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Simple groups of infinite matrices

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1 Abstract

One means of studying the algebraic structure of groups is to decompose them into smaller groups. As such the study of simple groups which cannot be so decomposed is an important part of group theory. We have proven that the group of infinite matrices on a finite field with a finite number of non-zero entries below the diagonal is topologically simple.

2 Introduction

To study the internal structure of groups, we can decompose them into smaller groups in such a way that the behaviour of the smaller groups determines the behaviour of the original group. As such, knowing about simple groups - groups which cannot be decomposed - becomes of great importance, which is why an effort was made to classify all finite simple groups. That classification was completed in the 1970s.

The extension of the classification of finite simple groups to infinite groups poses numerous difficulties - for instance when we work with infinite matrices we must restrict the matrices we use so that we have a well-defined matrix multiplication operation. Also, we cannot compute exact answers for products and inverses in many cases so we need some sort of approximation, which is where topology comes in.

We have managed to prove the existence of a topologically simple group of infinite matrices. The work in this project was done under the guidance of my supervisors George Willis and Colin Reid, who suggested the general direction of this proof while I filled in the details.

3 Background

The solution to Hilbert's fifth problem shows that every simple connected locally compact group is a simple real Lie group[2], and so a classification of the simple real Lie groups yields a classification of connected locally compact groups.

We do have a classification of a subset of infinite groups, namely the simple Lie groups which were classified by Wilhelm Killing and Elie Cartan in the 1890s[1].

This does not apply to disconnected (and totally disconnected) locally compact groups, and in fact we cannot classifiy finitely generated simple groups in general.

It is known that the projective special linear group $PSL(n, F_q)$ - defined as $PSL(n, F_q) = SL(n, F_q)/SZ(n, F_q)$, where F_q is the finite field of order q, is a simple group for $n \geq 3$, and also when $n = 2, q > 3$[3]. The proof of this uses the properties of transvections on $SL(n, F_q)$, and this idea is the basis of my proof of the topological simplicity of $AU(q)$.

4 Definitions

We denote the group of almost upper triangular matrices with elements belonging to the finite field $\mathbb{F}_q$ by $AU(q)$, the subgroup of those matrices with the non-zero entries below the diagonal having indices in $(m, n]$ by $AU_{m,n}(q)$, and the subgroup of matrices corresponding to scalar multiplications $AUZ(q) = \{kI : k \in \mathbb{F}_q^*\}$

We define the mapping $\psi_{m,n} : AU_{m,n}(q) \to GL(n-m, q)$ by

$$\psi_{m,n}(x) = (a_{ij}), a_{ij} = x_{(i+m)(j+m)}$$

which can be interpreted as 'cutting out' the part of the infinite matrix with indices between m and n (including n) into a finite matrix. Also we define $\phi_{m,n}$ by composing $\psi_{m,n}$ with the quotient homomorphism from $GL(n-m, q)$ to $PGL(n-m, q)$

Finally we also introduce a topology T on $AU(q)$ generated by the basis consisting of the sets

$$S(m, n, x) = \{y \in AU(q) : y_{ij} = x_{ij} \text{ for } m < i, j \leq n\}$$

5 Intermediate Results

Lemma 1. *$AU_{m,n}(q)$ is a subgroup of $AU(q)$ for all $m, n \in \mathbb{Z}$ where $m < n$.*

Proof. For any $x, y \in AU_{m,n}(q)$, we have that $x_{ij} = 0$ when $i > j$ and either $i > n$ or $j < m$, and similarly for y_{ij}. Then $(xy)_{ij} = \sum_{k \in \mathbb{Z}} x_{ik} y_{kj}$ so for $i > j$,
$(xy)_{ij} = \sum_{k \in [m,n]} x_{ik} y_{kj}$ which is 0 when $i > n$ or $j < m$ meaning that $xy \in AU_{m,n}(q)$.

□

Lemma 2. *$AU(q)$ with the topology T is a topological group.*

Proof. We need to show that the mapping $\pi : (g,h) \mapsto gh^{-1}$ is continuous. We pick $g_0, h_0 \in AU(q)$ and want to show that π is continuous at (g_0, h_0). A general neighbourhood for the topology T is $N_I = \{f \in AU(q) : f_{ij} = (g_0h_0^{-1})_{ij} \text{ for } -I \leq i,j \leq I\}$
Now $(g_0h_0^{-1})_{ij} = \sum_{k\in\mathbb{Z}}(g_0)_{ik}(h_0^{-1})_{kj}$ which is a finite sum for each i,j. For each i,j we define k_i, L_j such that

$$(g_0)_{ik} = 0 \text{ if } k < k_i$$

$$(h_0^{-1})_{kj} = 0 \text{ if } k > L_j$$

We also let $r = \min\{k_i : -I \leq i \leq I\}, c = \max\{L_j : -I \leq j \leq I\}$. Finally let

$$A = \{g \in AU(q) : g_{ij} = (g_0)_{ij}, r \leq i,j \leq c\}$$

$$B = \{h \in AU(q) : h_{ij} = (h_0)_{ij}, r \leq i,j \leq c\}$$

then A and B are open neighbourhoods of g_0 and h_0 and $AB^{-1} \subseteq N_I$. Therefore π is continuous at (g_0, h_0) and since g_0 and h_0 are arbitrary π is continuous everywhere and $AU(q)$ is a topological group.

□

Lemma 3. *The mapping $\psi_{m,n}$ is a group homomorphism.*

Proof. First, we show that $\psi_{m,n}$ preserves the group operation. Let $x, y \in AU_{m,n}(q)$ so $(xy)_{ij} = \sum_{k\in\mathbb{Z}} x_{ik}y_{kj}$. When $i \in [m,n], x_{ik} = 0$ for $k < m$ and when $j \in [m,n], y_{kj} = 0$ for $k > n$ so when $i,j \in [m,n]$,
$(xy)_{ij} = \sum_{k\in[m,n]} x_{ik}y_{kj}$ so the cutout portion of xy is simply the product of the cutout portions of x and y. This is also sufficient to show that $\psi_{m,n}$ actually maps into $GL(n-m,q)$, as if we had a matrix $x \in AU_{m,n}(q)$ with a singular cutout portion, then multiplying that with the cutout portion of x^{-1} would yield the identity (as that is the cutout portion of the identity in $AU(q)$), thereby contradicting the singularity of $\psi_{m,n}(x)$. Hence $\psi_{m,n}$ is a homomorphism.

□

Lemma 4. *$AU_{m,n}(q)/ker(\phi_{m,n})$ is simple whenever $gcd(n-m, q-1) = 1$ and $n-m > 3$.*

Proof. We have that $AU_{m,n}(q)$ is homomorphic to $GL(n-m,q)$, so the normal subgroups of $AU_{m,n}(q)/ker(\phi_{m,n})$ correspond to the normal subgroups of $GL(n-m,q)/GZ(n-m,q) =$

$PGL(n-m,q)$ by the Correspondence Theorem. Since $gcd(n-m,q-1)=1$, every element of $\mathbb{F}_q$ has an $(n-m)$th root and so $PGL(n-m,q)=PSL(n-m,q)$. As $PSL(n-m,q)$ is simple for $n-m>3$, this proves that $AU_{m,n}(q)/ker(\phi_{m,n})$ has no nontrivial normal subgroups and so is simple.

□

6 Result

Theorem 1. *$AU(q)/AUZ(q)$ is topologically simple.*

Proof. Suppose N is a closed normal subgroup that properly contains $AUZ(q)$. Then for a pair of integers $m<n$, we have a normal subgroup $N_{m,n}=N\cap AU_{m,n}(q)$ of $AU_{m,n}(q)$. If $AU_{m,n}/K_{m,n}(q)$ is simple (where $K_{m,n}(q)=AUZ(q)ker(\psi_{m,n})$) then either $N_{m,n}\leq K_{m,n}(q)$ or $N_{m,n}K_{m,n}(q)=AU_{m,n}(q)$. Since N is not contained in $AUZ(q)$, there are a and b such that $N_{a,b}$ is not contained in $K_{a,b}(q)$. We then select $a'<a$ and $b'>b$ such that $gcd(b'-a',q-1)=1$, meaning that $N_{a',b'}K_{a',b'}(q)=AU_{a',b'}(q)$. We can do this to get pairs (a'_i,b'_i) satisfying this property such that a'_i tends to $-\infty$ and b'_i tends to $+\infty$. Since $N_{a',b'}\supseteq AUZ(q)$ we have $N_{a'_i,b'_i}L_{a'_i,b'_i}=AU_{a'_i,b'_i}(q)$ for pairs (a'_i,b'_i) meaning that for every $g\in AU(q)$ and each index i there exists a $h_i\in N$ such that h_i agrees with g on the cutout square defined by $(a'_i,b'_i]$ so $h_i\in S(a'_i,b'_i,g)$. Since any neighbourhood U of g must contain at least one $S(a,b,g)$ we can pick m such that $a'_i<a$ and $b'_i>b$ for $i\geq m$ so $h_i\in U$ for all $i\geq m$ so h_i converges to g. This is true for all $g\in AU(q)$ and as N is closed, $N=AU(q)$ so $AU(q)/AUZ(q)$ is topologically closed. □

7 Conclusion

We have demonstrated the existence of a topologically simple infinite group, and can apply this method to investigate other groups of infinite matrices.

References

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[2] David Hilbert. Mathematical problems. *Bull. Amer. Math. Soc.*, 8(10):437–479, 1902.

[3] Joseph J. Rotman. *An introduction to the theory of groups.* Springer-Verlag, 1995.