Differentiability of quasiconformal reflections

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Introduction

Let \( h \) be an increasing homeomorphism of the real axis onto itself, for which there exists a constant \( M \) such that

\[
M^{-1} \leq \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq M,
\]

for all real \( x \) and \( t > 0 \). The Beurling-Ahlfors extension of \( h \) to the upper half plane is given by \( \phi(x, y) \), where

\[
\begin{align*}
\text{Re}(\phi) &= \frac{1}{2} \int_0^1 [h(x + ty) + h(x - ty)] \, dt \\
\text{Im}(\phi) &= \frac{r}{2} \int_0^1 [h(x + ty) - h(x - ty)] \, dt.
\end{align*}
\]

(1)

This is shown in [4] to be quasiconformal for any fixed \( r > 0 \). Let \( L \) be a Jordan curve through infinity which divides the plane into the regions \( \Omega \) and \( \Omega^* \). A reflection over \( L \) is a map which swaps \( \Omega \), \( \Omega^* \) and keeps points on \( L \) fixed. Using (1) Ahlfors proved the following lemma in [1, 2]:

**Lemma 1.** If \( L \) permits a \( K \)-quasiconformal reflection, then it also permits a \( C(K) \) quasiconformal reflection which changes Euclidean lengths at most by a factor \( C(K) \).

It is additionally claimed in [2, p. 48] that the reflection is differentiable. Differentiability away from \( L \) follows directly from the proof of the lemma, but differentiability on \( L \) does not. The aim of this report is to show that the parameter choice \( r = 2 \) and some extra assumptions about \( L \) are required to be sure the reflection is differentiable on \( L \).

The matrix of partial derivatives of a function \( g : \mathbb{C} \to \mathbb{C} \) at \( z = x + iy \) will be denoted \( D_z(g) \) or \( D_{(x,y)}(g) \). The symbols \( f, f^* \) will be used throughout to denote conformal maps from the upper and lower half plane to \( \Omega \) and \( \Omega^* \). Both maps extend to homeomorphisms of the boundary by Carathodory’s Theorem (on conformal mapping). The symbol \( \omega \) will be used to denote reflections over \( L \).

Reflections over curves

**Example 1.** A reflection cannot be differentiable over sharp corners

Let \( \Omega^* \) be the open first quadrant, \( \Omega \) be the interior of its complement, and let \( \omega \) be a reflection over \( L = \partial \Omega \). By definition \( \omega \) is differentiable at the origin if

\[
\lim_{h \to 0} \frac{\|\omega(h) - \omega(0) - D_0(\omega) \cdot h\|}{\|h\|} = 0.
\]

(2)

Taking limits along the positive real and positive imaginary axis shows that

\[
D_0(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Also, \( \omega \) fixes the origin and so (2) is equivalent to

\[
\lim_{h \to 0} \frac{\|\omega(h) - h\|}{\|h\|} = 0.
\]

(3)
If $h \in \Omega$ is in the lower left quadrant, the origin is closer to $h$ than any point of $\Omega^\star$. As $\omega(h) \in \Omega^\star$, it follows that

$$\|\omega(h) - h\| \geq \|h\|.$$ 

Taking $h \to 0$ in the lower left quadrant then contradicts (3), which proves $\omega$ is not differentiable at 0. For this to show that differentiability of the reflection in Lemma 1 cannot be guaranteed, it remains to show that $L$ permits a quasiconformal reflection. This follows from Theorem 2 (Appendix) with the constant $C = 1$.

**Example 2.** Formula for a reflection over a sharp corner

As in Example 1, let $\Omega^\star$ be the open first quadrant, $\Omega$ the open part of its complement. The conformal maps $f$, $f^\star$ from the upper and lower half planes to $\Omega$, $\Omega^\star$ are

$$f(z) = iz^{3/2} \quad f^{-1}(z) = (-iz)^{2/3}$$
$$f^\star(z) = iz^{1/2} \quad f^{\star-1}(z) = -z^2,$$

where the branch cut for $z^{2/3}$ is taken on the positive real axis, and the branch cut for $\sqrt{z}$ is taken on the negative real axis but with $\sqrt{re^{i\pi}} := re^{-i\pi/2}$. The boundary function is $h(x) = (f^{\star-1} \circ f)(x) = x^3$. Its Beurling-Ahlfors extension is

$$\phi(x, y) = x^3 + xy^2 + \frac{r^i}{2} \left( 3x^2y + \frac{y^3}{2} \right) = \rho^3 \left[ \cos \theta + \frac{r^i}{16} (9 \sin \theta + 5 \sin 3\theta) \right],$$

where $x = \rho \cos \theta$, $y = \rho \sin \theta$. Hence

$$j \circ \phi \circ f^{-1} = \rho^2 \left[ \cos \left( \frac{2\theta - \pi}{3} \right) + \frac{r^i}{16} \left( 5 \sin 2\theta - 9 \sin \left( \frac{2\theta - \pi}{3} \right) \right) \right].$$

And in $\Omega$, $\omega = f^\star \circ j \circ \phi \circ f^{-1}$, so

$$\omega \left( \rho e^{i\theta} \right) = i\rho \sqrt{\cos \left( \frac{2\theta - \pi}{3} \right) + \frac{r^i}{16} \left( 5 \sin 2\theta - 9 \sin \left( \frac{2\theta - \pi}{3} \right) \right)},$$

where $\theta \in \left[ \frac{\pi}{2}, 2\pi \right]$ and $\sqrt{\rho e^{-i\pi/2}} := \sqrt{\rho} \cdot e^{-i\pi/2}$. The behaviour of $\omega$ is shown in Figure 1.

**Example 3.** A reflection over the right branch of a hyperbola

Let $\Omega$ be the region to the left of the right branch of the hyperbola $x^2 - y^2 = a^2$, $\Omega^\star$ the region to the right. From [3, p. 97] the conformal maps are

$$f(z) = \begin{cases} 
  a \sqrt{\frac{1}{2} \left[ iz^3 + 3iz + 2 \right]} & \text{if } \text{Re}(z) \geq 0, \\
  a \sqrt{\frac{1}{2} \left[ -iz^3 - 3iz + 2 \right]} & \text{if } \text{Re}(z) \leq 0
\end{cases}$$
Figure 1: The image of three quarters of a square under $\omega$. The code used to generate this figure is given in the Appendix.
and

\[ f^*(z) = \sqrt{iz + a^2}, \quad (f^*)^{-1}(z) = i(a^2 - z^2). \]

The boundary function is

\[ h(x) = (f^*)^{-1} \circ f(x) = \frac{a^2}{2}(x^3 + 3x), \]

for all \( x \in \mathbb{R} \). With \( r = 2 \) the Beurling-Ahlfors extension of \( h \) is

\[ \phi(x + iy) = \frac{a^2}{2} \left[ x(x^2 + y^2 + 3) + \frac{iy}{2}(6x^2 + y^2 + 6) \right]. \]

The image of \( \Omega \) under \( \omega = f^* \circ j \circ \phi \circ f^{-1} \) is shown in Figure 2. For points \((x, y)\) on the hyperbola, the derivative of the reflection tends to

\[ D_{(x,y)}(\omega) = \frac{1}{x^2 + y^2} \begin{pmatrix} y^2 - x^2 & 2xy \\ 2xy & x^2 - y^2 \end{pmatrix}, \tag{4} \]

which is involutary \((D^2 = I)\). The tangent to the hyperbola at \((x, y)\) has direction \((y, x)\), which is an eigenvector of \( D \) with eigenvalue 1. The function \( \omega \) thus locally reflects points over the tangent to the curve; this can be seen in Figure 2. The formula (4) will later be evident and does not need to be shown directly.
Theorem 1. Let $z_0$ be a point on $L$, let the partial derivatives of $f$ extend continuously to a real neighbourhood of $x_0 = f^{-1}(z_0)$, and likewise for $f^*$. If $f_x(x_0, 0) \neq 0$, then $\omega$ is $C^1$ at $z_0$ only if $r = 2$.

Proof. In $\Omega$, the reflection $\omega$ is the composition of four functions $f^* \circ j \circ \phi \circ f^{-1}$. The proof will extend the partial derivatives of each of these to their boundaries, and then extend the partial derivatives of $\omega$ to $L$ via the chain rule.

By assumption $f_x$ is continuous inside the rectangle $R$ with vertices $x_0 \pm \epsilon, x_0 + i \pm \epsilon$ for some $\epsilon > 0$. Applying Theorem 3 (Appendix) to the real and imaginary parts of $f$, with

$$g(x, y) = C = \max_R |f_x(x, y)|$$

results in

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h, 0) - f(x_0, 0)}{h} = \lim_{h \to 0} \lim_{y \to 0} \frac{f(x_0 + h, y) - f(x_0, y)}{h} = \lim_{h \to 0} \lim_{y \to 0} \int_0^1 f_x(x_0 + ht, y) \, dt = f_x(x_0, 0),$$

with this holding similarly for $f^*$. Applying the chain rule to $h = f \circ f^{-1}$ gives

$$f'(x_0) = (f^*)'(h(x_0)) \cdot h'(x_0),$$

(5)

and so $(f^*)'(h(x_0)) \neq 0$ by the assumption $f'(x_0) \neq 0$. The limiting matrix $D_{x_0}(f) := \lim_{z \to x_0} D_z(f)$ has determinant $|f'(x_0)|^2 \neq 0$, so by the Inverse Function Theorem

$$D_{z_0}(f^{-1}) := \lim_{z \to z_0} D_z(f^{-1}) = (D_{x_0}(f))^{-1},$$

where $z \to z_0$ inside $\Omega$. This ensures that the partial derivatives of $f^{-1}$ extend continuously at $z_0$.

If $\phi$ is the extension of $h$ to the upper half plane then from [2, p. 43] its partial derivatives are

$$\text{Re}(\phi)_x = \frac{1}{2y} (h(x + y) - h(x - y)),
\text{Re}(\phi)_y = -\frac{1}{2y^2} \int_{x-y}^{x+y} h \, dt + \frac{1}{2y} (h(x + y) + h(x - y)),
\text{Im}(\phi)_x = \frac{r}{2y} [h(x + y) - 2h(x) + h(x - y)],
\text{Im}(\phi)_y = \frac{r}{2} \left[ -\frac{1}{y^2} \left( \int_x^{x+y} h \, dt - \int_x^{x-y} h \, dt \right) + \frac{1}{y} (h(x + y) - h(x - y)) \right].$$

Applying L’Hôpital’s rule gives the limiting matrix of partial derivatives

$$D_{(x_0, y)}(j \circ \phi) \to \begin{pmatrix} h'(x_0) & 0 \\ 0 & \frac{-r}{2} h'(x_0) \end{pmatrix} \text{ as } y \to 0^+.$$
where \( j \) is the conjugate map \( z \to \bar{z} \). Denote the real and imaginary parts of \( f^{-1} \) and \( f^* \) by \( u, v \) and \( u^*, v^* \) respectively. Then as \( z \to z_0 \) inside \( \Omega \),

\[
\begin{align*}
  f^{-1}(z) & \to f^{-1}(z_0) = x_0, \\
  j \circ \phi \circ f^{-1}(z) & \to h(x_0).
\end{align*}
\]

Thus

\[
D_z(f^{-1}) \to \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix},
\]

\[
D_{f^{-1}(z)}(j \circ \phi) \to \begin{pmatrix} h'(x_0) & 0 \\ 0 & -\frac{1}{\bar{z}} h'(x_0) \end{pmatrix},
\]

\[
D_{j \circ \phi \circ f^{-1}(z)}(f^*) \to \begin{pmatrix} u_x^*(h(x_0)) & u_y^*(h(x_0)) \\ v_x^*(h(x_0)) & v_y^*(h(x_0)) \end{pmatrix}.
\]

as \( z \to z_0 \) inside \( \Omega \). The reflection formula is

\[
\omega := \begin{cases} f^* \circ j \circ \phi \circ f^{-1} & \text{in } \Omega \cup \mathbb{L} \\ f \circ \phi^{-1} \circ j \circ f^{-1} & \text{in } \Omega^* \cup \mathbb{L}. \end{cases}
\]

Applying the chain rule at \( z \in \Omega \) yields

\[
D_z(\omega) = D_{j \circ \phi \circ f^{-1}(z)}(f^*) \cdot D_{f^{-1}(z)}(j \circ \phi) \cdot D_z(f^{-1}),
\]

\[
\to \begin{pmatrix} u_x^*(h(x_0)) & u_y^*(h(x_0)) \\ v_x^*(h(x_0)) & v_y^*(h(x_0)) \end{pmatrix} \begin{pmatrix} h'(x_0) & 0 \\ 0 & -\frac{1}{\bar{z}} h'(x_0) \end{pmatrix} \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix},
\]

which when squared becomes

\[
D_{z_0}(\omega)^2 = (u_x^2 + u_y^2) \cdot (u_x^2 + u_y^2) : h'(x_0)^2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Differentiating \( f^{-1} \circ f^* \circ h(x) = x \) at \( x = x_0 \) gives

\[
[u_x - iu_y] \cdot [u_x^* - iu_y^*] \cdot h'(x_0) = 1.
\]

Equating moduli and squaring results in

\[
D_{z_0}(\omega)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

that is, the limiting derivative matrix is its own inverse. By the Inverse Function Theorem, and the fact that \( \omega \) is its own inverse,

\[
D_{\omega(z)}(\omega) = (D_z(\omega))^{-1},
\]

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for $z$ not on $L$. As $\omega$ fixes points on $L$, combining (10) and (11) shows that $D_z(\omega) \to D_{z_0}(\omega)$ as $z \to z_0$ from inside $\Omega$ or $\Omega^*$. The formula $\det(AB) = \det(A) \det(B)$ combined with (7) implies that $r = 2$ is the only choice that makes $D_{z_0}(\omega)$ involutary, which means $\omega$ is not $C^1$ at $z_0$ for $r \neq 2$. We now show that $D_z(\omega) \to D_{z_0}(\omega)$ also holds as $z \to z_0$ along $L$. The form of (8) means $D_{z_0}(\omega)$ must reflect points over some axis. By (5) we know that $f''(h(x_0)) \neq 0$, so the tangent to $L$ at $z_0$ has direction $f''(h(x_0)) = \left(\frac{u^*_x}{u^*_y}\right)' = \left(\frac{u^*_x}{-u^*_y}\right)$. Multiplication gives

$$D_{z_0}(\omega) \cdot \begin{pmatrix} u^*_x \\ -u^*_y \end{pmatrix} = h'(x_0) \cdot \begin{pmatrix} u^*_x \cdot u_x + u^*_y \cdot u_y \\ u^*_x \cdot u_y - u^*_y \cdot u_x \\ -u^*_x \cdot u_x - u^*_y \cdot u_y \end{pmatrix} \cdot \begin{pmatrix} u^*_x \\ -u^*_y \end{pmatrix},$$

$$= h'(x_0) \cdot (u^*_x + u^*_y) \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Rearranging (9) gives

$$h'(x_0)(u_x - iu_y) = \frac{1}{u^*_x - iu^*_y}.$$

Equating real and imaginary parts of this results in

$$h'(x_0) \cdot u_x \cdot (u^*_x + u^*_y) = u^*_x, \quad h'(x_0) \cdot u_y \cdot (u^*_x + u^*_y) = -u^*_y,$$

and so

$$D_{z_0}(\omega) \cdot \begin{pmatrix} u^*_x \\ -u^*_y \end{pmatrix} = \begin{pmatrix} u^*_x \\ -u^*_y \end{pmatrix},$$

which means the axis of reflection is the tangent to $L$ at $z_0$. As $L$ has a continuously turning tangent at $z_0$, this proves $D_z(\omega) \to D_{z_0}(\omega)$ as $z \to z_0$ along $L$, and thus $D_{z_0}(\omega)$ defines a continuous extension of the partial derivatives of $\omega$ to $L$. It remains to show the extension matches the usual definition of partial derivatives, for example

$$\lim_{h \to 0} \frac{\omega(z_0 + h) - \omega(z_0)}{h} - \omega_x(z_0) = 0. \quad (12)$$

where $h$ is real. Assume without loss of generality that $L$ is not parallel to the $x$-axis at $z_0$. Then for $h$ small enough, $z_0 + h$ will not lie on $L$. Eq.(12) now follows by L'Hôpital's rule; $\omega_y(z_0)$ may be verified in the same way. As the partial derivatives of $\omega$ exist and are continuous in neighbourhood of $z_0$, $\omega$ is $C^1$ at $z_0$.

**Corollary 1.** If $L$ is Dini-smooth and $r = 2$, then $\omega$ is $C^1$ everywhere.

**Proof.** If $L$ is Dini-smooth and $r = 2$, the assumptions of Theorem 1 all hold by Theorem 4 (Appendix).

Every $C^1$ curve with Hölder continuous derivative is Dini-smooth, so a less general version of this is:

**Corollary 2.** If $L \in C^{1, \alpha}$ for some $0 < \alpha \leq 1$, then $\omega$ is $C^1$ everywhere for $r = 2$.  

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Acknowledgements
I would like to thank Prof. Michael Cowling for his time and effort spent supervising this project, and AMSI for funding and for organising the Big Day In.

Appendix

**Theorem 2.** A Jordan curve $L$ through $\infty$ permits a quasiconformal reflection if and only if there exists a constant $C$ such that
$$|\zeta_1 - \zeta_2| \leq C|\zeta_1 - \zeta_3|,$$
for any three points $\zeta_1, \zeta_2, \zeta_3$ on $L$ which follow each other in this order.

**Theorem 3.** Let $f_n$ be a sequence of real-valued measurable functions on a measure space $S$. Suppose that $f_n \to f$ pointwise and $f_n(x) \leq g(x)$ for all $x \in S$ and for all $n$, for some integrable function $g$. Then $f$ is integrable and
$$\lim_{n \to \infty} \int_S f_n(x) d\mu = \int_S f(x) d\mu = \int_S \lim_{n \to \infty} f_n(x) d\mu.$$

**Definition 1.** Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous. The modulus of continuity of $\gamma$ is
$$\omega(t) = \sup_{|x-y| \leq t} |\gamma(x) - \gamma(y)|.$$
Then $\gamma$ is called Dini-continuous if
$$\int_0^1 \frac{\omega(t)}{t} \, dt < \infty.$$
A $C^1$ curve with regular parametrisation $\gamma$ is Dini-smooth if $\gamma'$ is Dini-continuous.

**Theorem 4.** Let $f$ map $\mathbb{D}$ conformally onto the inner domain of the Dini-smooth Jordan curve $C$. Then $f'$ has a continuous extension to $\overline{\mathbb{D}}$ and
$$\frac{f(\zeta) - f(z)}{\zeta - z} \to f'(z) \neq 0 \quad \text{for} \quad \zeta \to z, \zeta, z \in \mathbb{D},$$
$$|f'(z_1) - f'(z_2)| \leq C \omega^*(\delta) \quad \text{for} \quad z_1, z_2 \in \overline{\mathbb{D}}, |z_1 - z_2| \leq \delta.$$

Theorem 4 and the definition of Dini-smooth are from [5, p. 48]. Theorem 2 is Ahlfors’ three point condition from [1]. The letter $D$ in Theorem 4 denotes the unit disc.

Mathematica code used to generate Figure 1

```math
a = 2;
arg2[z_] := Mod[Arg[z], -2*Pi, 2*Pi] / 3
ct2[z_] := Abs[z]^(2/3) * Exp[I*2*arg2[z]/3]
finv[z_] := ct2[-I*z]
fstar[z_] := I*Sqrt[z]
jphi[z_] := Abs[z]^2*Re[z] - a*I/2*(3*Re[z]^2*Im[z] + 1/2*Im[z]^3)
```
Mathematica code used to generate Figure 2

\[
\begin{align*}
\omega[z] & := N[fstar[jphi[finv[z]]]] \\
\text{omega2}[z] & := N[fstar[jphi[finv2[z]]]] \\
\end{align*}
\]

\[
\text{domain} = \text{ParametricPlot}[\]
\begin{align*}
&\text{Evaluate@Through[{Re, Im}[x + I*y]]} \text{Boole[} \\
&\quad x < 0 \text{ || } y < 0, \{x, -1, 1\}, \{y, -1, 1\}, \text{AspectRatio} \to 1, \\
&\quad \text{PlotRange} \to \text{All}, \text{PlotPoints} \to 80, \text{PlotStyle} \to \text{Red}, \\
&\quad \text{BoundaryStyle} \to \text{None}; \\
&\text{domain} = \text{ParametricPlot}[\]
\end{align*}
\]

\[
\begin{align*}
\text{image} & = \text{ParametricPlot}[\]
\end{align*}
\]

\[
\begin{align*}
&\text{image} = \text{ParametricPlot}[\]
\end{align*}
\]

\[
\begin{align*}
\text{upperimage} & = \text{ParametricPlot}[\]
\end{align*}
\]

\[
\begin{align*}
\text{lowerimage} & = \text{ParametricPlot}[\]
\end{align*}
\]

\[
\begin{align*}
\text{domain} & = \text{ParametricPlot}[\]
\end{align*}
\]
References


