Visualisation of subdifferentials

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Abstract
The aim of the report is to cover theoretical knowledge on subdifferentials of functions in $\mathbb{R}^n$ and to use this knowledge to create illustrations of subdifferentials in $\mathbb{R}^1$ and $\mathbb{R}^2$. Starting from an introductory on norms and derivatives, the report follows through with the following definitions of subdifferentials: convex subdifferentials, Fréchet subdifferential, Clarke subdifferential and limiting subdifferential, finishing with illustrations of subdifferentials of convex and non-convex functions. The report does not go into great details of the mathematics behind the subdifferentials and will keep to a few simple properties of the subdifferentials and methods used for calculations.
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1. Introduction

Functions can be non-differentiable and new tools are needed to work with such functions so that practical applications that need differentiation, such as optimisation, are still applicable. In particular, there is a whole field of mathematics known as non-smooth analysis; the study of non-differentiable functions. The subdifferential is the key tool for non-smooth optimisation. Explicitly the functions studied throughout this report are only from $\mathbb{R}^n$.

There are many different definitions of subdifferentials, however, the following three will only be covered in this report: convex, Fréchet, limiting and Clarke, which are denoted as follows:

1. Fréchet $\partial f(a)$
2. limiting $\bar{\partial} f(a)$
3. Clarke $\partial_c f(a)$.

The properties of the function and target of application will reflect the type of subdifferential needed.

Convex subdifferentials are applicable to convex functions; Fréchet, limiting, and Clarke subdifferentials are applicable to non-convex functions; although limiting and Clarke subdifferential are both applicable when the Fréchet subdifferential is empty. While each definition is different from each other, some of their properties do overlap, e.g. if a function is strictly differentiable, limiting and Clarke subdifferentials reduce to the derivative. Other properties that the Fréchet, limiting or Clarke possess are as follows:

1. If the function is convex, then the subdifferential reduces to the convex subdifferential.
2. If the function is differentiable then the Fréchet subdifferential reduces to the derivative.

For visualisation purposes, only functions where the differences of norms are calculated.

Definitions for norms and convex functions are taken from [2].
2. Norms

A norm denoted \( \| \cdot \| \) is a non-negative real-valued function.

- \( \| x \| \geq 0 \) & \( \| x \| = 0 \) if and only if \( x=0 \)
- \( \| -x \| = \| x \| \)
- \( \| tx \| = |t| \cdot \| x \| \) for scalar \( t \),
- \( \| x + y \| \leq \| x \| + \| y \| \)

Functions that will be investigated for visualising subdifferentials will include the difference of the following norms.

1. 1-norm: \( \| x \|_1 = \sum_{i=1}^{n} |a_i| \)
2. Euclidean norm (2): \( \| x \|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \)
3. \( \infty \)-norm: \( \| x \|_\infty = \max_{1 \leq i \leq n} |x_i| \)

The above norms are examples of the p norm; p-norm: \( \| x \|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \), \( p > 0 \)

Below is an illustration (Figure 1) of the level sets of the 1-norm (red), Euclidean norm (black) and the \( \infty \) norm (blue).

![Figure 1: Level curves of the unit norms](image-url)
3. Derivative

Geometrically, a derivative is the slope of a tangent line at a point $a$ to the graph of function $f$, where $f$ is only on $\mathbb{R}$. Figure 2 illustrates this.

This is the definition of the derivative from classical analysis, where $x$ is from $\mathbb{R}^n$ and $a$ is a number.

$$\nabla f(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Similarly the Fréchet derivative is defined as follows, $x^*$ is the Fréchet derivative and the function is on $\mathbb{R}^n$.

$$\lim_{x \to a} \frac{f(x) - f(a) - (x^*, x - a)}{\|x - a\|} = 0.$$

The derivative is strict if

$$\lim_{x' \to a \atop x' \neq x} \frac{f(x) - f(x') - (x^*, x - x')}{\|x - x'\|} = 0.$$

**Example 1** $f(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

An example where the derivative exists but is not strict

$$\nabla f(0) = 0$$

**Figure 3** $f(x) = x^2 \sin \left( \frac{1}{x} \right)$
4. Convex subdifferential
A convex function has a segment connecting any two points on its graph lies above the graph. In other words, for \( x \) and \( y \) in \( \mathbb{R}^n \), and \( t \) in the interval \([0,1]\),

\[
t f(x) + (1 - t)f(y) \geq f(tx + (1 - t)y).
\]

This inequality is known as the Jensen inequality.

If a function \( f \) is convex, then the convex subdifferential can be defined for any \( \mathbb{R}^n \).

**Definition 4.2** \( x^* \) is called the subgradient(s) and the whole set is known as the subdifferential. \( \partial f(a) = \{ x^* \in \mathbb{R}^n : f(x) - f(a) \geq \langle x^*, x - a \rangle \ \forall x \in \mathbb{R}^n \} \). The elements of \( d f(a) \) are called subgradients.

**Example 2** \( f(x) = |x| \).
Notice at the point 0 the derivative does not exist (can be defined for any \( a \) in \( \mathbb{R}^n \)), refer to Figure 4 below. The subdifferential, of \( f \) at 0 is the set \([-1,1]\).

\[
\partial f(0) = \{ x^* \in \mathbb{R}^n : |x| \geq x^* x, \forall x \in \mathbb{R}^n \}
\]

Let \( x > 0 \)

\[ x^* \leq 1 \]

Let \( x < 0 \)

\[ x^* \geq -1 \]

\[ \therefore \partial f(0) = [-1,1] \]

Show that the opposite is true!
Fix any \( x^* \in [-1,1] \). Then \( |x^*| \leq 1 \) For any \( x \in \mathbb{R} \),

\[ x^*(x - 0) = x^* x \leq |x^* x| = |x^*| \cdot |x| \leq |x| - |0| = f(x) - f(0). \]

\[ x^* \in \partial f(0) \). So \([-1,1] \subset \partial f(0) \]

![Figure 4 Illustration of subdifferential at 0](image_url)

![Figure 5 \( f(x) = |x|, \text{convex} \)](image_url)
Example 3 $f(x, y) = \sqrt{x^2 + y^2}$. The function is a norm, therefore it is convex.

\[
\partial f(0,0) = \{(x^*, y^*) \in \mathbb{R}^2 : f(x, y) - 0 \geq x^* x + y^* y, \forall (x, y) \in \mathbb{R}^2 \}
\]

\[
\sqrt{x^2 + y^2} \geq x^* x + y^* y
\]

\[
\partial f(0,0) = \{(x^*, y^*) | x^* + y^* = 1\}
\]

Figure 6 $f(x, y) = \sqrt{x^2 + y^2}$

Figure 7 for the function $(x, y) = \sqrt{x^2 + y^2}, \partial f(0,0) = \mathbb{B}$
5. Fréchet subdifferential

Fréchet subdifferential can be used when the function is:

$$\partial f(x) = \{ x^* \in \mathbb{R}^n : \liminf_{x \to a} \frac{f(x)-f(a)-(x^*,x-a)}{||x-a||} \geq 0 \}.$$ 

Elements of the subdifferential are called subgradients. They correspond to line ‘supporting’ the graph of $f$ at the point $a$ from below.

The Fréchet superdifferential can be defined in a similar way.

$$\partial^+ f(a) = \{ x^* \in \mathbb{R}^n : \limsup_{x \to a} \frac{f(x)-f(a)-(x^*,x-a)}{||x-a||} \leq 0 \}.$$ 

Example 4 Consider the function $f(x) = -|x|$. It is non-convex. The subdifferential at 0 is empty. However the superdifferential is not empty.

![Figure 8](image)

Figure 8 $f(x) = -|x|$, $\partial f(x) = [-1,1]$

Overall the Fréchet subdifferential has the following properties:

1. If the function is convex, the Fréchet subdifferential reduces to the convex subdifferential.
2. If the function is differentiable, then the Fréchet subdifferential reduces to the derivative.

The Sum rule is used throughout many calculations of Fréchet subdifferentials. The Sum coincides with the convex subdifferential:

$$\partial f(x) = \partial f_1(x) + \nabla f_2(x)$$

if $f_2$ is differentiable.
6. Limiting subdifferential and Clarke subdifferential

The limiting Fréchet subdifferential is the set of limits of the Fréchet subdifferential.

Given a lower semi-continuous function $f$, the limiting subdifferential at $a$ is defined as follows:

$$\tilde{\partial} f(a) = \left\{ x^* = \lim_{x_k \to a} x_k^*, \ x_k^* \in \partial f(x_k) \right\}.$$  

$\tilde{\partial} f(a)$ can be nonconvex. Refer to Example 5 below.

The limiting subdifferential is useful for when the Fréchet subdifferential is empty, e.g. $f(x) = -|x|$.

For locally Lipschitz functions, the Clarke subdifferential is the convex hull of the limiting subdifferential, Figure 10 illustrates the Clarke subdifferential, where the shaded in square is; Clarke subdifferential.

Properties of the limiting subdifferential and Clarke subdifferential are listed below:

1. The Fréchet subdifferential is a subset of the limiting subdifferential while the latter is a subset of the Clarke subdifferential:
   $$\partial f(a) \subset \tilde{\partial} f(a) \subset \partial_c f(a)$$

2. If $f$ is convex then $\tilde{\partial} f(a)$ and $\partial_c f(a)$ coincide with the convex subdifferential.

3. If the function is strictly differentiable at $a$.
   $$\tilde{\partial} f(a) = \partial_c f(a) = \{\nabla f(a)\}$$

Example 5: Referring to the function in Example 2, coincides with the convex subdifferential $\tilde{\partial} f(0) = \{-1,1\}$.

Example 6: With reference to Example 2 and 4 the functions $f(x) = |x|$ and $f(x) = -|x|$ the Clarke subdifferential at 0 for both function is non-empty and $\partial_c f(0) = [-1,1]$. 
7. Visualising the limiting subdifferential in $\mathbb{R}^2$

**Example 7:** $f(x, y) = |x| + |y|, (x, y) \in \mathbb{R}^2$

To calculate the Fréchet subdifferentials, the domain of the function is broken up into regions. Consider **Table 1** below:

<table>
<thead>
<tr>
<th>No</th>
<th>Regions</th>
<th>Fréchet subdifferentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x = 0, y = 0$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>$x &gt; 0, y &gt; 0$</td>
<td>${(1, -1)}$</td>
</tr>
<tr>
<td>3</td>
<td>$x &gt; 0, y &lt; 0$</td>
<td>${(1,1)}$</td>
</tr>
<tr>
<td>4</td>
<td>$x &lt; 0, y &gt; 0$</td>
<td>${(-1,1)}$</td>
</tr>
<tr>
<td>5</td>
<td>$x &lt; 0, y &lt; 0$</td>
<td>${(-1,-1)}$</td>
</tr>
<tr>
<td>6</td>
<td>$x &gt; 0, y = 0$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>7</td>
<td>$x &lt; 0, y = 0$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>8</td>
<td>$x = 0, y &gt; 0$</td>
<td>$[-1,-1] \times {1}$</td>
</tr>
<tr>
<td>9</td>
<td>$x = 0, y &lt; 0$</td>
<td>$[1,1] \times {1}$</td>
</tr>
</tbody>
</table>

**Table 1:** $f(x, y) = |x| + |y|$

The union of all the Fréchet subdifferentials for all regions will make limiting subdifferential at $(0,0)$.

Table 1 indicates calculations for the red segments on **Figure 11**. The blue segments correspond to the superdifferential.
Example 8: \( f(x, y) = 2 \max(|x|, |y|) - \sqrt{x^2 + y^2}, (x, y) \in \mathbb{R}^2 \)
Denote \( f_1(x, y) = 2 \max(|x|, |y|) \) and \( f_2(x, y) = \sqrt{x^2 + y^2} \)
Taking the union of all the results from Table 2 will form the limiting subdifferential at 0.

Figure 14

Since \( f_2(x, y) \) is differentiable everywhere but (0,0), therefore the sum rule is applicable everywhere but (0,0).

Using Table 2 and comparing it to Figure 12, the subdifferential can be visualised.
Table 2: Fréchet subdifferentials for \( f(x, y) = 2 \max\{|x|, |y|\} - \sqrt{x^2 + y^2} \)

<table>
<thead>
<tr>
<th>No</th>
<th>Regions</th>
<th>Fréchet subdifferential</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( x &gt; 0, \ -x &lt; y &lt; x )</td>
<td>( (2,0) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>2.</td>
<td>( x &lt; 0, \ x &lt; y &lt; -x )</td>
<td>( (-2,0) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>3.</td>
<td>( -y &lt; x &lt; y, \ y &gt; 0 )</td>
<td>( (0,2) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>4.</td>
<td>( y &lt; x &lt; -y, \ y &lt; 0 )</td>
<td>( (0,-2) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>5.</td>
<td>( x &gt; 0, \ y = x )</td>
<td>( \left{ \left( x^* - \frac{1}{\sqrt{2}}, y^* - \frac{1}{\sqrt{2}} \right) \mid x^* \geq 0, y^* \geq 0, x^* + y^* = 2 \right} )</td>
</tr>
<tr>
<td>6.</td>
<td>( x &lt; 0, \ y = x )</td>
<td>( \left{ \left( x^* + \frac{1}{\sqrt{2}}, y^* + \frac{1}{\sqrt{2}} \right) \mid x^* \geq 0, y^* \geq 0, y^* + x^* = -2 \right} )</td>
</tr>
<tr>
<td>7.</td>
<td>( x &gt; 0, \ y = -x )</td>
<td>( \left{ \left( x^* + \frac{1}{\sqrt{2}}, y^* - \frac{1}{\sqrt{2}} \right) \mid x^* \geq 0, y^* \geq 0, x^* - y^* = 2 \right} )</td>
</tr>
<tr>
<td>8.</td>
<td>( x &lt; 0, \ y = -x )</td>
<td>( \left{ \left( x^* - \frac{1}{\sqrt{2}}, y^* + \frac{1}{\sqrt{2}} \right) \mid x^* \geq 0, y^* \geq 0, -y^* + x^* = 2 \right} )</td>
</tr>
<tr>
<td>9.</td>
<td>( x = 0, \ y = 0 )</td>
<td>(</td>
</tr>
</tbody>
</table>

1. The graph \( f(x, y) = 2 \max\{|x|, |y|\} - \sqrt{x^2 + y^2} \) is symmetrical, therefore the illustration of the subdifferential will also be symmetrical.
2. Notice for \((x,y)\neq 0\) \( f_2 \) is differentiable.
   \[ \nabla f_2(x, y) = \frac{(x,y)}{\sqrt{x^2 + y^2}} \]
3. For the region \((y = |x|, \forall x)\) the subdifferentials will just be a segment. (refer to Table 2)
4. For the region \((x > 0, \ -x < y < x)\), the subdifferential will be a vector. When \((x,y)\) are changing, then they will form an arc of a circle. (refer to Table 2)
Figure 14 Illustration of subdifferential at 0
\[ f(x,y) = 2 \max(|x|,|y|) \ - \ \sqrt{x^2+y^2} \]
Example 9 \( f(x, y) = 2(|x| + |y|) - \sqrt{x^2 + y^2}, (x, y) \in \mathbb{R}^2 \)

Denote \( f_1(x, y) = 2(|x| + |y|) \) and \( f_2(x, y) = \sqrt{x^2 + y^2} \)

The sum rule can be used due to \( f_2 \) being differentiable everywhere but \((0,0)\). Refer to Table 3 for calculations.

<table>
<thead>
<tr>
<th>No</th>
<th>Regions</th>
<th>Fréchet subdifferentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x &gt; 0, \ y = 0 )</td>
<td>( {1} \times [-2,2] )</td>
</tr>
<tr>
<td>2</td>
<td>( x &lt; 0, \ y = 0 )</td>
<td>( {-1} \times [-2,2] )</td>
</tr>
<tr>
<td>3</td>
<td>( x = 0, \ y &gt; 0 )</td>
<td>( [-2,2] \times {1} )</td>
</tr>
<tr>
<td>4</td>
<td>( x = 0, \ y &lt; 0 )</td>
<td>( [-2,2] \times {-1} )</td>
</tr>
<tr>
<td>5</td>
<td>( x &gt; 0, \ y &gt; 0 )</td>
<td>( (2,2) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>6</td>
<td>( x &gt; 0, \ y &lt; 0 )</td>
<td>( (2,-2) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>7</td>
<td>( x &lt; 0, \ y &gt; 0 )</td>
<td>( (-2,2) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>8</td>
<td>( x &lt; 0, \ y &lt; 0 )</td>
<td>( (-2,-2) - \frac{(x,y)}{\sqrt{x^2 + y^2}} )</td>
</tr>
<tr>
<td>9</td>
<td>( x = 0, \ y = 0 )</td>
<td>( [-1,1] \times [-1,1] )</td>
</tr>
</tbody>
</table>

Table 3: Fréchet subdifferentials for \( f(x, y) = 2(|x| + |y|) - \sqrt{x^2 + y^2} \)
From Table 3 and Figure 15, the following results are formed:

1. The graph \( f(x,y) = 2(|x| + |y|) - \sqrt{x^2 + y^2} \) is symmetrical, therefore the illustration of the subdifferential will also be symmetrical.

2. Notice for \((x,y)\neq 0\) \( f(z) \) is differentiable.
   \[ \nabla f_z(x,y) = \frac{(x,y)}{\sqrt{x^2+y^2}} \]

3. For the regions \( y = 0, \forall x \) \& \( \forall y, x = 0 \) the subdifferentials will just be a segment. (refer to Table 3)

4. For the regions \( \forall x, \forall y \mid x \text{ and/or } y \neq 0 \), the subdifferential will be a vector. If \((x,y)\) are not set and are changing, then this will form an arc of a circle. (refer to Table 3)
Example 10: \( f(x, y) = 2 \max\{|x|, |y|\} - (|x| + |y|), (x, y) \in \mathbb{R}^2 \)

Denote \( f_1(x, y) = 2 \max\{|x|, |y|\} \) & \( f_2(x, y) = (|x| + |y|) \)

The sum rule is not applicable for this function, as neither \( f_1 \) or \( f_2 \) are differentiable.

<table>
<thead>
<tr>
<th>No</th>
<th>Regions</th>
<th>Fréchet subdifferentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x &gt; 0, \ y = 0 )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>2</td>
<td>( x &lt; 0, \ y = 0 )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>3</td>
<td>( x = 0, \ y &gt; 0 )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>4</td>
<td>( x = 0, \ y &lt; 0 )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>5</td>
<td>( x &gt; 0, \ y = x )</td>
<td>( {y^* = x^*</td>
</tr>
<tr>
<td>6</td>
<td>( x &lt; 0, \ y = x )</td>
<td>( {y^* = -x</td>
</tr>
<tr>
<td>7</td>
<td>( x &gt; 0, \ y = -x )</td>
<td>( {y^* = x^*</td>
</tr>
<tr>
<td>8</td>
<td>( x &lt; 0, \ y = -x )</td>
<td>( {y^* = -x</td>
</tr>
<tr>
<td>9</td>
<td>( x &gt; 0, \ -x &lt; y &lt; x )</td>
<td>( {1} x [1, -1] )</td>
</tr>
<tr>
<td>10</td>
<td>( x &lt; 0, \ x &lt; y &lt; -x )</td>
<td>( {-1} x [1, -1] )</td>
</tr>
<tr>
<td>11</td>
<td>( -y &lt; x &lt; y, \ y &gt; 0 )</td>
<td>( {1} x [-1,1] )</td>
</tr>
<tr>
<td>12</td>
<td>( y &lt; x &lt; -y, \ y &lt; 0 )</td>
<td>( {-1} x [-1,1] )</td>
</tr>
<tr>
<td>13</td>
<td>( x = 0, \ y = 0 )</td>
<td>((0,0))</td>
</tr>
</tbody>
</table>

Table 4
From Table 4 and Figure 18, the following results are formed:

1. The graph \( f(x, y) = 2\max\{|x|, |y|\} - (|x| + |y|) \) is symmetrical, therefore the illustration of the subdifferential will also be symmetrical.
2. The first four regions in Table 4 are empty.
3. For regions 5-12 the subdifferentials will just be a segment. (refer to Table 4)
4. At the point (0,0) the Fréchet subdifferential at (0,0) is (0,0).
**Example 11:** \( f(x, y) = 2\sqrt{x^2 + y^2} - (|x| + |y|), (x, y) \in \mathbb{R}^2 \)

Denote \( f_1(x, y) = 2\sqrt{x^2 + y^2} \) & \( f_2(x, y) = (|x| + |y|) \).

![Figure 21: \( f(x, y) = 2\sqrt{x^2 + y^2} - (|x| + |y|) \)](image1)

![Figure 22: Level set](image2)

<table>
<thead>
<tr>
<th>No</th>
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<th>Fréchet subdifferentials</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3</td>
<td>( x = 0, \ y &gt; 0 )</td>
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</tr>
<tr>
<td>4</td>
<td>( x = 0, \ y &lt; 0 )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>5</td>
<td>( x &gt; 0, \ y &gt; 0 )</td>
<td>( 2\left(\frac{(x, y)}{\sqrt{x^2 + y^2}}\right) - (1,1) )</td>
</tr>
<tr>
<td>6</td>
<td>( x &gt; 0, \ y &lt; 0 )</td>
<td>( 2\left(\frac{(x, y)}{\sqrt{x^2 + y^2}}\right) - (1,-1) )</td>
</tr>
<tr>
<td>7</td>
<td>( x &lt; 0, \ y &gt; 0 )</td>
<td>( 2\left(\frac{(x, y)}{\sqrt{x^2 + y^2}}\right) - (-1,1) )</td>
</tr>
<tr>
<td>8</td>
<td>( x &lt; 0, \ y &lt; 0 )</td>
<td>( 2\left(\frac{(x, y)}{\sqrt{x^2 + y^2}}\right) + (1,1) )</td>
</tr>
<tr>
<td>9</td>
<td>( x = 0, \ y = 0 )</td>
<td>(</td>
</tr>
</tbody>
</table>

*Table 5*
From Table 5 and Figure 21, the following results are formed:

1. The graph \( f(x, y) = 2\sqrt{x^2 + y^2} - (|x| + |y|) \) is symmetrical, therefore the illustration of the subdifferential will also be symmetrical.

2. Notice for \((x, y) \neq 0 \) \( f_1 \) is differentiable.
   \[
   \nabla f_1(x, y) = 2 \frac{(x, y)}{\sqrt{x^2 + y^2}}
   \]

3. For regions 1-4 the Fréchet subdifferential is empty.

4. For regions 5-8 the subdifferential will be a vector. When \((x, y)\) are changing, they will an arc of a circle. (refer to Table 5)

![Figure 22: Illustration of subdifferential](image)
**Example 12:** \( f(x, y) = 2\sqrt{x^2 + y^2} - \max\{|x|, |y|\}, (x, y) \in \mathbb{R}^2 \)

Denote \( f_1(x, y) = 2\sqrt{x^2 + y^2} \) & \( f_2(x, y) = \max\{|x|, |y|\} \).

![Figure 23: \( f(x, y) = 2\sqrt{x^2 + y^2} - \max\{|x|, |y|\} \)](image)

![Figure 24: Level set](image)

<table>
<thead>
<tr>
<th>No</th>
<th>Regions</th>
<th>Fréchet subdifferentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x &gt; 0, \ -x &lt; y &lt; x )</td>
<td>( 2 \left( \frac{(x, y)}{\sqrt{x^2 + y^2}} \right) - (1,0) )</td>
</tr>
<tr>
<td>2</td>
<td>( x &lt; 0, \ x &lt; y &lt; -x )</td>
<td>( 2 \left( \frac{(x, y)}{\sqrt{x^2 + y^2}} \right) - (-1,0) )</td>
</tr>
<tr>
<td>3</td>
<td>( -y &lt; x &lt; y, \ y &gt; 0 )</td>
<td>( 2 \left( \frac{(x, y)}{\sqrt{x^2 + y^2}} \right) - (0,1) )</td>
</tr>
<tr>
<td>4</td>
<td>( y &lt; x &lt; -y, \ y &lt; 0 )</td>
<td>( 2 \left( \frac{(x, y)}{\sqrt{x^2 + y^2}} \right) - (0,-1) )</td>
</tr>
<tr>
<td>5</td>
<td>( x &gt; 0, \ y = x )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>6</td>
<td>( x &lt; 0, \ y = x )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>7</td>
<td>( x &gt; 0, \ y = -x )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>8</td>
<td>( x &lt; 0, \ y = -x )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>9</td>
<td>( x = 0, \ y = 0 )</td>
<td>( [-1,1] \times [-1,1] )</td>
</tr>
</tbody>
</table>

*Table 6*
From Table 6 and Figure 23, the following results are formed:

1. The graph \( f(x, y) = 2\sqrt{x^2 + y^2} - \max\{ |x|, |y| \} \) is symmetrical, therefore the illustration of the subdifferential will also be symmetrical.

2. Notice for \((x,y) \neq 0\) \( f_1 \) is differentiable.

\[
\nabla f_1(x, y) = 2 \left( \frac{(x,y)}{\sqrt{x^2+y^2}} \right)
\]

3. For regions 5-8 the Fréchet subdifferential is empty.

4. For regions 1-4 the subdifferential will be a vector. When \((x,y)\) are changing, they will form a semi-circle moved by 1 in either the x direction or y. Refer to Table 6.

![Figure 25: Illustration of subdifferential](image-url)
Acknowledgements
I acknowledge Geogebra and Wolframalpha for the illustrations made throughout this report, Vera Roschina of RMIT University for her work in subdifferentials norms, David Yost for his help on several subdifferential calculations and welcomed advice.
References


