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Differentiability of quasiconformal reflections

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Introduction

Let h be an increasing homeomorphism of the real axis onto itself, for which there exists a constant M such that

$$M^{-1} \leq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leq M,$$

for all real x and $t > 0$. The Beurling-Ahlfors extension of h to the upper half plane is given by $\phi(x, y)$, where

$$\begin{aligned} \mathrm{Re}(\phi) &= \frac{1}{2}\int_0^1 [h(x+ty)+h(x-ty)]\, dt \\ \mathrm{Im}(\phi) &= \frac{r}{2}\int_0^1 [h(x+ty)-h(x-ty)]\, dt. \end{aligned} \tag{1}$$

This is shown in [4] to be quasiconformal for any fixed $r > 0$. Let L be a Jordan curve through infinity which divides the plane into the regions Ω and Ω^*. A reflection over L is a map which swaps Ω, Ω^* and keeps points on L fixed. Using (1) Ahlfors proved the following lemma in [1, 2]:

Lemma 1. *If L permits a K-quasiconformal reflection, then it also permits a $C(K)$ quasiconformal reflection which changes Euclidean lengths at most by a factor $C(K)$.*

It is additionally claimed in [2, p. 48] that the reflection is differentiable. Differentiability away from L follows directly from the proof of the lemma, but differentiability on L does not. The aim of this report is to show that the parameter choice $r = 2$ and some extra assumptions about L are required to be sure the reflection is differentiable on L.
The matrix of partial derivatives of a function $g : \mathbb{C} \to \mathbb{C}$ at $z = x + iy$ will be denoted $D_z(g)$ or $D_{(x,y)}(g)$. The symbols f, f^* will be used throughout to denote conformal maps from the upper and lower half plane to Ω and Ω^*. Both maps extend to homeomorphisms of the boundary by Carathodory's Theorem (on conformal mapping). The symbol ω will be used to denote reflections over L.

Reflections over curves

Example 1. *A reflection cannot be differentiable over sharp corners*

Let Ω^* be the open first quadrant, Ω be the interior of its complement, and let ω be a reflection over $L = \partial\Omega$. By definition ω is differentiable at the origin if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\omega(\mathbf{h}) - \omega(\mathbf{0}) - D_0(\omega)\cdot\mathbf{h}\|}{\|\mathbf{h}\|} = 0. \tag{2}$$

Taking limits along the positive real and positive imaginary axis shows that

$$D_0(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also, ω fixes the origin and so (2) is equivalent to

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\omega(\mathbf{h}) - \mathbf{h}\|}{\|\mathbf{h}\|} = 0. \tag{3}$$

If $\mathbf{h} \in \Omega$ is in the lower left quadrant, the origin is closer to $\mathbf{h}$ than any point of Ω^*. As $\omega(\mathbf{h}) \in \Omega^*$, it follows that

$$\|\omega(\mathbf{h}) - \mathbf{h}\| \geq \|\mathbf{h}\|.$$

Taking $\mathbf{h} \to \mathbf{0}$ in the lower left quadrant then contradicts (3), which proves ω is not differentiable at $\mathbf{0}$. For this to show that differentiability of the reflection in Lemma 1 cannot be guaranteed, it remains to show that L permits a quasiconformal reflection. This follows from Theorem 2 (Appendix) with the constant $C = 1$.

Example 2. *Formula for a reflection over a sharp corner*

As in Example 1, let Ω^* be the open first quadrant, Ω the open part of its complement. The conformal maps f, f^* from the upper and lower half planes to Ω, Ω^* are

$$\begin{aligned} f(z) &= iz^{3/2} \quad & f^{-1}(z) &= (-iz)^{2/3} \\ f^*(z) &= iz^{1/2} \quad & f^{*-1}(z) &= -z^2, \end{aligned}$$

where the branch cut for $z^{2/3}$ is taken on the positive real axis, and the branch cut for $\sqrt{z}$ is taken on the negative real axis but with $\sqrt{re^{i\pi}} := re^{-i\pi/2}$. The boundary function is $h(x) = (f^{*-1} \circ f)(x) = x^3$. Its Beurling-Ahlfors extension is

$$\begin{aligned} \phi(x, y) &= x^3 + xy^2 + \frac{ri}{2}\left(3x^2 y + \frac{y^3}{2}\right) \\ &= \rho^3 \left[\cos\theta + \frac{ri}{16}\left(9\sin\theta + 5\sin 3\theta\right)\right], \end{aligned}$$

where $x = \rho\cos\theta$, $y = \rho\sin\theta$. Hence

$$j \circ \phi \circ f^{-1} = \rho^2 \left[\cos\left(\frac{2\theta - \pi}{3}\right) + \frac{ri}{16}\left(5\sin 2\theta - 9\sin\left(\frac{2\theta - \pi}{3}\right)\right)\right].$$

And in Ω, $\omega = f^* \circ j \circ \phi \circ f^{-1}$, so

$$\omega\left(\rho e^{i\theta}\right) = i\rho\sqrt{\cos\left(\frac{2\theta - \pi}{3}\right) + \frac{ri}{16}\left(5\sin 2\theta - 9\sin\left(\frac{2\theta - \pi}{3}\right)\right)},$$

where $\theta \in [\frac{\pi}{2}, 2\pi]$ and $\sqrt{\rho e^{-i\pi}} := \sqrt{\rho} \cdot e^{-i\pi/2}$. The behaviour of ω is shown in Figure 1.

Example 3. *A reflection over the right branch of a hyperbola*

Let Ω be the region to the left of the right branch of the hyperbola $x^2 - y^2 = a^2$, Ω^* the region to the right. From [3, p. 97] the conformal maps are

$$f(z) = \begin{cases} a\sqrt{\frac{1}{2}\left[iz^3 + 3iz + 2\right]} & \text{if } \operatorname{Re}(z) \geq 0, \\ \overline{a\sqrt{\frac{1}{2}\left[-i\bar{z}^3 - 3i\bar{z} + 2\right]}} & \text{if } \operatorname{Re}(z) \leq 0 \end{cases}$$

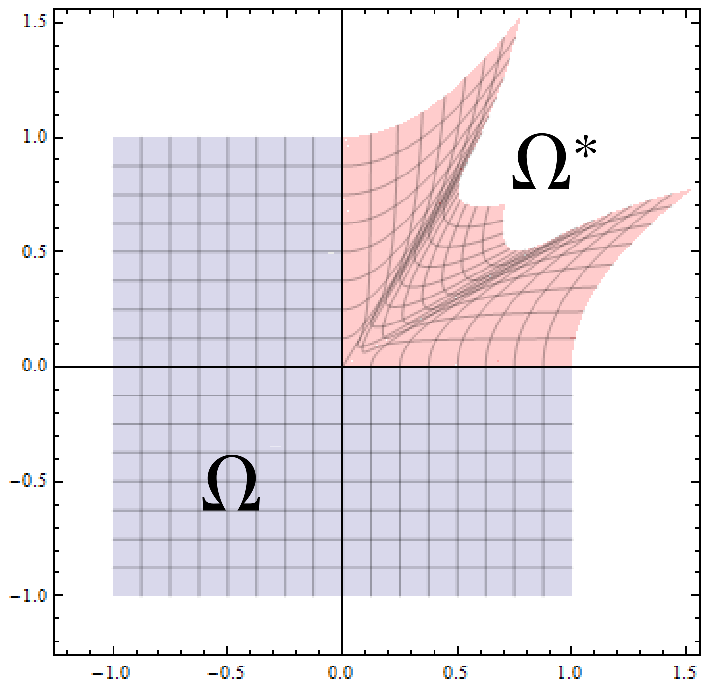

Figure 1: The image of three quarters of a square under ω. The code used to generate this figure is given in the Appendix.

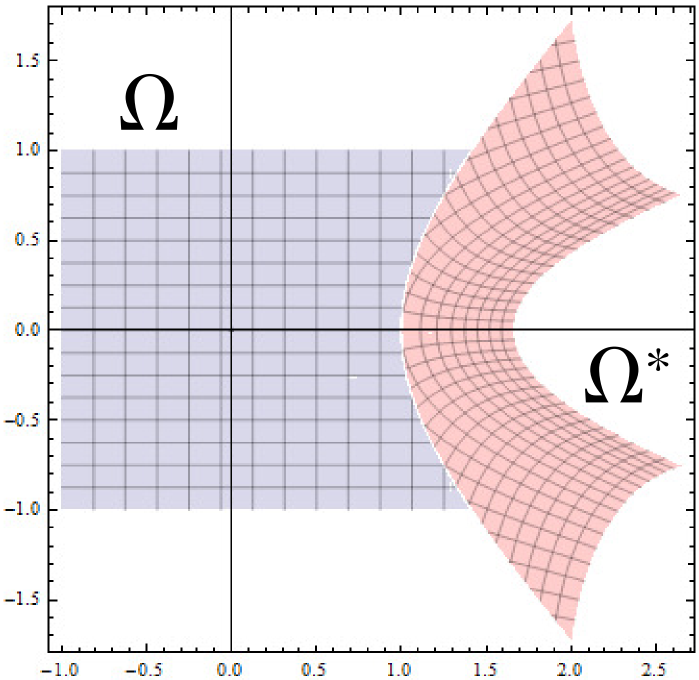

Figure 2: The reflection of the left side of a hyperbola onto the right. The code used to generate this figure is given in the Appendix.

and

$$f^*(z) = \sqrt{iz + a^2}, \quad (f^*)^{-1}(z) = i(a^2 - z^2).$$

The boundary function is

$$h(x) = (f^*)^{-1} \circ f(x) = \frac{a^2}{2}(x^3 + 3x),$$

for all $x \in \mathbb{R}$. With $r = 2$ the Beurling-Ahlfors extension of h is

$$\phi(x + iy) = \frac{a^2}{2}\left[x(x^2 + y^2 + 3) + \frac{iy}{2}(6x^2 + y^2 + 6)\right].$$

The image of Ω under $\omega = f^* \circ j \circ \phi \circ f^{-1}$ is shown in Figure 2. For points (x, y) on the hyperbola, the derivative of the reflection tends to

$$D_{(x,y)}(\omega) = \frac{1}{x^2 + y^2}\begin{pmatrix} y^2 - x^2 & 2xy \\ 2xy & x^2 - y^2 \end{pmatrix}, \tag{4}$$

which is involutary ($D^2 = I$). The tangent to the hyperbola at (x, y) has direction (y, x), which is an eigenvector of D with eigenvalue 1. The function ω thus locally reflects points over the tangent to the curve; this can be seen in Figure 2. The formula (4) will later be evident and does not need to be shown directly.

Theorem 1. *Let z_0 be a point on L, let the partial derivatives of f extend continuously to a real neighbourhood of $x_0 = f^{-1}(z_0)$, and likewise for f^*. If $f_x(x_0, 0) \neq 0$, then ω is C^1 at z_0 only if $r = 2$.*

Proof. In Ω, the reflection ω is the composition of four functions $f^* \circ j \circ \phi \circ f^{-1}$. The proof will extend the partial derivatives of each of these to their boundaries, and then extend the partial derivatives of ω to L via the chain rule.

By assumption f_x is continuous inside the rectangle R with vertices $x_0 \pm \epsilon$, $x_0 + i \pm \epsilon$ for some $\epsilon > 0$. Applying Theorem 3 (Appendix) to the real and imaginary parts of f, with

$$g(x, y) = C = \max_R |f_x(x, y)|$$

results in

$$\begin{aligned} f'(x_0) &:= \lim_{h \to 0} \frac{f(x_0 + h, 0) - f(x_0, 0)}{h} \\ &= \lim_{h \to 0} \lim_{y \to 0} \frac{f(x_0 + h, y) - f(x_0, y)}{h} \\ &= \lim_{h \to 0} \lim_{y \to 0} \int_0^1 f_x(x_0 + ht, y)\, dt \\ &= f_x(x_0, 0), \end{aligned}$$

with this holding similarly for f^*. Applying the chain rule to $h = f \circ f^{*-1}$ gives

$$f'(x_0) = (f^*)'(h(x_0)) \cdot h'(x_0), \tag{5}$$

and so $(f^*)'(h(x_0)) \neq 0$ by the assumption $f'(x_0) \neq 0$. The limiting matrix $D_{x_0}(f) := \lim_{z \to x_0} D_z(f)$ has determinant $|f'(x_0)|^2 \neq 0$, so by the Inverse Function Theorem

$$D_{z_0}(f^{-1}) := \lim_{z \to z_0} D_z(f^{-1}) = (D_{x_0}(f))^{-1},$$

where $z \to z_0$ inside Ω. This ensures that the partial derivatives of f^{-1} extend continuously at z_0. If ϕ is the extension of h to the upper half plane then from [2, p. 43] its partial derivatives are

$$\begin{aligned} \mathrm{Re}(\phi)_x &= \frac{1}{2y} \left(h(x + y) - h(x - y) \right), \\ \mathrm{Re}(\phi)_y &= -\frac{1}{2y^2} \int_{x-y}^{x+y} h\, dt + \frac{1}{2y} \left(h(x + y) + h(x - y) \right), \\ \mathrm{Im}(\phi)_x &= \frac{r}{2y} [h(x + y) - 2h(x) + h(x - y)], \\ \mathrm{Im}(\phi)_y &= \frac{r}{2} \left[-\frac{1}{y^2} \left(\int_x^{x+y} h\, dt - \int_{x-y}^{x} h\, dt \right) + \frac{1}{y} (h(x + y) - h(x - y)) \right]. \end{aligned}$$

Applying L'Hôpital's rule gives the limiting matrix of partial derivatives

$$D_{(x_0, y)}(j \circ \phi) \to \begin{pmatrix} h'(x_0) & 0 \\ 0 & \frac{-r}{2} h'(x_0) \end{pmatrix} \text{ as } y \to 0^+,$$

where j is the conjugate map $z \to \overline{z}$. Denote the real and imaginary parts of f^{-1} and f^* by u, v and u^*, v^* respectively. Then as $z \to z_0$ inside Ω,

$$\begin{aligned} f^{-1}(z) &\to f^{-1}(z_0) = x_0 \\ j \circ \phi \circ f^{-1}(z) &\to h(x_0). \end{aligned}$$

Thus

$$\begin{aligned} D_z(f^{-1}) &\to \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix} \\ D_{f^{-1}(z)}(j \circ \phi) &\to \begin{pmatrix} h'(x_0) & 0 \\ 0 & \frac{-r}{2} h'(x_0) \end{pmatrix}. \\ D_{j \circ \phi \circ f^{-1}(z)}(f^*) &\to \begin{pmatrix} u_x^*(h(x_0)) & u_y^*(h(x_0)) \\ v_x^*(h(x_0)) & v_y^*(h(x_0)) \end{pmatrix}, \end{aligned} \tag{6}$$

as $z \to z_0$ inside Ω. The reflection formula is

$$\omega := \begin{cases} f^* \circ j \circ \phi \circ f^{-1} & \text{in } \Omega \cup L \\ f \circ \phi^{-1} \circ j \circ f^{*-1} & \text{in } \Omega^* \cup L. \end{cases}$$

Applying the chain rule at $z \in \Omega$ yields

$$\begin{aligned} D_z(\omega) &= D_{j \circ \phi \circ f^{-1}(z)}(f^*) \cdot D_{f^{-1}(z)}(j \circ \phi) \cdot D_z(f^{-1}), \\ &\to \begin{pmatrix} u_x^*(h(x_0)) & u_y^*(h(x_0)) \\ v_x^*(h(x_0)) & v_y^*(h(x_0)) \end{pmatrix} \begin{pmatrix} h'(x_0) & 0 \\ 0 & \frac{-r}{2} h'(x_0) \end{pmatrix} \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix} \end{aligned} \tag{7}$$

as $z \to z_0$ inside Ω, by (6). Setting $r = 2$ and omitting arguments reduces this to

$$D_{z_0}(\omega) := \lim_{z \to z_0, z \in \Omega} D_{(z)}(\omega) = h'(x_0) \cdot \begin{pmatrix} u_x^* \cdot u_x + u_y^* \cdot u_y & u_x^* \cdot u_y - u_y^* \cdot u_x \\ \\ u_x^* \cdot u_y - u_y^* \cdot u_x & -u_x^* \cdot u_x - u_y^* \cdot u_y \end{pmatrix}, \tag{8}$$

which when squared becomes

$$D_{z_0}(\omega)^2 = (u_x^2 + u_y^2) \cdot (u_x^{*2} + u_y^{*2}) \cdot h'(x_0)^2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Differentiating $f^{-1} \circ f^* \circ h(x) = x$ at $x = x_0$ gives

$$[u_x - i u_y] \cdot \left[u_x^* - i u_y^*\right] \cdot h'(x_0) = 1. \tag{9}$$

Equating moduli and squaring results in

$$D_{z_0}(\omega)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{10}$$

that is, the limiting derivative matrix is its own inverse. By the Inverse Function Theorem, and the fact that ω is its own inverse,

$$D_{\omega(z)}(\omega) = (D_z(\omega))^{-1}, \tag{11}$$

for z not on L. As ω fixes points on L, combining (10) and (11) shows that $D_z(\omega) \to D_{z_0}(\omega)$ as $z \to z_0$ from inside Ω or Ω^*. The formula $\det(AB) = \det(A)\det(B)$ combined with (7) implies that $r = 2$ is the only choice that makes $D_{z_0}(\omega)$ involutary, which means ω is not C^1 at z_0 for $r \neq 2$. We now show that $D_z(\omega) \to D_{z_0}(\omega)$ also holds as $z \to z_0$ along L. The form of (8) means $D_{z_0}(\omega)$ must reflect points over some axis. By (5) we know that $f^{*'}(h(x_0)) \neq 0$, so the tangent to L at z_0 has direction $f^{*'}(h(x_0)) = \begin{pmatrix} u_x^* \\ v_x^* \end{pmatrix} = \begin{pmatrix} u_x^* \\ -u_y^* \end{pmatrix}$. Multiplication gives

$$\begin{aligned} D_{z_0}(\omega) \cdot \begin{pmatrix} u_x^* \\ -u_y^* \end{pmatrix} &= h'(x_0) \cdot \begin{pmatrix} u_x^* \cdot u_x + u_y^* \cdot u_y & u_x^* \cdot u_y - u_y^* \cdot u_x \\ u_x^* \cdot u_y - u_y^* \cdot u_x & -u_x^* \cdot u_x - u_y^* \cdot u_y \end{pmatrix} \cdot \begin{pmatrix} u_x^* \\ -u_y^* \end{pmatrix}, \\ &= h'(x_0) \cdot (u_x^{*2} + u_y^{*2}) \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix}. \end{aligned}$$

Rearranging (9) gives

$$h'(x_0)(u_x - iu_y) = \frac{1}{u_x^* - iu_y^*}.$$

Equating real and imaginary parts of this results in

$$h'(x_0) \cdot u_x \cdot (u_x^{*2} + u_y^{*2}) = u_x^*, \quad h'(x_0) \cdot u_y \cdot (u_x^{*2} + u_y^{*2}) = -u_y^*,$$

and so

$$D_{z_0}(\omega) \cdot \begin{pmatrix} u_x^* \\ -u_y^* \end{pmatrix} = \begin{pmatrix} u_x^* \\ -u_y^* \end{pmatrix},$$

which means the axis of reflection is the tangent to L at z_0. As L has a continuously turning tangent at z_0, this proves $D_z(\omega) \to D_{z_0}(\omega)$ as $z \to z_0$ along L, and thus $D_{z_0}(\omega)$ defines a continuous extension of the partial derivatives of ω to L. It remains to show the extension matches the usual definition of partial derivatives, for example

$$\lim_{h \to 0} \left[\frac{\omega(z_0 + h) - \omega(z_0)}{h} - \omega_x(z_0) \right] = 0. \tag{12}$$

where h is real. Assume without loss of generality that L is not parallel to the x-axis at z_0. Then for h small enough, $z_0 + h$ will not lie on L. Eq.(12) now follows by L'Hôpital's rule; $\omega_y(z_0)$ may be verified in the same way. As the partial derivatives of ω exist and are continuous in neighbourhood of z_0, ω is C^1 at z_0. □

Corollary 1. *If L is Dini-smooth and $r = 2$, then ω is C^1 everywhere.*

Proof. If L is Dini-smooth and $r = 2$, the assumptions of Theorem 1 all hold by Theorem 4 (Appendix). □

Every C^1 curve with Hölder continuous derivative is Dini-smooth, so a less general version of this is:

Corollary 2. *If $L \in C^{1,\alpha}$ for some $0 < \alpha \leq 1$, then ω is C^1 everywhere for $r = 2$.*

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Appendix

Theorem 2. *A Jordan curve L through ∞ permits a quasiconformal reflection if and only if there exists a constant C such that*

$$|\zeta_1 - \zeta_2| \le C|\zeta_1 - \zeta_3|,$$

for any three points $\zeta_1, \zeta_2, \zeta_3$ on L which follow each other in this order.

Theorem 3. *Let f_n be a sequence of real-valued measurable functions on a measure space S. Suppose that $f_n \to f$ pointwise and $f_n(x) \le g(x)$ for all $x \in S$ and for all n, for some integrable function g. Then f is integrable and*

$$\lim_{n\to\infty} \int_S f_n(x) d\mu = \int_S f(x) d\mu = \int_S \lim_{n\to\infty} f_n(x) d\mu.$$

Definition 1. *Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous. The modulus of continuity of γ is*

$$\omega(t) = \sup_{|x-y|\le t} |\gamma(x) - \gamma(y)|.$$

Then γ is called Dini-continuous if

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty.$$

A C^1 curve with regular parametrisation γ is Dini-smooth if γ' is Dini-continuous.

Theorem 4. *Let f map $\mathbb{D}$ conformally onto the inner domain of the Dini-smooth Jordan curve C. Then f' has a continuous extension to $\overline{\mathbb{D}}$ and*

$$\frac{f(\zeta) - f(z)}{\zeta - z} \to f'(z) \ne 0 \quad \textit{for} \quad \zeta \to z, \zeta, z \in \overline{\mathbb{D}},$$

$$|f'(z_1) - f'(z_2)| \le C\omega^*(\delta) \quad \textit{for} \quad z_1, z_2 \in \overline{\mathbb{D}}, |z_1 - z_2| \le \delta.$$

Theorem 4 and the definition of Dini-smooth are from [5, p. 48]. Theorem 2 is Ahlfors' three point condition from [1]. The letter $\mathbb{D}$ in Theorem 4 denotes the unit disc.

Mathematica code used to generate Figure 1

```
a = 2;
arg2[z_] := Mod[Arg[z], -2*Pi, 2 *Pi]
crt2[z_] := Abs[z]^(2/3) * Exp[I*2*arg2[z]/3]
finv[z_] := crt2[-I*z]
fstar[z_] := I*Sqrt[z]
jphi[z_] := Abs[z]^2*Re[z] - a*I/2*(3*Re[z]^2*Im[z] + 1/2*Im[z]^3)
```

```
omega[z_] := N[fstar[jphi[finv[z]]]]
image = ParametricPlot[
   Evaluate@(Through[{Re, Im}[omega[x + I*y]]] Boole[
       x < 0 || y < 0]), {x, -1, 1}, {y, -1, 1}, AspectRatio -> 1,
   PlotRange -> All, PlotPoints -> 80, PlotStyle -> Red,
   BoundaryStyle -> None];
domain = ParametricPlot[
   Evaluate@(Through[{Re, Im}[x + I*y]] Boole[
       x < 0 || y < 0]), {x, -1, 1}, {y, -1, 1}, AspectRatio -> 1,
   PlotPoints -> 80, BoundaryStyle -> None];
Rasterize[
 Show[domain, image, PlotRange -> {{-1.2, 1.5}, {-1.2, 1.5}}]]
```

Mathematica code used to generate Figure 2

```
a = 1; (*a determines the vertex of the hyperbola *)
w /. Simplify[Solve[(I*w^3 + 3*I*w + 2) == 2*z^2/a^2, w],
    Assumptions -> {a > 0}][[1]];
finv[z_] := (
  I (1 + I Sqrt[
      3] + (1 - I Sqrt[3]) (1 - z^2 + Sqrt[z^2 (-2 + z^2)])^(2/3)))/(
  2 (1 - z^2 + Sqrt[z^2 (-2 + z^2)])^(1/3));
finv2[z_] := (
  I (1 - I Sqrt[
      3] + (1 + I Sqrt[3]) (1 - z^2 + Sqrt[z^2 (-2 + z^2)])^(2/3)))/(
  2 (1 - z^2 + Sqrt[z^2 (-2 + z^2)])^(1/3));
jphi[z_] := (a^2/2)*(Re[z]*(Abs[z]^2 + 3) -
    I*Im[z]/2*(Abs[z]^2 + 5*Re[z]^2 + 6))
fstar[z_] := Sqrt[I*z + a^2]
omega[z_] := N[fstar[jphi[finv[z]]]]
omega2[z_] := N[fstar[jphi[finv2[z]]]]
upperimage =
  ParametricPlot[
   Evaluate@(Through[{Re, Im}[omega[x + I*y]]] Boole[
       x < a || (x^2 - y^2 < a^2)]), {x, 0, 2}, {y, 0, Sqrt[3]},
   AspectRatio -> 1, PlotRange -> All, PlotStyle -> Red,
   BoundaryStyle -> None, PlotPoints -> 20];
lowerimage =
  ParametricPlot[
   Evaluate@(Through[{Re, Im}[omega2[x + I*y]]] Boole[
       x < a || (x^2 - y^2 < a^2)]), {x, 0, 2}, {y, -Sqrt[3], 0},
   AspectRatio -> 1, PlotRange -> All, PlotStyle -> Red,
   BoundaryStyle -> None, PlotPoints -> 20];
domain = ParametricPlot[
   Evaluate@(Through[{Re, Im}[x + I*y]] Boole[
       x < a || (x^2 - y^2 < a^2)]), {x, -1, 2}, {y, -1, 1},
```

```
   AspectRatio -> 1, BoundaryStyle -> None, PlotPoints -> 20];
Rasterize[Show[domain, upperimage, lowerimage, PlotRange -> All]]
```

References

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