Edge Elimination for the Hamiltonian Cycle problem

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Abstract

The Hamilton cycle and travelling salesman problem are both highly studied problems in optimisation. We present a method relating these two problems, and use the travelling salesman problem to solve some Hamiltonian cycle problems.

Introduction

In this paper we will discuss a new approach and algorithm to solve the Hamiltonian cycle problem. In particular, we will look at using this algorithm to solve the Hamiltonian cycle challenge from Flinder's University.

Background

Hamiltonian Cycles

Given a graph $G = (V, E)$, the Hamiltonian cycle (or path) of G is a cycle that visits every vertex exactly once. Although this problem was originally defined by Hamilton, similar problems have been studied since (Euler, 1759). A Hamiltonian cycle need not be unique; in fact as shown in (Thomason, 1978), in a graph of which the vertices have only odd degrees, if one exists, a second one must exist. This problem was proven NP-Complete in (Garey & Johnson, 2002).

Travelling Salesman Problem

The travelling salesman problem (TSP) is one of the largest and most widely studied problems in all of computer science. It is an integer linear programming problem with vastly numerous applications. The travelling salesman problem is as follows: a salesman needs to visit an amount of cities and return home. He does not wish to visit the same city twice. What is the fastest way he can do this? In this context, fastest means shortest distance travelled. A more formal definition is to find the shortest Hamiltonian cycle for n cities. The TSP in particular is NP-Complete (Garey & Johnson, 2002).

How they relate: Transitive closure reduction

Intuitively, one can see how these two problems may related to each other, perhaps most obviously when finding the Hamiltonian cycle of a planar graph. There are several commonly used reduction techniques to transform a Hamiltonian problem to a TSP one. One commonly employed reduction for the Hamiltonian cycle problem given a graph $G = (V, E)$, is the standard reduction from (Gary & Johnson, 1979) in which the distance matrix, D, of the resulting TSP is defined as follows:

$$D_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 2 & \text{otherwise} \end{cases}$$

This is far from the only form of reduction used, in fact, all of our work in this paper will be using a reduction known as transitive closure (TC).

For a graph $G = (V, E)$ of the Hamiltonian problem and a TSP distance matrix D, the transitive closure reduction is defined as follows,

$$D_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ \delta_{ij} & \text{otherwise} \end{cases}$$

Where δ_{ij} is the shortest distance between i and j in the graph G. This can be found with Dijkstra's algorithm (Dijkstra, 1959).

Edge betweenness centrality

Betweenness centrality (Freeman, 1977), is a measure of how central a vertex or edge is to a graph. Given a graph $G = (V, E)$, the betweenness centrality of the edge e is

$$C(e) = \sum_{a \in V} \sum_{b \in V} \frac{\sigma_{ab}(e)}{\sigma_{ab}}$$

Where σ_{ab} is the number of shortest paths from a to b and $\sigma_{ab}(e)$ is the number of those paths that contain e.

Experiments

Flinder's Challenge data set

The Flinder's University proposed a challenge to solve 1001 Hamiltonian cycle problems (Haythorpe, 2015) in one year. This challenge spanned from September 2015 to September 2016. Although the challenge may be finished, the data set of Hamiltonian cycle problems is still a very large data set, and is quite useful for constructing techniques to solve Hamiltonian cycle problems in general. For our experiments, we ran our methods on the first 200 of the 1001 graphs.

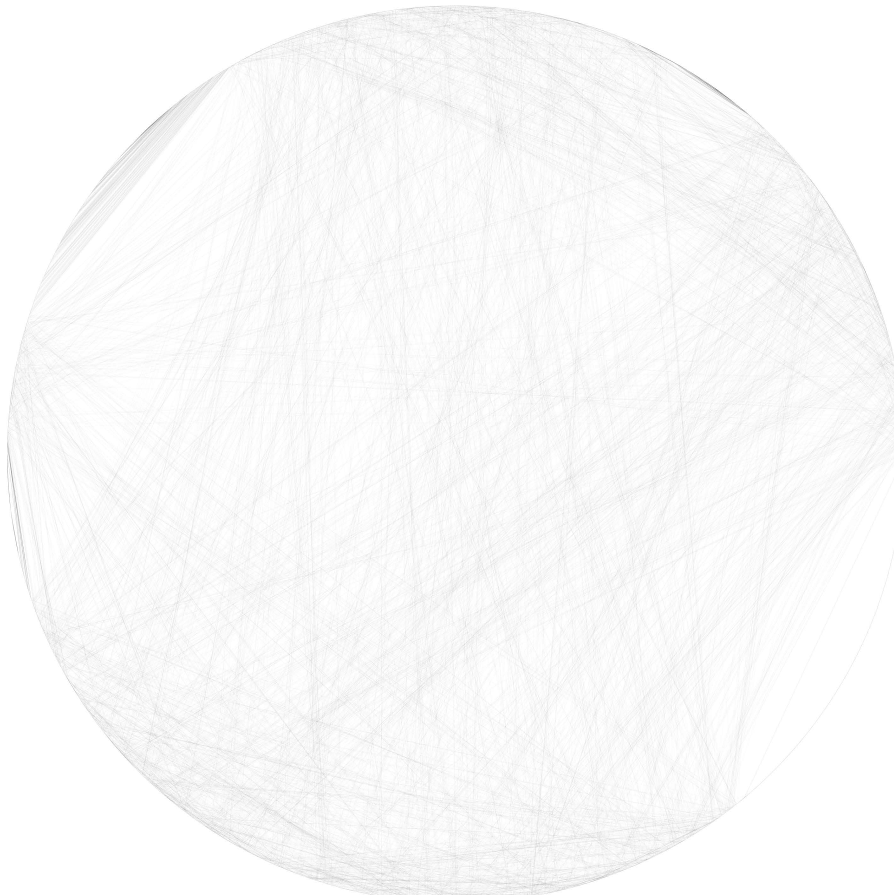

Figure 1: Graph998 from Flinder's Hamiltonian Cycle Problem

Concorde

Concorde (Applegate *et al.* , 2003, 2006) is a state-of-the-art TSP solver, widely used for its ability to find good and optimal solutions in extraordinary computation time. Concorde uses a combination of metaheuristics involving local searches, branch and cuts, Chained Lin-Kernighan, and other techniques to find its solutions. Concorde has solved many real world problems, including vehicle routing problems (Applegate *et al.* , 2002) and predicting protein functions (Johnson & Liu, 2006).

Single Edge elimination

When trying to solve the Hamiltonian cycle problem for a graph, G, with a reduction into the TSP problem, unless the number of edges is equal to or less than the number of vertices, there is an assumption that there are some edges in the graph G which are not in the solution, i.e. the Hamiltonian path. To this end, we came up with a simple algorithm to solve the Hamiltonian problems.

```
Concorde finished ← False;
while Concorde finished = False do
    remove an edge from G;
    transform G to TSP using TC reduction;
    run in Concorde for N seconds;
    if running time < N then
        Concorde finished ← True;
    end
    add the edge back into G;
end
```

Algorithm 1: Single edge elimination algorithms

For our initial single edge elimination, we chose the ordering of the graphs to be the default ordering from the data set, that is, ordered by ascending vertices. For our testing on the first 200 graphs, we chose an N of 40 seconds; this is a rather arbitrary choice, as even the larger graphs were solved in less than a second.

Edge betweenness centrality

We began with the preprocessing step of first sorting all the edges in each of the 1001 Hamiltonian graphs by descending order of edge betweenness centrality. Since edge betweenness centrality and sorting are calculated in polynomial time, the total computation time required for all 1001 graphs was very minimal. After the reordering, we used the Single edge elimination algorithm, eliminating edges using the new ordering, with N as 40 seconds.

Results

Using the method detailed above, so far we have been able to find solutions for over 200 of the 1001 graph Flinder's Hamiltonian Cycle Problem data set. For details on the results, refer to the appendix.

Discussion

Our edge elimination and reduction techniques work well on the problems we have given them, however, we still have to test their robustness on more complicated problems. One particular application could be the use in conjunction with already successful edge elimination methods for the travelling salesman problem, not just Hamiltonian cycles.

Conclusion

Our algorithms have shown that they can solve many Hamiltonian cycle problems in reasonable time, however they still possess some difficulties with more complex problems.

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