)| \leq c_p |\int_0^L f(x)dx - \sum_{k=1}^{n} \frac{L}{n} f(x_k)|,$$

where $c_p = p(\min_{x\in[0,L]} f(x))^{p-1}$, *and* $x_k = \frac{k}{n}L$.

Proof. We only prove the case when f is increasing, the proof of the decreasing case is the same.

Let $g(x) = c_p f(x) - f^p(x)$. Since f is increasing on the interval $[0, L]$, we have that $f'(x) \geq 0$. Therefore,

$$g'(x) = \frac{d}{dx}(c_p f(x) - f^p(x)) = c_p f'(x) - pf^{p-1}(x)f'(x) = f'(x)(c_p - pf^{p-1}(x)) \geq 0.$$

So $g(x)$ is increasing on $[x_k, x_{k+1}]$. Therefore, $g(x_{k+1}) \geq g(x)$, when $x \in [x_k, x_{k+1}] \subset [0, L]$. Since f is increasing, f^p is increasing, so for each k

$$|\int_{x_k}^{x_{k+1}} f(x)dx - \frac{L}{n}f(x_{k+1})| = \frac{L}{n}f(x_{k+1}) - \int_{x_k}^{x_{k+1}} f(x)dx$$
$$|\int_{x_k}^{x_{k+1}} f^p(x)dx - \frac{L}{n}f^p(x_{k+1})| = \frac{L}{n}f^p(x_{k+1}) - \int_{x_k}^{x_{k+1}} f^p(x)dx.$$

Note that $g(x)$ is continuous, so we have $\int_{x_k}^{x_{k+1}} g(x)dx = g(u_{k+1})(x_{k+1} - x_k)$ for some $u_{k+1} \in [x_k, x_{k+1}]$ by the Mean Value Theorem. Thus

$$\begin{aligned}
&c_p|\int_{x_k}^{x_{k+1}} f(x)dx - \frac{L}{n}f(x_{k+1})| - |\int_{x_k}^{x_{k+1}} f^p(x)dx - \frac{L}{n}f^p(x_{k+1})| \\
&= c_p[\frac{L}{n}f(x_{k+1}) - \int_{x_k}^{x_{k+1}} f(x)dx] - [\frac{L}{n}f^p(x_{k+1}) - \int_{x_k}^{x_{k+1}} f^p(x)dx] \\
&= \frac{L}{n}[g(x_{k+1}) - \int_{x_k}^{x_{k+1}} g(x)dx] = \frac{L}{n}[g(x_{k+1}) - g(u_{k+1})] \geq 0.
\end{aligned}$$

Therefore,

$$|\int_{x_k}^{x_{k+1}} f^p(x)dx - \frac{L}{n}f^p(x_{k+1})| \leq c_p|\int_{x_k}^{x_{k+1}} f(x)dx - \frac{L}{n}f(x_{k+1})|.$$

By the same procedure, we get the same results if f is decreasing in $[x_k, x_{k+1}]$. Therefore,

$$\begin{aligned}
|\int_0^L f^p(x)dx - \sum_{k=1}^n \frac{L}{n}f^p(x_k)| &\leq \sum_{k=1}^n |\int_{x_k}^{x_{k+1}} f^p(x_k)dx - \frac{L}{n}f^p(x_k)| \\
&\leq c_p \sum_{k=1}^n |\int_{x_k}^{x_{k+1}} f(x_k)dx - \frac{L}{n}f(x_k)| \\
&= c_p|\int_0^L f(x)dx - \sum_{k=1}^n \frac{L}{n}f(x_k)|
\end{aligned}$$

□

Corollary 3.3. *For a function* $f(x) = ax + b$, *the Riemann sum* $|\int_0^L f^p(x)dx - \sum_{k=1}^N \frac{L}{N}f^p(x_k)|$ *is bounded by* $\frac{aL^2}{2N}p[\min_{x\in[0,L]} f(x)]^{p-1}$.

Proof. This is a simple combination of the Lemma 3.1 and Theorem 3.2. □

The following is an application of the results above. We're going to apply them to find how thin is the division supposed to be to make any positive p fail to be a p-negative type for the space $\mathbb{R}^3$ in ℓ_∞ norm.

Theorem 3.4. *If $ax+y>0$ in $[-L,L]\times[-L,L]$, then*

$$|\int_{-L}^{L}\int_{-L}^{L}|ax+y|^p dxdy - \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{4L^2}{N^2}|ax_i+y_j|^p| \le c_p\frac{4aL^3}{N}.$$

where we partition the region into small squares of length $\frac{2L}{N}$.

Proof. We can choose a partition with the greatest error in one-dimensional case to give a upper error bound in high dimension.

$$\begin{aligned}
&|\int_{-L}^{L}\int_{-L}^{L}|ax+y|^p dxdy - \sum_{i=1}^{N}\frac{4L^2}{N^2}\sum_{j=1}^{N}|ax_i+y_j|^p| \\
&= \int_{-L}^{L}|\int_{-L}^{y}|ax+y|^p dx - \sum_{k=1}^{N_y}\frac{4L^2}{N^2}|ax_k+y|^p + \int_{y}^{L}|ax+y|^p dx - \sum_{k=N_y+1}^{N}\frac{4L^2}{N^2}|ax_k+y|^p|dy \\
&\le \int_{-L}^{L}|\int_{-L}^{L}(ax+y)^p dx - \sum_{k=1}^{N}\frac{4L^2}{N^2}(ax_k+y)^p|dy \\
&\le \int_{-L}^{L}c_p|\int_{-L}^{L}(ax+y)dx - \sum_{k=1}^{N}\frac{4L^2}{N^2}(ax_k+y)|dy \\
&\le \int_{-L}^{L}c_p\frac{a(2L)^2}{2N}dy \\
&\le c_p\frac{4aL^3}{N}
\end{aligned}$$

□

For the d_∞ metric and the measures supported on S_μ and S_ν that are used in [3], the expressions in (2.1) can be written out explicitly. For example

$$\iint_{\mathbb{R}\times\mathbb{R}} d(x,x')^p\,d\mu(x)d\mu(x') = 2\Big(\int_{-L}^{L}\int_{-L}^{L}|t-s|^p\,ds\,dt + \int_{-L}^{L}\int_{-L}^{L}\max(|t-s|,2)^p\,ds\,dt\Big). \quad (3.1)$$

By Theorem 3.4, we can now estimate the different between such an integral and the associated Riemann sum which is associated with the discrete p-negative type inequality. The terms where the integrand is bounded away from zero cause the least problem. For example

$$\epsilon_2 = \Big|\int_{-L}^{L}\int_{-L}^{L}\max\{|t-s|,1\}^p dxdy - \frac{4L^2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|t-s|^p\Big| \le \frac{2pL^3}{N}.$$

$$\epsilon_3 = \left| \int_{-L}^{L} \int_{-L}^{L} \max\{|t-s|, 2\}^p dxdy - \frac{4L^2}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |t-s|^p \right| \leq \frac{p 2^p L^3}{N}.$$

The first term on the right-hand side of (3.1) is more difficult as $|t-s|$ has zeroes in the region, and so we should consider the neighborhoods of zeroes seperately. We set apart the set $S = \{(t,s) || t-s| < l\}$ and consider the potentially maximal error. Since in one-dimensional case, the area that the curve surrounds (with the axis) is less than $l \cdot 2l = 2l^2$, if we choose the approximation l^2, the possible maximal error is $2l^2 - l^2 = l^2$. Therefore,

$$\begin{aligned}
\epsilon_1 &= | \int_{-L}^{L} \int_{-L}^{L} |t-s|^p dxdy - \frac{4L^2}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |t-s|^p | \\
&\leq \int_{-L}^{L} |c_p \frac{(2L-2l)^2}{2N} + l^2| dt \\
&\leq \int_{-L}^{L} |c_p \frac{2L^2}{N} + l^2| dt \\
&= c_p \frac{4L^3}{N} + 2l^2 L \\
&= p l^{p-1} \frac{4L^3}{N} + 2l^2 L.
\end{aligned}$$

Then we are going to find the minimum value of ϵ_1 with respect to l. Simple calculation leads us to the result that

$$\min_l \epsilon_1 = \left[\frac{p(1-p)L^2}{N} \right]^{2/(3-p)} \left(2L + \frac{4L}{1-p} \right)$$

when $l = \left[\frac{p(1-p)L^2}{N} \right]^{1/(3-p)}$.

It's also easy to verify that $\min_l \epsilon_1 \leq 4 \left(\frac{p}{N} \right)^{2/(3-p)} L^{(7-p)/(3-p)}$ when $p \in [0,1]$ expanding the expression above.

Doust, Sánchez and Weston [3] introduced the quantity

$$\Delta(L,p) = \frac{p^2+3p+2}{4p} \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} d(x,y)^p d\mu(x) d\nu(y) - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} d(x,y)^p d\mu(x) d\mu(y) \right).$$

For the d_∞ metric they showed that

$$\Delta(L,p) = 4Lp + 2^{p+1}p + 8L + 2^{p+1} - 4L2^p p - 8L2^p - p - 1$$

and then showed that $\Delta(1/p,p) < 0$.

We define $\delta(N,L,p) = \Delta(L,p) + \epsilon_1 + 2\epsilon_2 + \epsilon_3$. Following the same method, we let $L = \frac{1}{p}$ and get $\delta(N,1/p,p) = p(2^{p+1}-1) + (3-2^{p+1}) + \frac{8(1-2^p)}{p} + \epsilon_1 + 2\epsilon_2 + \epsilon_3$.

Since $\Delta(L,p)$ has nothing with N, we can easily get its maximal value $1 - 8\ln 2$ when $p \in [0,1]$. Therefore, $\delta(N,1/p,p) \leq 1 - 8\ln 2 + 4(\frac{1}{N})^{\frac{2}{3-p}} p^{\frac{-5+p}{3-p}} + 2\frac{2}{p^2 N} + \frac{2^p}{p^2 N}$. Let $N = \frac{4}{p^{\frac{5-p}{2}}}$, $\delta(\frac{4}{p^{\frac{5-p}{2}}}, 1/p, p) \leq 1 - 8\ln 2 + 1 + p^{\frac{1-p}{2}} + 2^p p^{\frac{1-p}{2}}/4$.

Given that $p^{\frac{1-p}{2}} \leq 1$, $2^p p^{\frac{1-p}{2}}/4 \leq 1/2$. So we get the result $\delta(\frac{4}{p^{\frac{5-p}{2}}}, 1/p, p) \leq 0$. Hence we conclude that we can prove any p fails to be the generalized roundness by choosing $N = 4p^{-\frac{5-p}{2}}$ points on the line $\{(t, \pm 1, 0) | -L \leq t \leq L\}$. For the simplification of the expression, we can choose $N = 4p^{-\frac{5}{2}}$ to satisfy all the possible value of $p \in (0, 1]$.

Theorem 3.5. *Suppose $p \in (0,1]$, that $L = 1/p$ and that $N > 4p^{-\frac{5}{2}}$. If the points $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ are chosen equally spaced along the line segments S_μ and S_ν respectively, then inequality (1.2) fails and hence p is not a generalized roundness exponent for $\ell_\infty^{(3)}$.*

4 The case $2 < q < \infty$

In the second part, we try to have an insight into how is the set like in $\ell_q^{(3)}$. It has been proved that $\ell_q^{(3)}$ has generalized roundness zero when $q \in [2, \infty]$. So it will be more specific if we can choose an approriate set of points so that any positive p fails to satisfy the condition of the generalized roundness. The challenge is to determine where these points should be in order to get the p-negative type inequality to fail for very small p.

Suppose that $x_1, \ldots, x_n, y_1, \ldots, y_n \in \ell_q^{(3)}$. Let

$$S_\ell(p) = \sum_{i,j=1}^{n} \{d(x_i, x_j)^p + d(y_i, y_j)^p\}, \qquad S_r(p) = 2\sum_{i,j=1}^{n} d(x_i, y_j)^p.$$

Now $S_\ell(0) = 2n(n-1)$ and $S_r(0) = 2n^2$ so certainly $S_\ell(0) < S_r(0)$. On the other hand, if the is a pair of points 'on one teams' which are further apart than any points in the same teams (that is $d(x_{i_0}, x_{j_0})$ is larger than any distance $d(x_i, y_j)$) then there will exist p_0 such that $S_\ell(p) > S_r(p)$ for $p > p_0$ and hence the maximal p-negative type will be less than or equal to p_0. As in [3] we tried to find such bounds using the measure version of the inequality rather than the sum versions.

Note that when we deal with the case of $q = \infty$, we use four parallel lines in $\mathbb{R}^3$. It's like a ellipsoid in the ℓ_∞ norm of unit height and long enough length. So we apply this idea to the current case, namely, using the similar ellipsoid of ℓ_q norm. To cut down on the number of variable quantities we concentrated first on the case when $q = 4$. Following the method and notation of [3], it's sufficient to show that $I_0 + I_2 > 2I_1$ (see below). We picked two sets $S_\mu = \{(x, y, 0) | \frac{x^4}{a^4} + y^4 = R^4\}$, $S_\nu = \{(x, 0, z) | \frac{x^4}{a^4} + z^4 = R^4\}$. By calculation and rearrangement, the quantities that we need to estimate are

$$I_0 = \int_{-L}^{L}\int_{-L}^{L} \{|t-s|^q + [(R^q - \frac{t^q}{a^q})^{1/q} - (R^q - \frac{s^q}{a^q})^{1/q}]^q\}^{p/q} dsdt$$

$$I_1 = \int_{-L}^{L}\int_{-L}^{L} \{|t-s|^q + [((R^q - \frac{t^q}{a^q})^{1/q})^q + ((R^q - \frac{s^q}{a^q})^{1/q})^q\}^{p/q} dsdt$$

$$I_2 = \int_{-L}^{L}\int_{-L}^{L} \{|t-s|^q + [(R^q - \frac{t^q}{a^q})^{1/q} + (R^q - \frac{s^q}{a^q})^{1/q}]^q\}^{p/q} dsdt$$

where L denotes the length of part of the ellipsoid used in this case, $0 \leq L \leq a$.

Unlike the case when $q = \infty$, the above integrals can not be calculated analytically, so we tried calculating these numerically, to determine the largest value of p when $I_0 + I_2 > 2I_1$. By programming experiments, R doesn't matter at all, behaving just like a strech. Therefore, for convenience, we let $R = 1$. And the experiments show that when $L \to R$, p gets relatively smaller, coinciding with the $p = \infty$ case that we should use almost all the length of the ellipsoid. Then we let $L = R$. When a is large enough, for example, 50, this critical value of p converges to a certain number, 0.34. It is much smaller than the p value we get using other previous methods.

We did experiments with other $q \in (2, \infty)$. The critical values of p that appeared had a surprising pattern:

q	critical p
3	$0.51 \approx \frac{1}{2}$
4	$0.34 \approx \frac{1}{3}$
5	$0.26 \approx \frac{1}{4}$
6	$0.21 \approx \frac{1}{5}$
7	$0.17 \approx \frac{1}{6}$
$\vdots$	$\vdots$
10	$0.12 \approx \frac{1}{9}$

Since we set 0.01 as the pace of our programming experiments, it's likely the smallest p we can get is $p = \frac{1}{q-1}$ by all of these outputs above. It's no more than a conjecture right now since it lacks theoretical proof given the complicated expressions. But hopefully, it will be proved to be a correct result. In any case, it appears that the construction in [3] does not work for the case $2 < q < \infty$ and so some new ideas will be needed.

In the constructions so far, the points were chosen equally spaces along a one dimensional curve, which is equivalent to choosing Lebesgue measure on those curves. We decided therefore to try changing the measure by multplying Lebesgue measure by some weight function. We try exerting a bunch of weight functions on the measure, believing a change of measure may help perform unexpectedly better. However, to our disappointment, these attempts are all in vain without giving a prospective direction.

5 Open problems

When it comes to the further developments, I reckon that letting $f = F_1 - F_2$ where F_1, F_2 are both continuously increasing functions by theorems in real analysis, and trying to construct an approximation of F_1^p, F_2^p may lead to a sharper bound in the first part of the research, since we have no need to sacrifice some dimension when dealing with the region where 0 value takes over, as the difference of a large enough constant doesn't affect the continuity and monotonicity of F_1 or F_2 at all.

And for the second part, attempting to exert *some* unrevealing extraordinary weight functions probably gives a remarkable improvement of the results. As we've proved the results directly that this metric space has no positive p-negative type, there's bound to be a kind of point selection which makes the result straightforward.

References

[1] J. Bretagnolle, D. Dacunha-Castelle and J. L. Krivine, *Lois stables et espaces L^p*, Ann. Inst. H. Poincaré **2** (1966), 231–259.

[2] A. Cayley, *On a theorem in the geometry of position*, Cambridge Mathematical Journal **II** (1841), 267–271.

[3] I. Doust, I., S. Sánchez, and A. Weston, *A direct proof that $\ell_\infty^{(3)}$ has generalized roundness zero*, Expo. Math., doi:10.1016/j.exmath.2014.06.001 (2014).

[4] P. Enflo, *On a problem of Smirnov*, Ark. Mat. **8** (1969), 107–109.

[5] A. Koldobsky, *Schoenberg's problem on positive definite functions*, (English translation in St. Petersburg Math. J. **3** (1992), 563-570), Algebra and Analysis **3** (1991), 78–85.

[6] C. J. Lennard, A. M. Tonge and A. Weston, *Generalized roundness and negative type*, Mich. Math. J. **44** (1997), 37–45.

[7] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. **100** (1928), 75–163.

[8] K. Menger, *New foundation of Euclidean geometry*, Amer. J. Math. **53** (1931), 721–745.

[9] K. Menger, *Géométrie Générale*, Mémor. Sci. Math., no. 124, Académie des Sciences de Paris, Gauthier-Villars (1954).

[10] J. K. Misiewicz, *Positive definite norm dependent functions on ℓ^∞*, Statist. Probab. Lett. **8** (1989), 255–260.

[11] P. Nickolas and R. Wolf, *Distance geometry and quasihypermetric spaces. I.*, Bull. Aust. Math. Soc. **80** (2009), 1–25.

[12] I. J. Schoenberg, *Remarks to Maurice Frechet's article "Sur la définition axiomatique d'une classe d'espaces distanciés vectoriellement applicable sur l'espace de Hilbert."*, Ann. Math. **36** (1935), 724–732.

[13] I. J. Schoenberg, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. Math. **38** (1937), 787–793.

[14] I. J. Schoenberg, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. **44** (1938), 522–536.