Knots and Sticks

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Abstract

Knot theory is an important area of mathematics with an abundance of properties and uses yet to be discovered. This report will introduce a few elementary ideas of knot theory to give a basic understanding of what the study of knots is about. It will then delve deeper into the idea of the stick number and explore some proofs relevant to stick knots.


1 Introduction

Knot theory is an interesting area of mathematics that does not rely upon a mathematical background to fully grasp the ideas; one must merely have a mind for visualising three dimensional objects. The study of knots in a mathematical context was first introduced by Lord Kelvin in the 1800’s [5] as a proposed model for the structure of atoms, where each distinct knot would represent a distinct atom. Although proven wrong, this area has evolved and progressed greatly over the past two centuries and has many practical applications, particularly in synthetic chemistry [10]. In essence, knot theory is the study of the properties and cataloguing of mathematical knots. When one thinks of a knot one generally imagines tying their shoelaces, however a mathematical knot is not defined by its "knottiness" but rather by the topological properties of the knot embedding in three dimensions. Through this report the reader will gain a brief understanding of the workings of knot theory and why it is important to study. The reader will then be introduced to the idea of stick knots and the stick number, and a special category of knots known as torus knots and how one can go about finding stick numbers for these knots.

2 A Brief Introduction to Knot Theory

To understand knot theory, one must understand what it means to be a mathematical knot. This gives rise to the first definition:

Definition 2.1. A knot $\mathcal{K}$ is a knotted closed loop in 3 dimensions [2].

However, a more mathematical, and possibly more useful, definition may be given as follows:
Definition 2.2. A knot $K$ is a non-self intersecting embedding of $S^1$ (where $S^1$ is the one-sphere, more commonly known as a circle) in $\mathbb{R}^3$ [2].

By this definition, the most basic of knots is the unknot which consists of a simple closed circle in 3 dimensions. This gives rise to the question however: How do we determine that a given knot is not actually just the unknot?

Definition 2.3. A projection of a knot $K$ is a 2 dimensional projection of $K$ that is injective up to a finite number of crossing points.

![Figure 2.1: Two projections of the unknot](image)

To answer this question one needs to understand how a knot can be manipulated and moved in space without changing its key properties. To do this one is allowed to perform what is known as a Reidemeister move [9], of which there are three types, on any projection of a knot. The first Reidemeister move allows you to untwist a 'free' loop [figure 2.2]. The second allows you to move a string of the knot over another as long as it is 'free'. Finally, the third allows us to extend a string over crossings as seen in figure 2.2.

To understand knots one must first be able to determine if distinct knots actually do exist, and with these movements not altering the knot at all we are able to define an equivalence for knots that will allow us to address this idea.
Definition 2.4. We define two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ to be equivalent if there exists an ambient isotopy from one knot to the other. In this case we say that $\mathcal{K}_1$ and $\mathcal{K}_2$ belong to the knot class $[\mathcal{K}]$.

It follows from this that, since Reidemeister moves do not change the knot type, we can find equivalence with the following theorem.

Theorem 2.5. Two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ are equivalent if there exists a set of Reidemeister moves that transforms $\mathcal{K}_1$ into $\mathcal{K}_2$ when applied to $\mathcal{K}_1$ [2].

Example 2.1. The above knots [figure 2.1] are equivalent to each other. Note the knot on the left may be untwisted (by a type III then type II Reidemeister move) until the simple unknot is left.

Now that this idea of equivalence has been determined it can be shown that there are in fact a multitude of knots aside from the unknot, the next simplest being the trefoil knot (figure 2.3). One way to catalogue these numbers is using the idea of the crossing number.

Definition 2.6. The crossing number $c(\mathcal{K})$ of a knot $\mathcal{K}$ is the least number of crossings found in any projection of $\mathcal{K}$.

Example 2.2. • The unknot has crossing number 0 [Figure 2.3].

- The trefoil knot has crossing number 3 [Figure 2.3].
• There is no knot with crossing number 2

To further classify knots, one would like to know what are the other defining properties of the given knots. For instance, is it possible to make the given knot by attaching previously known knots together? This gives rise to the idea of composition of knots and prime knots.

**Definition 2.7.** The composition of two knots $K_1$ and $K_2$, denoted $K_1 \# K_2$, is formed by cutting open both knots along a free edge (i.e. avoiding crossings) and gluing the ends together.

**Definition 2.8.** We call a knot that cannot be broke down into a composition of knots a *prime knot*. Similarly we define a knot the can be broken down into a composition of prime knots a *composite knot*.

**Example 2.3.**

• Composing any knot $K$ with the unknot returns $K$.

• By composing knots we are able to obtain an array of different and complicated knots.
Many prime knots are known and catalogued but there are still many yet to be discovered, with the difficulty of telling knots apart growing exponentially as the number of crossings for a given knot increases. One way of determining whether a knot is distinct is by observing the various invariants of the given knot. These invariants include properties such as: the crossing number; various polynomials (such as the Jones Polynomial or the Alexander Polynomial); the unknotting number (the minimum number of crossings needed to change in order to make the knot into the unknot); and the stick number.

3 Sticks And Knots

The notion of the stick number is an interesting aspect of a knot. The two following definitions give an overview to the concept of the stick number:

**Definition 3.1.** A stick knot, or polygon knot, is a knot in 3 dimensions made up of rigid line segments (‘sticks’) [2].

It can be seen from this that every knot has an embedding that is in fact a stick knot. This stick representation of a knot can be constructed quite simply. First of all, take a tabular neighbourhood of the knot in $\mathbb{R}^3$. Inside this tabular neighbourhood the knot can be approximated by a piecewise linear curve. By doing this, the curve is replaced by a series of rigid sticks and hence becomes a stick knot. It is important to note however that there is a minimum number of lines needed to respect the correct crossings of the knot, and so we have the next definition.

**Definition 3.2.** The stick number $s(K)$ of a knot $K$ is the least number of rigid sticks required to make a knot in 3 dimensions [2].

**Example 3.1.** • The stick number of the unknot is 3 (you can make it out of a triangle of sticks)
• The stick number of the trefoil knot is 6. See Figure 3.1.

Figure 3.1: The Stick number of the trefoil Knot is 6

**Theorem 3.3.** *For any non-trivial knot $K$, $s(K) \geq 6$.*

*Proof.* It is impossible to make a closed loop with 1 stick. It is similarly impossible to make a closed loop with 2 sticks. With 3 sticks you can make the unknot by placing them in a triangle. There are two ways to place 4 sticks in space (see figure 3.2) and clearly each is isotopic to the unknot. With 5 sticks, simply project vertically down one of the sticks so that it projects to a point. We are now left with the 4 stick case. Finally, with 6 sticks it is possible to craft a knot as seen in figure 3.1.

Figure 3.2: The two possible embeddings of 4 sticks in space

Stick numbers are of particular interest in the area of synthetic chemistry as DNA is made up of rigid sticks of things like sugar and phosphorus [10]. As cyclic molecules can be created by
these rigid sticks attached so that they loop back to the beginning [10], and different knot types represent different molecules, realizing the stick number of a knot could make the construction of such molecules much simpler. Although this is the case, in practice it proves quite difficult to find the stick number of a knot as one must prove that it is impossible to construct the knot with less sticks. Many have tried to find ways to simplify this process by creating bounds and working from these to evaluate the stick number. One such bound is as follows:

**Theorem 3.4.** Negami’s [1] bounds for the stick number are given as:

\[
\frac{5 + \sqrt{25 + 8(c(\mathcal{K}) - 2)}}{2} \leq s(\mathcal{K}) \leq 2c(\mathcal{K})
\]

**Proof.** **Left side:** Take any stick knot \( \mathcal{K}' \) that realises the stick number of \( \mathcal{K} \) and let \( s(\mathcal{K}) = n \). Project \( \mathcal{K}' \) along one of the edges so that edge projects down to a vertex, this will leave us with \( n-1 \) sticks. Notice how each stick may cross at most \( n-4 \) other sticks as it will never cross itself or the stick adjacent to it (the sticks glued to the ends of it). By adding together the crossings for each stick we find that:

\[
\frac{1}{2}(n - 1)(n - 4) \leq c(\mathcal{K}') = c(\mathcal{K})
\]

Note that we divide by 2 because each crossing will have been counted twice. Rearranging this yields:

\[
n^2 - 5n - 2(c(\mathcal{K}) - 2) \leq 0
\]
Which implies the result.

**Right side:** Suppose $K^*$ is a projection of the knot $K$ that realises the minimum crossing number $c(K) = x$. Project $K^*$ onto a plane so that it is self intersecting. At each crossing, place a vertex. Results from graph theory state that we can replace the curved edges from the knot by straight edges without changing the graph, and by deleting doubled up edges we are left with a simple 4 regular graph. Further results from graph theory state that, in this situation, we are left with twice as many edges as we have vertices, namely, $2x$ edges. We can then alter this graph slightly, converting it to three dimensions, and eliminating the self intersections and connecting the corresponding sticks. In doing so, we are left with a stick representation of the knot such that:

$$ s(K) \leq 2x = 2c(K) $$

Implying the intended result. (This proof was adapted from Negami’s, for further reading, refer to [1]).

**Example 3.2.**

- So far, the trefoil knot is the only known knot to realise the equality for the right hand bound.

- There are no known knots that realise the left hand bound.

Note that the above theorem gives bounds on the stick number and hence, in some cases, provides a way to find them. Others, such as Huh and Oh [11], have worked to improve these bounds. In 2010 Huh and Oh successfully proved the following:

**Theorem 3.5.** Let $K$ be a non-trivial knot. Then $s(K) \leq \frac{3}{2}(c(K) + 1)$ [11].
Proof. Omitted, see [11].

Stick numbers have been found for many different types of knots, including some infinite classes under certain regulating conditions. Due to work in this area some interesting results have been found, but before delving into the following theorems we will need a few definitions and lemmas.

**Definition 3.6.** Given any three adjacent sticks $s_a, s_b, s_c$ we define the skew of $s_b$ to be clockwise if the minimal angle (the angle less that $\pi$) between $s_a$ and $s_c$ when projected along $s_b$ is obtained by rotating clockwise from $s_a$ to $s_b$. We define the skew to be anticlockwise similarly. If the project angle between $s_a$ and $s_c$ is 0 or $\pi$ the $s_b$ is said to have both skews as either could easily be obtained by moving the end of $s_a$ slightly either way [figure 3.3].

**Definition 3.7.** A free vertex $v$ of a stick knot $K$ is a vertex with the property that, when you enclose $K$ completely within a sphere and shrink it until it comes into contact with $K$, $v$ will be the first point of contact with the sphere. A knot may have multiple free vertices.

![Figure 3.3: Stick $s_b$ has a clockwise skew](image)

**Lemma 3.8.** A bijective linear transformation $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $\det([F]) > 0$ will preserve knot type and stick number when applied to any stick knot $K$. 

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PROOF OUTLINE. As $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has $\det([F]) > 0$ it belongs to the path connected subset of the General Linear Group $GL(3, \mathbb{R})$ containing the identity. Because of this, the transformation matrix $[F]$ can be continuously deformed to the identity, which implies there is a continuous isotopy from $K$ to $F(K)$. Hence, knot type and stick number are retained through linear transformation by matrices of positive determinant as it directly relates to an isotopic transformation from one knot to the other. 

Lemma 3.9. Given a non-trivial stick representation of a knot $K$ and a stick $S_z$ from the representation, there exists a linear transformation on $\mathbb{R}^3$ that preserves the knot type and the number of sticks that sends the two adjacent sticks, $S_x$ and $S_y$, to sticks orthogonal to $S_z$, and also orthogonal to each other when projected along $S_z$.

Proof. Take a stick representation of a non-trivial knot $K$ and a stick $S_z$ from the representation. Let the two adjacent sticks be $S_x$ and $S_y$. If $S_x, S_z$ and $S_y$ are coplanar, move the end of $S_x$ not attached to $S_z$ slightly so they no longer lie in the same plane. Let $\mathcal{P}_x$ be the plane spanned by $S_x$ and $S_z$ and let $Q_x : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation that maps to the identity in $\mathcal{P}_x$ and sends $S_y$ to a stick orthogonal to $\mathcal{P}_x$. This will preserve the knot type by Lemma 2.7. Now take the plane $\mathcal{P}_y$ spanned by $S_z$ and $Q_x(S_y)$. Define a similar linear transformation as before $Q_y$ that sends $S_x$ to a stick orthogonal to $\mathcal{P}_y$. Once again this will preserve knot type. Let $S_x^* = Q_y(S_x)$ and $S_y^* = Q_x(S_y)$. The result will be a knot representation of $K$ with the same number of sticks with $S_z$ orthogonal to both $S_x^*$ and $S_y^*$ and $S_y^*$ orthogonal to $S_x^*$ when projected along $S_z$.

Theorem 3.10. For any two knots $K_1$ and $K_2$, $s(K_1 \# K_2) \leq s(K_1) + s(K_2) - 3$ [3]

Proof. Take a minimal stick representation of $K_1$ and a stick $S_z$ from the representation and apply Lemma 3.9 to $S_z$. Rotate the new representation of $K_1$ until $S_z$ is parallel to the $z$–axis, $S_x$ is
parallel to the $x$–axis and $S_y$ is parallel to the $y$–axis. Now transform $\mathcal{K}_1$ so that the line containing $S_z$ does not intersect $\mathcal{K}_1$ in any other place. Now dilate $\mathcal{K}_1$ in the direction perpendicular to $S_z$ until there exists a cylinder of arbitrary radius and infinite height that contains only $S_z$ and portions of $S_x$ and $S_z$ from $\mathcal{K}_1$ [figure 3.4].

Now take a minimal stick representation of $\mathcal{K}_2$. Take any free vertex of $\mathcal{K}_2$ and let the sticks meeting at it be $S_a$ and $S_c$. Separate $S_a$ and $S_c$ by moving one away from the other and add a new stick $S_b$ such that $S_b$ has the same skew as $S_z$ [figure 3.5]. Now apply Lemma 3.9 to $S_b$. Dilate this representation of $\mathcal{K}_2$ until $|S_b| = |S_z|$ and place it inside the cylinder made big enough to contain $\mathcal{K}_2$ so that $S_a$ lines up with $S_x$, $S_c$ lines up with $S_y$ and $S_b$ and $S_z$ occupy the same space. Remove both $S_b$ and $S_z$ and extend $S_a$ to $S_x$ so it becomes one stick and $S_c$ to $S_y$ so that it also becomes one stick. We have now successfully constructed a stick representation of $\mathcal{K}_1 \# \mathcal{K}_2$ and removed 4 sticks, however as we had to add a stick to $\mathcal{K}_2$ to allow the composition, we arrive at the intended inequality.

\[ \square \]

Figure 3.5: Separation of a free vertex
The above theorem relates to the composition of knots, assuring us that we can always lose three sticks when constructing the composite knot. An interesting class of knots is the continuous composition of trefoil knots. This knot class holds an interesting property pertaining to the skew of the outside edge sticks of the knot, and due to this the following theorem allows us to find the stick number of any number of composition of trefoil knots.

Theorem 3.11. Let \( nT \) denote the composition of \( n \) trefoil knots. Then \( s(nT) = 2n + 4 \) \cite{3}

Proof. Omitted, see \cite{3} for further reading.

The proof of this theorem is similar to one presented in the next section (Theorem 4.3) and so we will no go into it here. Another interesting class of knots that the stick number is known for is a special class of torus knots, which we will investigate now.
4 Torus Knots and Sticks

Torus knots have many interesting properties, and produce some of the infinite class of knots for which the stick number is known. In this section we will introduce two theorems about the stick number of particular classes of torus knots, and we will proceed to prove the second.

Definition 4.1. A torus knot $T_{p,q}$ is a knot that can be wrapped about a torus in space. They can be constructed with the following algorithm [2]:

- Take $p$ and $q$ as coprime numbers (if they are not coprime you get more than one knot).
- Mark $p$ points on the inside equator of the torus and $p$ points on the outside equation of the torus denoted $x_i$ and $y_i$ respectively, $i = 1, 2, ..., p$.
- Connect each of the corresponding points along the underside of the torus by a line (i.e. $x_i$ to $y_i$).
- Finally, connect each point $y_i$ along the outside equator to the point $x_{(i+p) \mod p}$ on the inside equator along the top of the torus. Repeat this until each point has been connected on the topside of the torus.

Example 4.1. The trefoil knot is the torus knot $T_{3,2}$.

Another way to visualise a torus knot is not as a knot, but as straight lines on a flat sheet of paper. As a torus has the same local geometry as a plane, one can visualise the flat torus as a rectangle with the top edge connected to the bottom and the left edge connected to the right edge. If you were to do this you could place a knot on the torus as follows:

- Place $p$ points on left hand side of the rectangle and $p$ points on the right.
Connect each point \( x_i \) \((i = 1, 2, \ldots, p)\) on the left hand side to the point \( y_{i+q \mod p} \) up from it on the right hand side. When making this connection do not pass over any previously drawn lines, and if you pass through the top of the rectangle, continue the line from the corresponding position on the bottom of the rectangle.

When this is done correctly, you should get a picture as seen in Figure 4.2. Doing this helps with visualisation of the torus, as well as allowing us to fairly simply distinguish links from knots (simply by following the corresponding lines). This visualisation could be useful when working through the following theorems and proofs, which will now be introduced.

**Theorem 4.2.** If \( 2 \leq p < q \leq 2p \), then \( s(T_{p,q}) = 2q \) [4]

*Proof.* Omitted.

**Theorem 4.3.** \( s(T_{p,p-1}) = 2p \) [3].

Before we can prove this we must introduce the idea of the bridge number of a knot and the total curvature of a knot. We will also have to introduce a few helpful theorems and lemmas to assist in the proof.
Definition 4.4. The bridge number of a knot $\mathcal{K}$, denoted $b(\mathcal{K})$, is defined to be the minimal number of local maxima in any direction of any projection of $\mathcal{K}$. In other words, it is defined as:

$$b(\mathcal{K}) = \min_{\mathcal{K}' \in [\mathcal{K}]} \min_{v \in \mathbb{R}^3} p_v(\mathcal{K}')$$

Where $p_v(\mathcal{K}')$ counts the local maxima in the direction of $v$. [2, 6]

Definition 4.5. The total curvature of a knot $\mathcal{K}$, denoted $C(\mathcal{K})$, is the integral that gives the total turning of the tangent vector as we move along the knot in a particular direction [8]. In particular, the total curvature of a stick knot can be thought of as the sum of the smallest angles at each vertex of the knot [3].

Theorem 4.6. $C(\mathcal{K}) \geq 2\pi(b(\mathcal{K}))$ [8, 3].

Proof. Omitted.
**Theorem 4.7.** \( b(T_{m,n}) = n \) for \( n < m \) and \( n \geq 2 \) \([7, 3]\). Moreover, \( b(T_{p,p-1}) = p - 1 \).

**Proof.** Omitted \( \square \)

**Lemma 4.8.** Let \( K_{\mathbb{R}^2} \) denote a 2 dimensional projection of a stick knot \( K \) which is considered to have flat crossings, so that it intersects itself. Similarly, let \( K_{\mathbb{R}^3} \) be a three dimensional representation of a stick knot \( K \), then:

\[
\forall \epsilon > 0, \exists K_{\mathbb{R}^3} \text{ such that } C(K_{\mathbb{R}^3}) < C(K_{\mathbb{R}^2}) + \epsilon \ [3].
\]

**Proof.** Given a 2 dimensional stick knot \( K_{\mathbb{R}^2} \) and an \( \epsilon > 0 \), take \( x \) to be the number of crossings in the projection. Now, at each crossing, bend the stick that is to be the over-strand of the crossing up by an angle of \( \frac{\epsilon}{8x} \) a short distances \( y \) from the crossing. Now over where the crossing occurs, bend the stick down by an angle of \( \frac{\epsilon}{4x} \) and extend the stick until it meets the original stick again, at a distance of \( y \) from the crossing. By this construction the crossing has gained \( \frac{\epsilon}{8x} + \frac{\epsilon}{4x} + \frac{\epsilon}{8x} = \frac{\epsilon}{2x} \) in total curvature [figure 4.3]. Therefore, we have a 3 dimensional stick representation such that \( C(K_{\mathbb{R}^3}) = C(K_{\mathbb{R}^2}) + \frac{\epsilon}{2x} \) as we have done this at every crossing. More simply, we have \( C(K_{\mathbb{R}^3}) = C(K_{\mathbb{R}^2}) + \frac{\epsilon}{2} < C(K_{\mathbb{R}^2}) + \epsilon \) as desired. \( \square \)

![Figure 4.3: Increasing the total curvature by \( \frac{\epsilon}{2x} \)](image-url)
Lemma 4.9. For a $p,p-1$ torus knot $T_{p,p-1}$, $s(T_{p,p-1}) \leq 2p$ [3].

Proof. (By construction) Create two unit circles $C_1$ and $C_2$ in the xy-plane with centres at $(0, 0, 0)$ and $(0, 0, 1)$ respectively. Place $p$ equispaced points $m_i$ counterclockwise around $C_1$ and connect each point $m_i$ to the point projectively opposite it on $C_2$. Given this construction, the lines will form a double cone with vertex at $(0, 0, \frac{1}{2})$. Now fix the endpoints, but rotate $C_2$ by a small amount $\epsilon$. Let the new endpoints found of $C_2$ be $n_i$ such that $n_i$ connects to $m_i$. The lines given by this connection of points will now lie on a hyperboloid (as a hyperboloid is a ruled surface). Now connect each point $n_i$ to $m_{i-1} \mod(p)$. This construction will produce another hyperboloid with a wider centre than the original. Take the torus generated by the sub-annuli of the two hyperboloids and you will be left with a stick representation of a torus knot that passes one full turn around the maridian and a $\frac{p-1}{p}$ turn around the longitude, producing the $T_{p,p-1}$ knot with $2p$ sticks.

With these ideas we can now establish a proof for Theorem 4.3.

Proof of 3.2. By lemma 4.9 we know that for a $T_{p,p-1}$ knot, $s(T_{p,p-1}) \leq 2p$ and so it suffices to show that we cannot construct the knot with less sticks. From Theorems 4.6 and 4.7 we know that $C(T_{p,p-1}) \geq 2\pi(b(T_{p,p-1})) = 2\pi(p - 1)$.

Assume $s(T_{p,p-1}) = 2p - 1 < 2p$. Therefore, if we project along one stick so that it projects to a point we get a two dimensional representation, $T_{\mathbb{R}^2}$, with $s(T_{\mathbb{R}^2}) = 2p - 2$ sticks. Since each stick can contribute less than $\pi$ to the total curvature of the knot, we have that $C(T_{\mathbb{R}^2}) < \pi(2p - 2) = 2\pi(p - 1)$. We know from Lemma 4.8 that there exists a three dimensional representation of $T_{p,p-1}$, denoted $T_{\mathbb{R}^3}$, such that the total curvature is within $\epsilon$ of the 2 dimensional representation.
Therefore, take $\epsilon = \frac{1}{2}(2\pi(p - 1) - C(T_{R^2}) > 0$ and we get:

\[
C(T_{R^2}) < C(T_{R^2}) + \epsilon
= C(T_{R^2}) + \frac{1}{2}(2\pi(p - 1) - C(T_{R^2})
= \frac{1}{2}(2\pi(p - 1) + C(T_{R^2})
< \frac{1}{2}(2\pi(p - 1) + 2\pi(p - 1))
< 2\pi(p - 1)
\]

Which contradicts Theorems 4.6 and 4.7, hence contradicting our assumption, which implies the result that $s(T_{p,p-1}) = 2p.$

5 Conclusion

There is still so much to be discovered in knot theory that can be achieved by anyone with an open mind and drive to do so. The applications in mathematics and over the sciences are endless, and due to this, knot theory makes for an interesting area for future study. Whether looking at the geometrical and topological properties found through the study of knots or looking at a simpler aspect such as the stick number, knot theory allows for exceptional mental development and is an invigorating entry to an abstract area of mathematics for people with base mathematical understanding. With the possible applications of this area of mathematics, knot theory will most definitely play a interesting role in the world of science in times to come.
6 References


