Parallelogram Polyominoes, Partitions and Polynomials

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Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.
1 Introduction

1.1 Knot theory

Knot theory is a subfield of topology, concerned with mathematical knots. Informally, a knot is the result of tangling a string and joining the ends together. We consider two knots to be the same if one can be molded into the other without cutting or tearing. Finding ways to tell knots apart is a central problem in knot theory, and is usually investigated via the study of knot invariants.

A knot invariant is some quantity or mathematical object that depends only on a given knot, and not any particular representation. An example of a knot invariant is the Jones polynomial, which assigns a Laurent polynomial to a knot. For example the Jones Polynomial of the trefoil knot is $q^{-1} + q^{-3} - q^{-4}$. Thus if we wish to determine the identity of some knot, we could calculate its Jones Polynomial, and if it were not $q^{-1} + q^{-3} - q^{-4}$, we would know that it is not the trefoil knot. There is currently no known method to always distinguish one knot from another.

The formal definition of a knot is as follows.

Definition 1. A knot is a piecewise-linear continuous embedding $\gamma : S^1 \to S^3$.

We will often abuse terminology by referring to the image of the knot as the knot itself.
1.2 The 3d index

The 3d index assigns to certain ideal triangulations of an orientable 3-manifold with toroidal boundary a function \( f : \mathbb{Z}^2 \rightarrow \mathbb{Z}((q^{1/2})) \). It was first defined by [1] and is a conjectured invariant of the underlying manifold. Since knot complements are examples of such 3-manifolds, the invariance of the 3d index would function as a knot invariant.

1.3 The tetrahedron index

The 3d index in terms of the tetrahedron index.

**Definition 2.** The tetrahedron index is a function \( I_\Delta : \mathbb{Z}^2 \rightarrow [[q^{1/2}]] \), defined by

\[
I_\Delta(m, e) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n+1}{2} - m(n+\frac{1}{2})}}{(q)_n(q)_{n+e}}
\]

where \((q)_n\) is the Pochhammer symbol:

\[
(q)_n = \prod_{i=1}^{n} (1 - q^i)
\]

The index is well-defined for all integer values \( m \) and \( e \), since the \( q^{\frac{n+1}{2} n} \) term ensures that the coefficients are well-defined.

The tetrahedron index satisfies a number of identities.

- Duality:
  \( I_\Delta(m, e) = I_\Delta(-e, -m) \)

- Triality:
  \( I_\Delta(m, e) = (-1)^e q^{\frac{e}{2}} I_\Delta(e, -e - m) = (-1)^m q^{\frac{m}{2}} I_\Delta(-e - m, m) \)

- Quadratic:
  \[
  \sum_{e \in \mathbb{Z}} I_\Delta(m, e)I_\Delta(m, e + c)q^e = \delta_{c,0}
  \]

- Pentagon:
  \[
  I_\Delta(m_1 - e_2, e_1)I_\Delta(m_2 - e_1, e_2) = \sum_{e_3 \in \mathbb{Z}} q^{e_3} I_\Delta(m_1, e_1 + e_3)I_\Delta(m_2, e_2 + e_3)I_\Delta(m_1 + m_3, e_3)
  \]

In [2], Garoufalidis observed that the coefficients of \( I_\Delta(0, 1)/I_\Delta(0, 0) \) are positive. This positivity can in fact be explained via polyomino enumeration.
2 Polyomino enumeration

2.1 Polyominoes

We now introduce the notion of a polyomino.

**Definition 3.** A polyomino is a connected subset of the infinite square grid. We consider two polyominoes to be the same if one can be translated into the other.

![Figure 2: Some Polyominoes](image)

We also introduce generating functions, the central tool we use to enumerate polyominoes.

**Definition 4.** Suppose we have some class of polyomino, and $P_{t,l}$ is the number of polyominoes in this class that consist of $t$ squares and contain exactly $l$ blocks in their left-most column. Then $\sum_{t=0}^{\infty} \sum_{l=0}^{\infty} P_{t,l} s^t q^l$ is the generating function for this class of polyomino.

For example, the generating function for polyominoes consisting of a single column of squares is $sq + s^2q^2 + s^3q^3 + \cdots = \frac{sq}{1-sq}$. In [3] a general method for enumerating various classes of polyominoes using generating functions is provided. Suppose $X(s, q) \in \mathbb{Z}[[s, q]]$ is the generating function for some class of polyominoes. The hope is that we can leverage the combinatorial nature of the polyominoes in order to derive a functional equation of the form

$$X(s, q) = a(s, q) + b(s, q)X(1, q) + c(s, q)X(sq, q).$$

(1)

Bousquet-Mélou provides the general solution to this equation,

$$X(1, q) = \frac{A(1, q)}{1 - B(1, q)}$$

(2)
where

\[ A(1, q) = \sum_{n=0}^{\infty} a(sq^n, q) \prod_{i=0}^{n-1} c(sq^i, q) \]

\[ B(1, q) = \sum_{n=0}^{\infty} b(sq^n, q) \prod_{i=0}^{n-1} c(sq^i, q) \]

### 2.2 Enumeration of parallelogram polyominoes

To explain the positivity Garoufalidis observed we require the notion of a parallelogram polyomino.

**Definition 5.** Select a point in the integer lattice, and select another point that is above and to the right of it. Draw two non-intersecting paths from the first to the second, with the restriction that each path only moves up and to the right. The polyomino that these two paths border is a parallelogram polyomino.

![Some Parallelogram Polyominoes](image)

Figure 3: Some Parallelogram Polyominoes

We shall demonstrate Bousquet-Mélou’s enumeration method by proving the following theorem.

**Theorem 1.** Let \( P(s, q) \) be the generating function for parallelogram polyominoes. Then

\[ P(1, q) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)} \frac{q^n}{(q)_n (q)_{n+1}} \]  \( \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)} \frac{1}{(q)_n^2} \). \( \text{(3)} \)

**Proof.** We start by performing various transformations on the parallelogram polyominoes and recording the transformations algebraically.
We start with a parallelogram polyomino.

\[ P(s, q) \]

We add a single block in the bottom left corner.

\[ sqP(1, q) \]

We add blocks above above and below the new one.

\[ \frac{sq}{(1-sq)^2} P(1, q) \]

This last step creates some polyominoes that are not parallelogram, due to the added blocks at the top. We now construct these violations exactly.
We start with a parallelogram polyomino.

We duplicate the left most column.

We add a single block on top of the duplicated column.

We add blocks above and below.

Thus \( sqP(sq, q) \) only counts parallelogram polyominoes, but it does not count all of them. Observe that we have not constructed parallelogram polyominoes that consist only of a single column. Fortunately these are easily enumerated as \( 1 + sq + s^2q^2 + s^3q^3 + \cdots = \frac{1}{1-sq} \). Thus \( \frac{sq}{(1-sq)^2}P(1, q) - \frac{sq}{(1-sq)^2}P(sq, q) + \frac{1}{1-sq} \) is the generating function for parallelogram polyominoes, and this gives us our functional equation,

\[
P(s, q) = \frac{1}{1-sq} + \frac{sq}{(1-sq)^2}P(1, q) - \frac{sq}{(1-sq)^2}P(sq, q).
\]

Observe that this is in the form of the general equation above, with \( a(s, q) = \frac{1}{1-sq} \), \( b(s, q) = \frac{sq}{(1-sq)^2} \), and \( P(1, q) \) the generating function for monominoes.
\( \frac{sq}{(1-sq)^2} \), and \( c(s, q) = \frac{-sq}{(1-sq)^2} \). Hence we can compute an explicit expression for \( P(1, q) \).

\[
A(s, q) = \sum_{n=0}^{\infty} a(sq^n, q) \prod_{i=0}^{n-1} c(sq^i, q)
= \sum_{n=0}^{\infty} \frac{1}{1-sq^{n+1}} \prod_{i=0}^{n-1} \frac{-sq^{i+1}}{(1-sq^{i+1})^2}
A(1, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n(q)_{n+1}}
\]

\[
B(s, q) = \sum_{n=0}^{\infty} b(sq^n, q) \prod_{i=0}^{n-1} c(sq^i, q)
= \sum_{n=0}^{\infty} \frac{sq^{n+1}}{(1-sq^{n+1})^2} \prod_{i=0}^{n-1} \frac{-sq^{i+1}}{(1-sq^{i+1})^2}
B(1, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}(n+1)(n+2)}}{(q)_{n+1}^2}
1 - B(1, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2}.
\]

And so

\[
P(1, q) = \frac{A(1, q)}{1 - B(1, q)} = \frac{\sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n(q)_{n+1}}}{\sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2}}
\tag{5}
\]

As required.

In fact \( A(1, q) = I_{\Delta}(0, 1) \) and \( 1 - B(1, q) = I_{\Delta}(0, 0) \). Hence the surprising result,

\[
\frac{I_{\Delta}(0, 1)}{I_{\Delta}(0, 0)} = P(1, q).
\tag{6}
\]

Clearly the coefficients of \( P(1, q) \) must be positive since they count objects. This explains the positivity Garoufalidis observed, and also provides us with an alternative way of studying the tetrahedron index, through the enumeration of polyominoes.

2.3 Interpretation for \( m, e \)

In fact, the coefficients of \( \frac{I_{\Delta}(m,e)}{I_{\Delta}(0,0)} \) are always non-negative or always non-positive. We will prove this shortly, but we first need the following definition.
Definition 6. A young diagram is a polyomino consisting of decreasing columns from left to right, with each column starting on the same row.

![Some young diagrams.](image)

Figure 4: Some young diagrams.

Young diagrams represent partitions of numbers, where each column is one number in the partition.

Theorem 2. Let $R_{m,e}(s,q)$ be the generating function for parallelogram polyominoes with an $m$ by $m+e$ rectangle contained in the top-right corner, where $m > 0$ and $e > -m + 1$. Then

$$(-1)^m q^{\frac{m}{2}(e+m-1)} \frac{\mathcal{I}_\Delta(m,e)}{\mathcal{I}_\Delta(0,0)} = R_{m,e}(1,q)$$

(7)

Proof. Note that the functions $b$ and $c$ will be the same as in the $(0,1)$ case, since their only purpose is to ensure that the ”paralleogramness” is preserved. Thus we examine how the $m$ by $m+e$ rectangle affects the function $a$. Recall that the purpose of $a$ is to re-build the polyominoes that aren’t constructed by $b$ and $c$. In the $(0,1)$ case this was simply all single column polyominoes, but the $(m,e)$ case is more complicated. To illustrate, let $m = 3$ and $e = 1$, then for instance the following polyominoes are not constructed by $b$ and $c$, and hence must be constructed by $a$. 

![Illustration of polyominoes.](image)
These blocks beneath the rectangles are exactly young diagrams with the restriction that the rows do not exceed \( m \) blocks in length. Young diagrams of this type are enumerated by

\[
\prod_{i=0}^{m} \left( \frac{1}{1 - sq^i} \right), \tag{8}
\]

So accounting for the rectangle we find that

\[
a(s, q) = s^{m+e} q^{m(m+e)} \prod_{i=0}^{m} \left( \frac{1}{1 - sq^i} \right). \tag{9}
\]

This gives us our function equation for \( R_{m,e} \),

\[
R_{m,e}(s, q) = s^{m+e} q^{m(m+e)} \prod_{i=0}^{m} \left( \frac{1}{1 - sq^i} \right) + \frac{sq}{(1 - sq)^2} R_{m,e}(1, q) - \frac{sq}{(1 - sq)^2} R_{m,e}(sq, q). \tag{10}
\]

Using the lemma we construct \( A \),

\[
A(s, q) = \sum_{n=0}^{\infty} a(s q^n, q) \prod_{i=0}^{n-1} c(sq^i, q)
= \sum_{n=0}^{\infty} (s q^n)^{m+e} q^{m(m+e)} \prod_{i=0}^{m} \left( \frac{1}{1 - sq^{i+n}} \right) \prod_{i=0}^{n-1} \frac{-sq^{i+1}}{(1 - sq^{i+1})^2}
\]

\[
A(1, q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)+(n+m)(m+e)}}{(q)_n(q)_{n+m}}
= q^{m(m+e)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)+(n+m)(m+e)}}{(q)_n(q)_{n+m}}
= q^{m(e+m)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)+(e+m)(n+\frac{m}{2})}}{(q)_n(q)_{n+m}}
= q^{m(e+m)} \mathcal{I}_\Delta(-e - m, m)
= (-1)^m q^{-m(e+m)} \frac{m}{\pi} \mathcal{I}_\Delta(-e - m, m)
= (-1)^m q^{-m(e+m)} \frac{m}{\pi} \mathcal{I}_\Delta(m, e),
\]

where the final line is the triality index. Since \( B \) is unaffected by \( a \), as above \( B(1, q) = 1 - \mathcal{I}_\Delta(0, 0) \). Therefore,

\[
R_{m,e}(1, q) = \frac{A(1, q)}{1 - B(1, q)} = (-1)^m q^{-m(e+m-1)} \frac{\mathcal{I}_\Delta(m, e)}{\mathcal{I}_\Delta(0, 0)}, \tag{11}
\]

As required.
2.4 An alternative enumeration

We could also enumerate parallelogram polyominoes with an \( m + e \) by \( m \) rectangle in the top right. In this case,

\[
a(s, q) = s^m q^{m(m+e)} \prod_{i=1}^{m+e} 1 - sq^i
\]

So

\[
A(s, q) = \sum_{n=0}^{\infty} a(sq^n, q) \prod_{i=0}^{n-1} c(sq^i, q)
= \sum_{n=0}^{\infty} (sq^n)^m q^{m(m+e)} \prod_{i=1}^{m+e} 1 - (sq^n)q^i \prod_{i=0}^{n-1} \frac{-sq^{i+1}}{(1 - sq^{i+1})^2}
= \sum_{n=0}^{\infty} s^m q^{nm+m(m+e)} \prod_{i=1}^{n-1} 1 - sq^{n+1} \prod_{i=0}^{n-1} \frac{-sq^{i+1}}{(1 - sq^{i+1})^2}
\]

\[
A(1, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)+nm+m(m+e)}}{(q)_n(q)_{n+m+e}}
= q^m \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)+m(n+\frac{(m+e)}{2})}}{(q)_n(q)_{n+m+e}}
= q^m \mathcal{I}_\Delta(-m, m+e)
\]

Though of course, there should be the same number of parallelogram polyominoes with an \( m \) by \( m+e \) rectangle in the top-right corner as there are with an \( m+e \) by \( m \) rectangle in the top-right corner. Thus we have that

\[
q^{\frac{m(m+e)}{2}} \mathcal{I}_\Delta(-e - m, m) = q^{\frac{m(m+e)}{2}} \mathcal{I}_\Delta(-m, m+e)
\]

\[
\mathcal{I}_\Delta(-e - m, m) = \mathcal{I}_\Delta(-m, m+e)
\]

Which can be reparameterised as \( \mathcal{I}_\Delta(-a, -b) = \mathcal{I}_\Delta(b, a) \), where \( a = m + e \) and \( b = m \), which is exactly the duality identity. This actually proves the duality index for \( a > 0, b > 0 \).
3 Extending the tetrahedron index

We have demonstrated that polyomino enumeration can be a useful technique in proving statements about the tetrahedron index. Furthermore, expressions similar to the tetrahedron index appear elsewhere in the literature, such as in [4]. Thus we will demonstrate how this technique can be used with a more general example.

**Theorem 3.** Let

\[
J(a_1, a_2, \ldots, a_m) = \sum_{n=0}^{\infty} (-1)^n q^{m(n+1)+mn} \prod_{i=1}^{m} (q)_{n+a_i}.
\] (12)

Then \(J(a_1, a_2, \ldots, a_m)/J(0, 0, \ldots, 0)\) has only non-negative coefficients.

**Proof.** We first describe the class of polyomino we will enumerate.

**Definition 7.** Let \(a \in \mathbb{N}\). A \(p\)-\(\alpha\)-polyomino is a polyomino satisfying the following:

- The right side of the polyomino consists of a young diagram, the longest row of which consists of \(p\) squares.
- There is a diagonal line of squares North-West of the corner block of this young diagram.
- Columns extend down from this diagonal line, with the restriction that no column extends lower than any column to the right of it.

![Figure 5: Two 4-\(\alpha\)-Polyominoes](image)

Figure 5: Two 4-\(\alpha\)-Polyominoes
Definition 8. Let \( \mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{N}^m \). An \( \mathbf{a} \)-\( \alpha \)-tuple is an \( m \)-tuple where the \( i \)-th element of a tuple is an \( (a_i) \)-\( \alpha \)-polyomino, and the tuple has the following additional conditions:

- The total number of columns in the polyomino not including the young diagram is the same in every polyomino in the tuple.
- The downward extension of two adjacent columns in the left of the polyominos is not the same in all polyominos in the tuple.

The second condition is easiest to understand once we construct the polyominos.

![Figure 6: A (1, 2, 3) – \( \alpha \)-tuple.](image)

Let \( \mathbf{a} = (a_1, a_2, \ldots, a_m) \) be arbitrary. Let \( X_{\alpha}(s, q) \) be the generating function for \( \mathbf{a} \)-\( \alpha \)-tuples. We once again seek a recurrence relation.
We start with an m-tuple \( X_a(s, q) \) We add one block to each polyomino. 

\[ s^m q^m X_a(1, q) \] We add blocks below in each polyomino in various configurations.

\[ \frac{s^m q^m}{(1-sq)^m} X_a(1, q) \] We add blocks below in each polyomino in various configurations.

This constructs some polyominoes that violate the restrictions, and so as in the parallelogram case we construct them exactly.
We start with an m-tuple

\[ X(s, q) \]

We duplicate the column and add a single block to each column

\[ s^m q^m X_a(sq, q) \]

We add blocks below the duplicated column

\[ \frac{s^m q^m}{(1-sq)^m} X_a(sq, q) \]

This creates the violations. Since the expansion of \( \frac{1}{(1-sq)^m} \) starts with a 1, the tuple with only duplicated columns counts as a violation, this is why we require the second condition on the tuples.

We now enumerate the young-diagrams on the right side. As discussed above, the generating function for young diagrams with a maximum row-length of \( a_i \) is \( \prod_{j=1}^{a_i} \frac{1}{1-sq^j} \). Hence the possible combinations of three-tuples of young diagrams are given by \( \prod_{i=1}^{m} \prod_{j=1}^{a_i} \frac{1}{1-sq^j} \). Thus we finally have our functional equation,

\[
X_a(s, q) = \frac{1}{\prod_{i=1}^{m} \prod_{j=1}^{a_i} (1-sq^j)} + \frac{s^m q^m}{(1-sq)^m} X_a(1, q) - \frac{s^m q^m}{(1-sq)^m} X_a(sq, q). \tag{13}
\]
We now compute the solution:

\[
A(s, q) = \sum_{n=0}^{\infty} a(sq^n, q) + \prod_{i=0}^{n-1} c(sq^i, q)
\]

\[
= \sum_{n=0}^{\infty} \prod_{i=1}^{m} \prod_{j=1}^{q^n} \left( 1 - (sq^n)q^j \right) \prod_{i=0}^{n-1} \frac{-(sq^i)q^m_i}{(1 - (sq^i)q)^m}
\]

\[
= \sum_{n=0}^{\infty} \prod_{i=1}^{m} \prod_{j=1}^{q^n} \left( 1 - sq^{n+j} \right) \prod_{i=0}^{n-1} \frac{-s^m q^{m(i+1)}}{(1 - sq^{i+1})^m}
\]

\[
A(1, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^m n(n+1)}{\prod_{i=1}^{m} (q)_{n+a_i}}
\]

\[
A(1, q) = \sum_{n=0}^{\infty} \prod_{i=1}^{m} \prod_{j=n+1}^{n+a_i} (1 - q^j)(q)_m
\]

And,

\[
B(s, q) = \sum_{n=0}^{\infty} b(sq^n, q) \prod_{i=0}^{n-1}
\]

\[
= \sum_{n=0}^{\infty} \frac{s^m q^{m(n+1)}}{(1 - sq^{n+1})^m} \prod_{i=0}^{n-1} \frac{-s^m q^{m(i+1)}}{(1 - sq^{i+1})^m}
\]

\[
B(1, q) = \sum_{n=0}^{\infty} \frac{q^{m(n+1)}}{(1 - q^{n+1})^m} \frac{(-1)^n q^m n(n+1)}{(q)_m}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n q^m (n+1)(n+2)}{(q)_m}
\]

\[
= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n q^m n(n+1)}{(q)_m}
\]

And so the result follows. □

References

