Geometry of Minkowski space

Zackary D. Burton

Supervised by
Dr. G. Cairns and Dr. Y. Nikolayevsky
La Trobe University
1. Introduction

Minkowski geometry can be investigated by identifying $\mathbb{R}^2$ with the set $\{x + iy : x, y \in \mathbb{R}\}$ whilst imposing the condition $i^2 = 1$. Equipped with this product, $\mathbb{R}^2$ is denoted by $\mathbb{M}$ and is the main focus of investigation throughout this paper.

For vectors $u = x_1 + iy_1$ and $v = x_2 + iy_2$, we define the following, in analogy with the Euclidean case:

- **Conjugation:** $\bar{u} := x_1 - iy_1$
- **Inner product:** $\langle u, v \rangle := \mathbb{R}(u \bar{v})$
- **Length:** $\|u\| := \langle u, u \rangle$
- **Distance:** $d(u, v) := \|u - v\|$

Key differences between Euclidean and Minkowskian geometry can be seen by comparing these operations with their Euclidean analogues.

**Example 1.1.** Considering the above-defined distance between vectors $u = i$ and $v = 2i$, we find some major differences between the two geometries. In Euclidean geometry, the distance would equal 1, but if we consider the same vectors in $\mathbb{M}$, we have

\[
d(u, v) = \|u - v\| = (u - v)(\bar{u} - \bar{v}),
\]

\[
= u\bar{u} - u\bar{v} - v\bar{u} + v\bar{v},
\]

\[
= -i^2 + 2i^2 - 2i^2 - 4i^2,
\]

\[
= -1. \quad \text{(as } i^2 = 1)\]

**Example 1.2.** We can take vectors $u = 0$ and $v = n(1 + i)$, where $n \in \mathbb{R}$, which in Euclidean geometry gives the (squared) distance as $2n^2$, which is non-zero unless $n = 0$. But, if we consider the same vectors again in $\mathbb{M}$, we find

\[
d(u, v) = \|u - v\| = \|v\| = v\bar{v},
\]

\[
= n^2 - (ni)^2,
\]

\[
= 0. \quad \text{(as } i^2 = 1)\]

Within the Minkowski setting, distance is often referred to as pseudo-distance, and $\mathbb{M}$ is referred to as a pseudo-Euclidean space. This is because, as the above results indicate, the Minkowski distance as defined can take negative values; there are also pairs of non-equal points whose pseudo-distance is equal to zero.

**Definition 1.3.** A vector is called light-like if it has length zero.
If we consider the aforementioned definition of inner product in the Minkowski setting, we say two vectors are **orthogonal** if their inner product equals zero.

For the proof of the next lemma, recall the definition of span: if \( S \subseteq \mathbb{R}^2 \), then \( \text{span}(S) \) denotes the real subspace of \( \mathbb{R}^2 \) generated by \( S \).

**Lemma 1.4.** Let \( z \) be a non-zero vector in \( \mathbb{M} \). Then a vector \( w \) is orthogonal to \( z \) if and only if \( w \in \text{span}\{iz\} \).

**Proof.** Assume first that \( w = k(iz) \) for some \( k \in \mathbb{R} \). Then
\[
\langle z, w \rangle = \Re(z\bar{w}),
\]
\[
= \Re(zk(iz)),
\]
\[
= \Re(-ikzz).
\]
As \( k \in \mathbb{R} \) and \( zz \in \mathbb{R} \), it is clear that \(-ikzz\) is imaginary, so \( \Re(-ikzz) = 0 \). Hence, if \( w \in \text{span}(\{iz\}) \), then \( w \) is orthogonal to \( z \).

Conversely, assume that an arbitrary vector \( w \) gives \( \langle z, w \rangle = 0 \). So, letting \( z := a + ib \) and \( w := x + iy \), where \( a, b, x, y \in \mathbb{R} \), we have
\[
\Re(z\bar{w}) = 0,
\]
\[
\implies \Re((a + ib)(x - iy)) = 0,
\]
\[
\implies ax - by = 0,
\]
\[
\implies ax = by.
\]
Now, since \( z \neq 0 \), we know that \( a \neq 0 \) or \( b \neq 0 \). So, considering the case \( a \neq 0 \), we have
\[
x = \frac{by}{a}.
\]
From this, we find
\[
w \in \left\{ \frac{by}{a} + iy : y \in \mathbb{R} \right\},
\]
\[
= \text{span}\left( \left\{ \frac{b}{a} + i \right\} \right),
\]
\[
= \text{span}(\{b + ia\}).
\]
Similarly, the case where \( b \neq 0 \) gives the same solution set for \( w \). So, as
\[
\text{span}(\{b + ia\}) = \text{span}(\{i(a + ib)\}),
\]
\[
= \text{span}(\{iz\}),
\]
we know that \( w \in \text{span}(\{iz\}) \). Hence, \( w \) is orthogonal to \( z \) iff \( w \in \text{span}(\{iz\}) \). \( \square \)
Notation 1.5. Given two points \(a, b\), we denote the line segment between \(a\) and \(b\) by \([a, b]\). Note that we allow \(a = b\), which gives \([a, b] = \{a\}\).

In the Euclidean setting, the midpoint of a line segment \([a, b]\) is the coordinate-wise arithmetic mean of the endpoints. We take this to be our definition for the midpoint of a line segment within \(\mathbb{M}\).

Definition 1.6. The midpoint \(m(a, b)\) of \([a, b]\) is defined by \(m(a, b) = \frac{1}{2}(a + b)\).

Remark 1.7. If, in Minkowski geometry, we instead defined the midpoint of a line segment \([a, b]\) to be the point on \([a, b]\) that is equidistant from each endpoint, there would be no unique midpoint for any light-like line segment. To show this, we can consider Example 1.2, which shows that the Minkowski distance between any two points on a light-like line is equal to zero.

Definition 1.8. The Minkowski-perpendicular bisector of a line segment \([a, b]\) is the line passing through \(m(a, b)\) whose direction is orthogonal to the direction of \([a, b]\). We denote this line by \([a, b]^\perp\). Note that this is well-defined if and only if \(a \neq b\).

Theorem 1.9. Let \(a, b\) be distinct points. Then a point \(z\) lies on \([a, b]^\perp\) iff \(z\) is Minkowski-equidistant from \(a\) and \(b\).

Proof. As translation preserves Minkowski distance, it suffices to consider a line segment of the form \([a, -a]\), where \(a \neq 0\).

First assume that \(z \in [a, -a]^\perp\). By Lemma 1.4, we have \(z = \lambda i a\) for some \(\lambda \in \mathbb{R}\). Now,
\[
d(z, a) = \|z - a\|, \\
= (z - a)(\overline{z} - \overline{a}), \\
= (\lambda ia - a)(-\lambda i \overline{a} - \overline{a}), \\
= -\lambda^2 a \overline{a} - \lambda i a \overline{a} + \lambda i a \overline{a} + a \overline{a}, \\
= a \overline{a}(1 - \lambda^2),
\]
and
\[
d(z, -a) = \|z + a\|, \\
= (z + a)(\overline{z} + \overline{a}), \\
= (\lambda ia + a)(-\lambda i \overline{a} + \overline{a}), \\
= -\lambda^2 a \overline{a} + \lambda i a \overline{a} - \lambda i a \overline{a} + a \overline{a}, \\
= a \overline{a}(1 - \lambda^2), \\
= d(z, a).
\]
This shows that each point on \([a, -a]^\perp\) is equidistant from \(a\) and \(-a\).
Conversely, assume that \( d(z, a) = d(z, -a) \). Then

\[
\|z - a\| = \|z + a\|,
\]

\[
\implies (z - a)\overline{(z - a)} = (z + a)\overline{(z + a)},
\]

\[
\implies (z - a)\overline{(z - a)} = (z + a)(\overline{z} + \overline{a}),
\]

\[
\implies z\overline{z} - z\overline{a} - a\overline{z} + a\overline{a} = z\overline{z} + z\overline{a} + a\overline{z} + a\overline{a},
\]

\[
\implies 2z\overline{a} + 2a\overline{z} = 0,
\]

\[
\implies z\overline{a} + a\overline{z} = 0,
\]

\[
\implies \Re(z\overline{a}) = 0,
\]

\[
\implies \langle z, a \rangle = 0.
\]

By Lemma 1.4, \( z \) lies on the line through the origin which is orthogonal to \([a, -a] \). Hence, every point equidistant from \( a \) and \(-a\) lies on \([a, -a] \). \( \square \)

As we have characterised orthogonality and perpendicular bisection within \( \mathbb{M} \), we will now extract definitions from the Euclidean setting and observe the changes when they are translated into Minkowskian geometry.

2. Shaping the plane

Within Euclidean geometry, the definition of a circle centred at a point \( a \) is the set of points that are some fixed distance from \( a \). If we take this definition and translate it into \( \mathbb{M} \), we find, that for each \( r \in \mathbb{R} \), the Minkowski circle of radius \( r \) centred at a point \( a \in \mathbb{M} \) is given by the set

\[
\{ z \in \mathbb{M} : \|z - a\| = r \}.
\]

Hence, for \( r \neq 0 \), the Minkowski circle centred at the origin is given by the Euclidean hyperbola \( \{ x + iy : x^2 - y^2 = r \} \). Also, for \( r = 0 \), the Minkowski circle centred at the origin is given by the set \( \{ x \pm ix : x \in \mathbb{R} \} \).

**Definition 2.1.** The Minkowski circumcentre of a triangle is the centre of the circle passing through each of its vertices.

**Theorem 2.2.** Let \( T \) be a non-degenerate triangle in Minkowski space with no light-like sides. Then \( T \) has a unique circumcentre.

**Proof.** Let the triangle \( T \) have vertices \( a, b \) and \( c \). As \([a, b]\) and \([a, c]\) are not parallel, \([a, b] \) and \([c, a] \) are not parallel, so they must intersect at a point, say \( p \). By Theorem 1.9, \( p \) is equidistant from \( a, b \) and \( c \). Hence \( a, b \) and \( c \) all lie on a circle centred at \( T \). Since any circumcentre of \( T \) must lie in \([a, b] \) \( \cap [b, c] \) \( \cap [c, a] \), and \( p \) is the only point in this set, the circumcentre is unique. \( \square \)
So, for the following conditions, we find three distinct regions for circles centred at the origin within Minkowski space.

![Minkowski circles](image)

**Figure 1.** Minkowski circles

We now give some more results which will be used in the proofs of our main theorems.

**Lemma 2.3.** Let $a$, $b$ and $c$ be points with $a \neq b$. Then the line segment $[m(a, c), m(b, c)]$ is parallel to $[a, b]$.

**Proof.** First note that the direction of $[a, b]$ is given by $b - a$. Taking the midpoints of $[a, c]$ and $[b, c]$, i.e., $m(a, c)$ and $m(b, c)$, we find that the line segment between these points has the direction

$$m(b, c) - m(a, c) = \frac{1}{2}(b + c) - \frac{1}{2}(a + c),$$

$$= \frac{1}{2}(b - a).$$

As the direction of this line segment is a non-zero multiple of $b - a$, it is evident that the line segment $[m(a, c), m(b, c)]$ is parallel to $[a, b]$. \qed

**Theorem 2.4.** Let $x, y, z$ be points with $x \neq y$. Then $[x, z]$ and $[y, z]$ are orthogonal (i.e., $\langle x - z, y - z \rangle = 0$) iff $z$ lies on the circle with diameter $[x, y]$.

**Proof.** Since $x \neq y$, we can assume that $z \neq x$ so that $[x, z]^\perp$ exists.

Assume that $\langle x - z, y - z \rangle = 0$ and let $c := m(x, y)$. Then $[m(x, z), c]$ and $[y, z]$ are parallel by Lemma 2.3 and hence $[x, z]$ and $[m(x, z), c]$ are orthogonal. As $c \in [x, z]$, $c$ is equidistant from $x$ and $z$ by Theorem 1.9. Since $c$ is equidistant from $x$ and $y$, we have that $z$ lies on the circle with diameter $[x, y]$.

Conversely, assume $z$ lies on the circle with diameter $[x, y]$. Then $c := m(x, y)$ is equidistant from $x$ and $z$, so, by Theorem 1.9, $[x, z]$ and $[m(x, z), c]$ are orthogonal. By Lemma 2.3, we know that $[m(x, z), c]$ and $[z, y]$ are parallel, whence $[x, z]$ and $[y, z]$ are orthogonal. \qed
**Notation 2.5.** Let $a$ be a point and let $L$ be a non-degenerate line segment that is not light-like. We denote by $\pi(a, L)$ the orthogonal projection from $a$ onto the extension of $L$ to an infinite line.

**Definition 2.6.** Let $a, b$ and $c$ be the vertices of a non-degenerate triangle. The *altitude* of a vertex $a$ is the (infinite) line through $a$ and $\pi(a, [b, c])$.

**Definition 2.7.** The *Minkowski orthocentre* of a non-degenerate triangle with no light-like sides is the point at which the altitudes of each vertex intersect.

The existence of a unique orthocentre now follows easily from Snapper’s Theorem. As this is a theorem within affine geometry, the result is independent of whether we use the Euclidean or Minkowski (pseudo-)metric.

**Snapper’s Theorem.** Let $T$ be a non-degenerate triangle with vertices $a_1$, $a_2$, $a_3$ and centroid $g$. Let $m_i$ denote the midpoint of the side opposite $a_i$. Let $p$ be an arbitrary point, let $N_i$ be a straight line through $p$ and $m_i$, and let $L_i$ denote the straight line through $a_i$ parallel to $N_i$. Then the lines $L_1$, $L_2$ and $L_3$ pass through a common point $q$, and the points $p$, $q$ and $g$ are collinear.

**Proof.** A full proof is given in [1]. The idea is to let $g = 0$ and consider the images of the lines $N_i$ under the map $z \mapsto -2z$, and use the fact that $g$ trisects each $[a_i, m_i]$. □

**Theorem 2.8.** Let $T$ be a non-degenerate triangle with no light-like sides. Then $T$ has a unique orthocentre.

**Proof.** Note that we use the notation in the statement of Snapper’s Theorem.

First assume that $p \neq m_i$ for each $i$. Take $p$ to be the circumcentre of $T$. Then $L_i$ is the altitude of $a_i$, so the altitudes intersect at $q$, which is unique because $p$ is unique.

Now, assume $p = m_i$ for some $i$; say $m_1$. By Theorem 2.4, the sides $[a_1, a_2]$ and $[a_1, a_3]$ are orthogonal, so the altitudes of $a_2$ and $a_3$ pass through $a_1$. Also, the altitude of $a_1$ passes through $a_1$ by definition. So, the orthocentre of $T$ is $a_1$, and is unique because the orthocentre is the intersection of three pairwise non-parallel lines. □

**Definition 2.9.** A *Minkowski rectangle* is a non-degenerate quadrilateral such that each pair of adjacent sides is orthogonal.

**Remark 2.10.** We know by Lemma 1.4 that an orthogonal to a line is uniquely determined, so it is evident that opposite sides of a Minkowski rectangle are parallel. Hence, a rectangle within $\mathbb{M}$ is a (Euclidean) parallelogram.

**Definition 2.11.** The *mean centre* of the points $a_1, \ldots, a_n$ is the point $\frac{1}{n}(a_1 + \cdots + a_n)$. The mean centre of a polygon is defined to be the mean centre of its vertices.
The following is a well-known fact in Euclidean space.

**Lemma 2.12.** The diagonals of a parallelogram intersect each other at their midpoints. Moreover, this point is the mean centre of the parallelogram.

As we define the midpoint of a line segment to be the same for both Euclidean and Minkowski geometries, this lemma holds true in $\mathbb{M}$.

Analogous to the way we write $[a, b]$ to mean the line segment between points $a$ and $b$, we now extend this to a finite sequence of points to give a polygon.

**Notation 2.13.** For points $a_1, a_2, \ldots, a_n$, we write

$[a_1, a_2, \ldots, a_{n-1}, a_n] := [a_1, a_2] \cup [a_2, a_3] \cup \ldots \cup [a_{n-1}, a_n] \cup [a_n, a_1]$.

**Lemma 2.14.** For a non-degenerate triangle $[a, b, c]$, $[a, m(a, b), m(b, c), m(c, a)]$ is a parallelogram whose mean centre is $\frac{1}{4}(2a + b + c)$.

**Proof.** By Lemma 2.3, $[m(a, b), m(b, c)]$ is parallel to $[c, a]$, and by transitivity is parallel to $[a, m(c, a)]$. Similarly, $[a, m(a, b)]$ is parallel to $[m(c, a), m(b, c)]$. Thus, we have that $[a, m(a, b), m(b, c), m(c, a)]$ is a parallelogram. Its mean centre is

$$\frac{1}{4} \left( a + \frac{1}{2}(a+b) + \frac{1}{2}(b+c) + \frac{1}{2}(c+a) \right) = \frac{1}{4}(2a + b + c),$$

as required.

**Lemma 2.15.** Let $R$ be a Minkowski rectangle. Then there exists a Minkowski circle centred at the mean centre of $R$ passing through each vertex.

**Proof.** Let $R = [a, b, c, d]$. By Theorem 2.4, $b$ and $d$ lie on the circle with diameter $[a, c]$. Its centre is $m(a, c)$, which is the mean centre of $R$ by Lemma 2.12.

**Lemma 2.16.** The midpoints of the sides of a quadrilateral form a parallelogram.

**Proof.** Consider the quadrilateral $[a, b, c, d]$ (where $a, b, c$ and $d$ are pairwise distinct). From Lemma 2.3, we know that $[m(a, b), m(b, c)]$ is parallel to $[a, c]$, and $[m(a, d), m(d, c)]$ is parallel to $[a, c]$, so by transitivity $[m(a, b), m(b, c)]$ and $[m(a, d), m(d, c)]$ are parallel. By symmetry, $[m(b, a), m(a, d)]$ and $[m(b, c), m(c, d)]$ are parallel.

From the proof of Lemma 2.16, we observe the following:

**Corollary 2.17.** Let $Q$ be a quadrilateral with no light-like sides or diagonals. If the diagonals of $Q$ are orthogonal, then the midpoints of each side form a rectangle.
3. Why seven eight nine

With all of the preliminaries in place we can now prove our main theorems.

**Eight-Point Circle Theorem.** Let \( Q = [a, b, c, d] \) be a quadrilateral with no light-like sides or diagonals and with \( a, b, c \) and \( d \) pairwise distinct. If the diagonals of \( Q \) are orthogonal, then there exists a circle, centred at the mean centre of \( Q \), passing through the eight points

\[
\begin{align*}
    m(a, b), & \quad \pi(m(a, b), [c, d]), \\
    m(b, c), & \quad \pi(m(b, c), [d, a]), \\
    m(c, d), & \quad \pi(m(c, d), [a, b]), \\
    m(d, a), & \quad \pi(m(d, a), [b, c]).
\end{align*}
\]

**Proof.** Combining Lemma 2.15 with Corollary 2.17, we know that there is a circle \( C \) centred at the mean centre of \( Q \) passing through the midpoints of each side of \( Q \). Now, it is clear from the definitions that \([m(a, b), \pi(m(a, b), [c, d])]\) and \([\pi(m(a, b), [c, d]), m(c, d)]\) are orthogonal. So, \( \pi(m(a, b), [c, d]) \) lies on the circle with diameter \([m(a, b), m(c, d)]\), which is exactly the circle \( C \). By symmetry, the remaining three points lie on \( C \). \( \square \)

**Nine-Point Circle Theorem.** Let \( T = [a, b, c] \) be a non-degenerate triangle with no light-like sides, and let \( h \) be the orthocentre of \( T \). Then there exists a circle, centred at the mean centre of \( a, b, c \) and \( h \), passing through the nine points

\[
\begin{align*}
    m(a, b), & \quad m(a, h), & \quad \pi(a, [b, c]), \\
    m(b, c), & \quad m(b, h), & \quad \pi(b, [c, a]), \\
    m(c, a), & \quad m(c, h), & \quad \pi(c, [a, b]).
\end{align*}
\]

**Proof.** Let \( n \) be the mean centre of \( a, b, c \) and \( h \).

Assume first that \( h \in \{a, b, c\} \); say \( h = a \). Then we have the following

\[
\begin{align*}
    \pi(b, [c, a]) = \pi(c, [a, b]) = m(a, h) = a, \\
    m(a, b) = m(b, h), \\
    m(c, a) = m(c, h),
\end{align*}
\]

and also, \([a, b]\) and \([c, a]\) are orthogonal. So, by Lemma 2.14 and Lemma 2.15, there is a circle \( C' \) passing through \( a = m(a, h), m(a, b), m(b, c), \) and \( m(c, a) \). The centre of \( C' \) is

\[
\frac{1}{4}(2a + b + c) = \frac{1}{4}(h + a + b + c) = n.
\]

Now, by Theorem 2.4, we have that \( \pi(a, [b, c]) \) lies on \( C' \). So, given the repetition of points, this completes the proof for the case where \( h \in \{a, b, c\} \).
Now, assume that \( h \notin \{a, b, c\} \), and consider the quadrilateral \([a, b, c, h]\). By the Eight-Point Theorem, there is a circle \( C \) centred at \( n \) passing through \( m(a, b) \), \( m(b, c) \), \( m(a, h) \), and \( m(c, h) \). By symmetry, we may interchange \( a \) and \( b \), which gives a circle \( C' \) centred at \( n \) passing through \( m(a, b) \), \( m(a, c) \), \( m(b, h) \), and \( m(c, h) \). As \( C' \) and \( C \) share the same centre and have two points in common, \( C' = C \). Now, the points \( m(b, c) \) and \( \pi(a, [b, c]) \) both lie on the line through \( b \) and \( c \), and the points \( \pi(a, [b, c]) \) and \( m(a, h) \) lie on the altitude of \( a \), so by definition of orthocentre, we have that \([m(b, c), \pi(a, [b, c])]\) and \([\pi(a, [b, c]), m(a, h)]\) are orthogonal. By Theorem 2.4, \( \pi(a, [b, c]) \in C \), and by symmetry, the remaining two points lie on \( C \).

\[ \square \]

**Figure 2.** Minkowski nine-point circle
REFERENCES