Dirac Operators on Manifolds and Applications to Physics

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Abstract

This work was completed in the summer of 2017-18 as part of an AMSI vacation research scholarship under the supervision of Adam Rennie and Alan Carey. The aim of the project was to study differential operators on manifolds and applications to physics, in particular to understand the APS boundary conditions presented by Bar and Strohmaier in [2] and apply them to the case of a cylinder. We further attempt to combine the work in [2] with that of Connes’ in [3] and develop some possible candidates for a trace in relativistic quantum mechanics.

1 Introduction

This report develops the prerequisite material for studying Dirac operators on Lorentzian manifolds and applies the theory to the case of a cylinder with APS boundary conditions as found in [2].

In Section 2 we develop the theory of elliptic differential operators on manifolds using unbounded operator theory for Hilbert spaces. In particular, we extend the notion of an adjoint to a differential operator and prove some results relating to boundary conditions and self-adjointness.

In Section 3 we develop the algebraic prerequisites required for understanding Dirac operators. The section is focused on Clifford algebras, modules and bundles with a number of examples. The concepts of spinor bundles and connections are also introduced.

In Section 4, we define a Dirac operator for a vector bundle over a manifold and provide numerous Euclidean examples, as well as the Dirac operator on a Riemannian cylinder.

In Section 5, the theory developed in the previous sections is utilised to determine the Dirac operator on a Lorentzian cylinder. The eigenvalues for the operators are computed, as well as some candidates for a trace in an attempt to combine the work in [2] and that of Connes’ in [3].

2 Elliptic Differential Operators

The material in this section is mostly based on [5, Chapter 10], with some elements drawn from [10] and [7]. We develop the theory of differential operators on smooth vector bundles over smooth manifolds. For a refresher on vector bundles and manifolds, see [4] and [6].

2.1 First Order Differential Operators

Definition 2.1. Let $M$ be a smooth manifold and let $p : E \to M$ be a smooth vector bundle over $M$. Let $C^\infty(M, E)$ be the space of smooth sections of $E$. A first order linear differential operator on $E$ is a complex linear map $D : C^\infty(M, E) \to C^\infty(M, E)$ which satisfies the following two properties:

1. If $u_1$ and $u_2$ are smooth sections of $E$ which agree on an open set $U \subseteq M$ then $Du_1$ and $Du_2$ agree on $U$, and
2. For each coordinate patch $U \subseteq M$, if we choose coordinates $x_j$ in $U$ and a trivialisation for the bundle $E$ over $U$, then $D$ can be represented in local coordinates by

$$Du = \sum_j A_j \frac{\partial u}{\partial x_j} + Bu,$$

where the $A_j$ and $B$ are smooth matrix valued functions on $U$.

We define the support of $D$ to be the complement of the largest open set $U$ for which the restriction of $D$ to $U$ is zero.

The functions $A_j$ and $B$ above are dependent on the bundle trivialisation, however retain their form. Let $\Phi_U : E_U \to U \times \mathbb{C}^k$ and $\Phi_V : E_V \to V \times \mathbb{C}^k$ be bundle trivialisations. The transition function $g_{UV} : U \cap V \to \text{GL}_k(\mathbb{C})$ is defined as

$$g_{UV} = \Phi_U \circ \Phi_V^{-1}.$$

Given coordinates $\varphi_U : U \to \mathbb{R}^n$ and $u \in C^\infty(M, E|U)$, we have for $x \in \mathbb{R}^n$ that

$$\Phi_U(Du)(x) = \sum_j A_j(x) \frac{\partial}{\partial x_j} \Phi_U(u) \circ \varphi_U^{-1}(x) + B \Phi_u(u) \circ \varphi_U^{-1}(x).$$

Let $y = \varphi_V \circ \varphi_U^{-1}(x)$ and compute that

$$g_{UV}(\Phi_U(Du))(\varphi_V^{-1}(x)) = \Phi_V(Du)(\varphi_V^{-1}(y))$$

$$= \sum_j \left( g_{UV} A_j \frac{\partial}{\partial x_j} (\Phi_U \circ \Phi_V^{-1} \circ \Phi_V u) \right) \left( \varphi_V^{-1} \circ \varphi_V \circ \varphi_U^{-1}(x) \right)$$

$$+ \left( g_{UV} B \Phi_U \circ \Phi_V^{-1} \circ \Phi_V u \right) \left( \varphi_V^{-1} \circ \varphi_V \circ \varphi_U^{-1}(x) \right)$$

$$= \sum_j \left( g_{UV} A_j \frac{\partial g_{UV}}{\partial x_j} \left( \varphi_V^{-1}(x) \right) (\Phi_V(u)) \left( \varphi_V^{-1}(y) \right) \right)$$

$$+ \sum_j g_{UV} A_j g_{UV} \frac{\partial y}{\partial x_j} \frac{\partial \Phi_V(u)}{\partial y} \circ \varphi_V^{-1}(y) + (g_{UV} B g_{UV} \Phi_V(u))(\varphi_V^{-1}(y))$$

$$= \sum_j A_j \frac{\partial \Phi_V(u)}{\partial y} \left( \varphi_V^{-1}(y) \right) + Bu \left( \varphi_V^{-1}(y) \right).$$

From this expression we see that in the new trivialisation we still have a differential operator, however the functions $A_j$ and $B$ have changed.

**Definition 2.2.** Define the symbol of $D$, $\sigma_D : T^*M \to \text{End}(E)$ by

$$\sigma_D(x, \xi) = i \sum_j A_j \xi_j.$$

**Remark 2.3.** Definition 2.1 only refers to first order differential operators. We may more generally consider an order $k$ differential operator by replacing condition (2) with

$$Du = \sum_{|\alpha| \leq k} A_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha},$$
where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) is a multi-index and \( |\alpha| = \sum_{j=1}^{n} \alpha_j \). We may also extend Definition 2.2 to an order \( k \) operator as

\[
\sigma_D(x, \xi) = i^k \sum_{|\alpha|=k} A_{\alpha} \xi^\alpha.
\]

Definition 2.1 can also be extended to differential operators between different vector bundles \( E \) and \( F \) as 

\[
D : C^\infty(M, E) \to C^\infty(M, F),
\]

where in local coordinates \( D \) is given by the same formula as above. The symbol is defined in this case in the same manner, however \( \sigma_D : T^*M \to \text{Hom}(E, F) \). Note also that from the definition, it is clear that if \( D_1 \) and \( D_2 \) are differential operators, then

\[
\sigma_{D_1 \circ D_2}(x, \xi) = \sigma_{D_1}(x, \xi) \circ \sigma_{D_2}(x, \xi).
\]

A number of differential operators are already familiar from vector calculus and Riemannian geometry.

**Example 2.4.** Recall that the Laplace-Beltrami operator \( \Delta : C^\infty(M, E) \to C^\infty(M, E) \) is given in local coordinates as (for \( f \in C^\infty(M, E) \))

\[
\Delta f = \frac{1}{\sqrt{g}} \sum_{j,k} \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{jk} \frac{\partial f}{\partial x_k} \right) = \sum_{j,k} \left( g^{jk} \frac{\partial^2 f}{\partial x_j \partial x_k} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x_j} \frac{\partial f}{\partial x_k} \right).
\]

So we calculate that for \( (x, \xi) \in T^*M \),

\[
\sigma_D(x, \xi) = i^2 \sum_{j,k} g^{jk} \xi_j \xi_k = -\|\xi\|^2.
\]

**Definition 2.5.** If the vector bundle \( E \) has a Hermitian metric \((\cdot, \cdot)\) and \( M \) has a smooth measure \( \mu \), then we may define an inner product on \( C^\infty_c(M, S) \) by

\[
\langle u, v \rangle = \int_M (u(x), v(x)) \, d\mu(x).
\]

Using this we define \( L^2(M, S) \) to be the completion of \( C^\infty_c(M, S) \) relative to this inner product.

**Proposition 2.6.** Let \( D : C^\infty(M, E) \to C^\infty(M, E) \) be a differential operator. There is a unique differential operator \( D^\dagger : C^\infty(M, E) \to C^\infty(M, E) \) such that

\[
\langle Du, v \rangle = \langle u, D^\dagger v \rangle
\]

for all \( u, v \in C^\infty_c(M, E) \). The symbols of \( D \) and \( D^\dagger \) are related by the formula

\[
\sigma_{D^\dagger}(x, \xi) = \sigma_D(x, \xi)^*.
\]

We call \( D^\dagger \) the adjoint of \( D \).
Proof. By using a partition of unity, it suffices to show existence in the case where $D$ is supported in a single coordinate patch. Then we compute that on this patch $U$ we have

$$
(Du,v) = \int_U \left( \sum_j A_j \frac{\partial u}{\partial x_j} + Bu, v \right) \, dx
$$

$$
= \sum_j \int_U \left( A_j \frac{\partial u}{\partial x_j}, v \right) \, dx + \int_U (Bu, v) \, dx
$$

$$
= \sum_j \int_U \left( \frac{\partial u}{\partial x_j}, A_j^* v \right) \, dx + \int_U (u, B^* v) \, dx
$$

$$
= \sum_j \int_U \left[ \frac{\partial}{\partial x_j} \left( u, A_j^* v \right) - \left( u, \frac{\partial}{\partial x_j} \left[ A_j^* v \right] \right) \right] \, dx + \int_U (u, B^* v) \, dx
$$

$$
= -\sum_j \int_U \left( u, \frac{\partial A_j^* v}{\partial x_j} + A_j^* \frac{\partial v}{\partial x_j} \right) \, dx + \int_U (u, B^* v) \, dx
$$

$$
= \int_U \left( u, \sum_j -A_j^* \frac{\partial v}{\partial x_j} + \left[ B^* - \sum_j \frac{\partial A_j}{\partial x_j} \right] v \right) \, dx,
$$

where the second last equality follows from the fact that $D$ is zero in a neighbourhood of the boundary of $U$.

For uniqueness, suppose that $D_1$ and $D_2$ are both formal adjoints for $D$. Define $L = D_1 - D_2$ and compute that for any $u, v \in C^\infty_c(M,E)$ we have

$$
\langle u, Lv \rangle = \langle u, (D_1 - D_2)v \rangle
$$

$$
= \langle u, D_1 v \rangle - \langle u, D_2 v \rangle
$$

$$
= \langle Du, v \rangle - \langle Du, v \rangle
$$

$$
= 0.
$$

Since the range of $L$ is orthogonal to every $u \in C^\infty_c(M,E)$, we have $L = 0$ as required. Note also that from the above formula for $D^\xi$, we see that for $(x, \xi) \in T^* M$ we have

$$
\sigma_{D^\xi}(x, \xi) = i \sum_j -A_j^* \xi_j
$$

$$
= \left( i \sum_j A_j \xi_j \right)^*
$$

$$
= \sigma_D(x, \xi)^*.
$$

Example 2.7. Let $M_g$ denote the operator of multiplication by $g$ in $C^\infty(M,S)$. Then we calculate that for
any smooth section $u$ of $E$

$$([D, M_g]) u = D(gu) - gDu$$

$$= \sum_j \left( A_j \frac{\partial (gu)}{\partial x_j} \right) + Bgu - g \left( \sum_j A_j \frac{\partial u}{\partial x_j} + Bu \right)$$

$$= \sum_j \left( A_j \frac{\partial g}{\partial x_j} u + A_j g \frac{\partial u}{\partial x_j} - gA_j \frac{\partial u}{\partial x_j} \right) + Bgu - gBu$$

$$= \left( \sum_j A_j \frac{\partial g}{\partial x_j} \right) u.$$ 

Evaluating this computation at $x \in M$, we have

$$i[D, M_g]u(x) = \sigma_D(x, dg)u(x).$$

Henceforth we shall refer to a first order differential operator as simply a differential operator.

### 2.2 Symmetric and Self-Adjoint Differential Operators

In this section we develop the Hilbert space theory of differential operators, although much care is needed since these operators are usually not bounded. We consider a differential operator $D$ as an unbounded operator initially defined on $C^\infty_c(M, E) \subset L^2(M, E)$. We begin with some technical results.

**Lemma 2.8.** Let $D : C^\infty_c(M, E) \to C^\infty_c(M, E)$ be a differential operator. Then $D$ is closable.

**Proof.** Suppose that $v_j \to 0$ in $C^\infty_c(M, E)$ and $Dv_j \to w \in L^2(M, E)$. For any $u \in C^\infty_c(M, E)$ we have that

$$\langle u, w \rangle = \lim_{j \to \infty} \langle u, Dv_j \rangle$$

$$= \lim_{j \to \infty} \langle D^\dagger u, v_j \rangle$$

$$= \langle D^\dagger u, 0 \rangle$$

$$= 0.$$ 

Since $w$ is orthogonal to $C^\infty_c(M, E)$, a dense subspace of $L^2(M, E)$, we have that $w = 0$ and so $D$ is closable. 

The overline indicating closure of an operator will be omitted when the distinction between the two is not necessary. Recall that an operator $T$ is symmetric if $\langle Tu, v \rangle = \langle u, Tv \rangle$ for all $u, v \in \text{Dom}(T)$. For a differential operator on a manifold without boundary this amounts to being self-adjoint, and these are the primary operators of interest to us.

**Definition 2.9.** Let $D : C^\infty_c(M, E) \to C^\infty_c(M, E)$ be a symmetric differential operator. The minimal domain of $D$ is $\text{Dom}(D)$ and the maximal domain of $D$ is $\text{Dom}(D^*)$.

Note that if $D$ is symmetric then $D \subseteq \overline{D} \subseteq D^*$ and all three of these operators may be different from one another.
Example 2.10. Consider the manifold $M = (0,1)$. Let $\text{Dom}(D) = C_c^\infty(M, \mathbb{C})$ and define the differential operator $D : \text{Dom}(D) \to L^2([0,1])$ by $D = i \frac{d}{dx}$. Recall that

$$\text{Dom}(D^*) = \{ \xi \in L^2([0,1]) : \text{there exists } \eta \in L^2([0,1]) \text{ s.t. for all } \psi \in \text{Dom}(D), \langle D\psi, \xi \rangle = \langle \psi, \eta \rangle \}. $$

Then we may check that $\xi(x) = x$ is in $\text{Dom}(D^*)$, since for any $\psi \in \text{Dom}(D)$,

$$\langle D\psi, \xi \rangle = i \int_0^1 \frac{d\psi}{dx} x \, dx = i \int_0^1 \frac{d}{dx} (\psi(x)x) \, dx - i \int_0^1 \psi(x) \, dx = (\psi, -i).$$

However $\xi \notin \text{Dom}(D)$. To see this, first fix $u \in \text{Dom}(D)$ and calculate that

$$\langle Du, 1 \rangle = i \int_0^1 \frac{du}{dx} \, dx = 0,$$

then extend by continuity in the graph norm ($\|\xi\|_{Gr} = \|\xi\| + \|D\xi\|$) to see that this holds for all $u \in \text{Dom}(D)$ also. Now note that

$$\langle D\xi, 1 \rangle = i \int_0^1 1 \, dx = i \\ \\
\neq 0.$$ 

Hence $D^*$ and $D$ are distinct operators.

The example above suggests that failure of an operator to be self-adjoint is related to boundary conditions on a non-compact manifold $M$. We will make this more precise soon, but some technical results are required.

Elements of the proof of the following result were adapted from [10, Proposition 3.2].

Lemma 2.11. Let $K \subset M$ be compact. For sufficiently small $t > 0$, there exist operators $F_t : L^2(K, E) \to L^2(M, E)$ such that

1. $\|F_t\| < 1$,

2. for each $u \in L^2(K, E)$, $F_t u \to u$ in $L^2(M, E)$ as $t \to 0$,

3. for each $u \in L^2(K, E)$, $F_t u$ is smooth, with compact support, and

4. the commutator $[D, F_t]$ extends to a bounded operator from $L^2(K, E)$ to $L^2(M, E)$, whose norm is bounded independent of $t$.

We call such a family Friedrichs mollifiers on $M$. 

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Proof. Fix $\varepsilon > 0$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth positive function with compact support and total mass 1 with $\text{supp}(\varphi) \subseteq B_1(0)$. There are many choices of such a function, one of which is $\varphi(x) = c_n e^{-\frac{1}{1-|x|^2}}$ on $B_1(0)$ extended to 0 on $\mathbb{R}^n$, where $c_n$ is a normalising constant. Define $F_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by $F_t u(x) = t^{-n} \int_{\mathbb{R}^n} u(y) \varphi\left(\frac{x-y}{t}\right) \, dy$. Note that $F_t u(x) = (u * \varphi_t)(x)$, where $\varphi_t(x) = t^{-n} \varphi\left(\frac{x}{t}\right)$.

To estimate the norm, we apply Young’s inequality to see that

\[
\|F_t u\|_2 = \|u * \varphi_t\|_2 \\
\leq \|u\|_2 \|\varphi_t\|_1 \\
= \|u\|_2.
\]

Hence taking the supremum over all $u$ with $\|u\|_2 = 1$, we have that $\|F_t\| \leq 1$.

Next, we aim to show $L^2$ convergence. To this end, we first show that $F_t w \to w$ uniformly for $w \in C_c(\mathbb{R}^n)$. Note that $\text{supp}(\varphi_t) \subseteq B_1(0)$ and so we may find $s > 0$ such that $\sup \{|w(x) - w(y)| : |y| \leq t\} < \varepsilon$ for $t < s$. So fix $t < s$ and calculate that

\[
|F_t w(x) - w(x)| = \left| \int_{\mathbb{R}^n} w(x-y) \varphi_t(y) \, dy - \int_{\mathbb{R}^n} w(x) \varphi_t(y) \, dy \right| \\
\leq \int_{\mathbb{R}^n} |w(x-y) - w(x)| \varphi_t(y) \, dy \\
\leq \sup \{|w(x-y) - w(x)| : |y| \leq t\} \\
< \varepsilon.
\]

Find $v \in C_c(\mathbb{R}^n)$ such that $\|u - v\| < \frac{\varepsilon}{3}$. Applying property 1 we have that

\[
\|F_t w - F_t v\|_2 = \|F_t (u - v)\|_2 \\
\leq \|u - v\|_2 \\
< \frac{\varepsilon}{3}.
\]

Choose $t > 0$ such that $\|F_t v - v\|_2 < \frac{\varepsilon}{3}$ (possible since uniform convergence implies $L^2$ convergence) and calculate that

\[
\|F_t u - u\|_2 \leq \|F_t u - F_t v\|_2 + \|F_t v - v\|_2 + \|u - v\|_2 \\
\leq \frac{2\varepsilon}{3} + \|F_t v - v\| \\
< \varepsilon.
\]

Hence $F_t u \to u$ in $L^2$.

By definition, we have that $\text{supp}(F_t u) \subseteq \{x+y : x \in \text{supp}(u) \text{ and } y \in B_1(0)\}$ which is bounded. Hence
supp($F_t u$) is compact. To check smoothness, we calculate that

$$
\lim_{h \to 0} \frac{F_t u(x + he_i) - F_t u(x)}{h} = \lim_{h \to 0} \int_{\mathbb{R}^n} \frac{\varphi_t(x + he_i - y) - \varphi_t(x - y)}{h} u(y) \, dy \\
= \int_{\mathbb{R}^n} \frac{\partial \varphi_t(x - y)}{\partial x_i} u(y) \, dy \\
= u \ast \frac{\partial \varphi}{\partial x_i}.
$$

So we see inductively that $F_t u$ is smooth, since $\varphi_t$ is smooth. Note also that in a similar manner to the above calculation, we can show that $\frac{\partial}{\partial x_j} (u \ast \varphi_t) = \frac{\partial u}{\partial x_j} \ast \varphi_t = u \ast \frac{\partial \varphi_t}{\partial x_j}$. So we calculate the commutator for any $u \in C_c^\infty(\mathbb{R}^n)$ as

$$
[D, F_t] u(x) = D(F_t u) - F_t(Du) \\
= \sum_j A_j \frac{\partial}{\partial x_j} (u \ast \varphi_t) + B(u \ast \varphi_t) - \sum_j \left( A_j \frac{\partial u}{\partial x_j} \right) \ast \varphi_t - (Bu \ast \varphi_t) \\
= \sum_j \left( A_j \left( u \ast \frac{\partial \varphi_t}{\partial x_j} \right) + B(u \ast \varphi_t) - \sum_j \left( A_j \frac{\partial u}{\partial x_j} \right) \ast \varphi_t - (Bu \ast \varphi_t) \right),
$$

where we have moved the derivative from $u$ to $\varphi$ as discussed before the calculation. We now write the convolution in integral form and apply integration by parts, giving

$$
[D, F_t] u(x) = \sum_j \left( A_j(x) \int_{\mathbb{R}^n} u(y) \frac{\partial \varphi_t}{\partial x_j}(x - y) \, dy - \int_{\mathbb{R}^n} A_j(y) \frac{\partial u}{\partial y_j} \varphi_t(x - y) \, dy \right) \\
+ B(x) \int_{\mathbb{R}^n} u(y) \varphi_t(x - y) \, dy - \int_{\mathbb{R}^n} B(y) u(y) \varphi_t(x - y) \, dy \\
= \sum_j \int_{\mathbb{R}^n} \left( (A_j(x) - A_j(y)) u(y) \frac{\partial \varphi_t}{\partial x_j}(x - y) - A_j(y) \frac{\partial}{\partial y_j} (u(y) \varphi_t(x - y)) \right) \, dy \\
+ \int_{\mathbb{R}^n} (B(x) - B(y)) u(y) \varphi_t(x - y) \, dy.
$$

We now rearrange terms and use the fact that $\frac{\partial \varphi_t}{\partial x_j}(x - y) = -\frac{\partial \varphi_t}{\partial y_j}(x - y)$ to obtain

$$
[D, F_t] u(x) = \sum_j \int_{\mathbb{R}^n} (A_j(x) - A_j(y)) u(y) \frac{\partial \varphi_t}{\partial x_j}(x - y) \, dy + \int_{\mathbb{R}^n} (B(x) - B(y)) u(y) \varphi_t(x - y) \, dy \\
= \sum_j \frac{1}{t} \int_{\mathbb{R}^n} (A_j(x) - A_j(y)) u(y) \frac{\partial \varphi}{\partial x_j} \left( \frac{x - y}{t} \right) \, dy + \int_{\mathbb{R}^n} (B(x) - B(y)) u(y) \varphi_t(x - y) \, dy.
$$

Using local coordinates and a partition of unity, it is possible to graft this onto an arbitrary manifold to construct a family of Friedrichs mollifiers there. The proof and technique for this are quite involved and will not be presented here.

We will need another technical lemma before being able to prove the result about ‘boundary conditions’, however the proof is omitted. The proof can be found in [5, Lemma 1.8.1].

**Lemma 2.12.** Let $T$ be a closable operator on a Hilbert space $\mathcal{H}$. Then $u \in \mathcal{H}$ belongs to the minimal domain if and only there is a sequence $(u_j)$ in the domain of $T$ such that $u_j \to u$ while $\|Tu_j\|$ remains bounded.
**Lemma 2.13.** Let $D$ be a symmetric differential operator on a bundle $E$ over a manifold $M$ and suppose that $u$ is a compactly supported element of $L^2(M, E)$. Then $u$ belongs to the minimal domain of $D$ if and only if it belongs to the maximal domain.

**Proof.** We use the Friedrichs' mollifiers developed in Lemma 2.11. Suppose $u \in \text{Dom}(D^*)$ is supported in $K$. We know that $F_t u \to u$ as $t \to 0$ and $F_t u$ is compactly supported and smooth. We also have that

$$DF_t u = F_t Du + [D, F_t]u,$$

which is bounded in norm by Lemma 2.11. Hence by Lemma 2.12, we have that $u$ belongs to the maximal domain if and only if belongs to the minimal domain.

**Corollary 2.14.** Every compactly supported symmetric differential operator on an open manifold is essentially self-adjoint.

**Proof.** Applying Lemma 2.13, we see that $\text{Dom}(D) = \text{Dom}(D^*) = \text{Dom}(D^*)$. Hence $D$ is self-adjoint and $D$ is essentially self-adjoint.

3 **Algebraic Prerequisites**

We wish to work with a specific class of differential operators, called Dirac operators. However before defining Dirac operators, we must first develop some algebraic prerequisites from the theory of Clifford algebras and vector bundles. The material in this section is primarily found in [7, Chapter I].

3.1 **Clifford Algebras**

**Definition 3.1.** Let $V$ be a finite dimensional vector space and $q : V \times V \to \mathbb{C}$ a non-degenerate inner product. Define an ideal $J_q$ in the tensor algebra $T(V) = \bigoplus_{n \geq 0} V \otimes^n$ generated by elements of the form $v \otimes v + q(v, v)1$ for $v \in V$. We define the Clifford algebra $\text{Cliff}(V, q)$ to be the quotient

$$\text{Cliff}(V, q) = T(V)/J_q.$$

Note that $\text{Cliff}(V, q)$ is generated by 1 and $v, w \in V$ satisfying

$$v \cdot w + w \cdot v = -2q(v, w).$$

Hence if $v$ and $w$ are orthogonal they anticommute in $\text{Cliff}(V, q)$. Clifford algebras also satisfy a universal property as follows.

**Lemma 3.2.** If $A$ is a unital associative algebra and $c : V \to A$ is a linear map satisfying

$$c(v)c(v) = -q(v, v)1_A$$

for all $v \in V$, there exists a unique algebra homomorphism $\tilde{c} : \text{Cliff}(V, q) \to A$ extending $c$. 

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Proof. Firstly, there exists a unique algebra homomorphism \( \varphi : T(V) \to A \) such that \( \varphi \circ \iota = c \), where \( \iota : V \to T(V) \) is the inclusion map, defined by \( \varphi(v_1 \otimes \cdots \otimes v_n) = c(v_1)\cdots c(v_n) \). Next, check that for \( v \in V \) we have

\[
\varphi(v \otimes v + q(v,v)) = \varphi(v \otimes v) + \varphi(q(v,v)) = c(v) \cdot c(v) + q(v,v)1_A
\]

\( = 0. \)

Hence \( \varphi \) vanishes on \( J_q \) and so descends uniquely to \( \tilde{c} \) on \( \text{Cliff}(V,q) \).

Example 3.3. Note that if \( A \in O(V,q) = \{ A : V \to V : q(Av,Av) = q(v,v) \text{ for all } v \in V \} \), then we may define a linear map \( c : V \to \text{Cliff}(V,q) \) by \( c(v)w = Av \cdot w \). Since this satisfies

\[
c(v)c(v) = Av \cdot Av = -q(Av,Av) = -q(v,v).
\]

Hence by Lemma 3.1, \( c \) extends uniquely to an automorphism \( \tilde{c} : \text{Cliff}(V,q) \to \text{Cliff}(V,q) \). If we consider the map \( \alpha : V \to V \) defined by \( \alpha(v) = -v \), we have an involution on \( \text{Cliff}(V,q) \) and thus we may decompose into the eigenspaces for \( \pm 1 \) as

\[
\text{Cliff}(V,q) = \text{Cliff}^0(V,q) \oplus \text{Cliff}^1(V,q).
\]

Since \( \alpha(v_1 \cdots v_n) = (-1)^n v_1 \cdots v_n \) we see that \( \text{Cliff}(V,q) \) decomposes into sums of even and odd products of vectors. Since \( \alpha \) is an algebra homomorphism, we see that

\[
\text{Cliff}^i(V,q) \cdot \text{Cliff}^j(V,q) \subseteq \text{Cliff}^{i+j}(V,q),
\]

where the addition of indices is mod 2. Algebras satisfying this property are said to be \( \mathbb{Z}_2 \)-graded.

3.2 Clifford Modules and Bundles

Definition 3.4. A \( \mathbb{C} \)-representation of the Clifford algebra \( \text{Cliff}(V,q) \) is an \( \mathbb{R} \)-algebra homomorphism \( \varphi : \text{Cliff}(V,q) \to \text{End}_\mathbb{C}(W) \) into the algebra of complex linear transformations of a finite dimensional complex vector space \( W \). A vector space \( W \) together with a \( \text{Cliff}(V,q) \) representation is called a Clifford module.

For \( v \in \text{Cliff}(V,q) \) and \( w \in W \), we write

\[
v \cdot w = \varphi(v)(w)
\]

and refer to this as Clifford multiplication of \( w \) by \( v \). We may define \( \mathbb{R} \)-representations and \( \mathbb{H} \)-representations in a similar manner.
Definition 3.5. A \( \mathbb{C} \)-representation is called reducible if there exists a decomposition \( W = W_1 \oplus W_2 \) such that \( \varphi(v)(W_j) \subset W_j \) for \( j = 1, 2 \) and all \( v \in \text{Cliff}(V,q) \). In this case we may write \( \varphi = \varphi_1 \oplus \varphi_2 \). A representation is irreducible if there are no proper invariant subspaces. Two representations are equivalent if there exists a \( \mathbb{C} \)-linear isomorphism \( F : W_1 \rightarrow W_2 \) such that \( F \varphi_1(v)F^{-1} = \varphi_2(v) \) for all \( v \in \text{Cliff}(V,q) \).

It is a non trivial fact that every complex \( \text{Cliff}(V,q) \) module decomposes uniquely as a sum of irreducibles.

Example 3.6. Define \( \varphi : \text{Cliff}(V,q) \rightarrow \text{Cliff}(V,q) \) by \( \varphi(v)w = v \cdot w \). Then we see that \( \text{Cliff}(V,q) \) is trivially a \( \text{Cliff}(V,q) \) module.

Example 3.7. For a more involved example, consider the exterior algebra \( \Lambda^* V \). Let \( (v_1, \ldots, v_n) \) be an orthonormal basis for \( V \). Define exterior multiplication by \( v_j \) as \( e_j \) and interior multiplication by \( v_j \) as \( i_j \). Recall that \( e_ji_k + i_ke_j = \delta_{jk} \). Define \( \varphi_j = e_j - i_j \) and calculate that
\[
\varphi_j \circ \varphi_k + \varphi_k \circ \varphi_j = (e_j - i_j)(e_k - i_k) + (e_k - i_k)(e_j - i_j)
\]
\[
= e_je_k - e_ji_k - i_je_k + i_je_k - e_ke_j + i_ke_j + i_je_k - i_ke_j + i_je_k - i_ke_j
\]
\[
= -2\delta_{jk}.
\]
So for a fixed \( v_j \), define a linear map \( \varphi : V \rightarrow \text{End}(\Lambda^* V) \) by \( \varphi(v) = \varphi_j(v) \). We can uniquely lift this to \( \tilde{\varphi} : \text{Cliff}(V) \rightarrow \text{End}(\Lambda^* V) \) by the universal property (Lemma 3.1). Hence \( \Lambda^* V \) is a \( \text{Cliff}(V,q) \) module.

We now wish to extend the properties of Clifford algebras to vector bundles, in order to apply the results to manifolds and their tangent and cotangent bundles.

Recall that by Example 3.3, for any \( A \in \text{SO}_n \) we have a unique extension to \( \text{Cliff}(\mathbb{R}^n) \). Hence we have a representation \( \rho_n : \text{SO}_n \rightarrow \text{Aut}(\text{Cliff}(\mathbb{R}^n)) \) defined by lifting the transformation to the Clifford algebra. This representation will allow us to construct bundles of Clifford algebras.

Definition 3.8. The Clifford bundle of the oriented Riemannian vector bundle \( E \) over the manifold \( M \) is the bundle
\[
\text{Cliff}(E) = P_{\text{SO}(S)} \times_{\rho_n} \text{Cliff}(\mathbb{R}^n).
\]

Note that we could also have defined \( \text{Cliff}(E) \) as the quotient bundle
\[
\text{Cliff}(E) = T(E)/I(E),
\]
where \( T(E) = \bigoplus_{n \geq 0} E \otimes \mathbb{R}^n \) and \( I(E) \) is the bundle of ideals whose fibre at \( x \in M \) is the two-sided ideal \( I(E_x) \) in \( T(E_x) \) generated by elements \( v \otimes v + \|v\|^2 \) for \( v \in E_x \). From this definition we see that \( \text{Cliff}(E) \) is a bundle of Clifford algebras over \( M \). By defining fibrewise multiplication for sections \( u, v : M \rightarrow \text{Cliff}(E) \) as
\[
(u \cdot v)_x = u(x)v(x)
\]
we may give an algebra structure to the space of sections, \( \Gamma(E) \). Note that many notions for Clifford algebras carry through to the theory of Clifford bundles. For example, if we consider the bundle automorphism
\[
\tilde{\alpha} : \text{Cliff}(E) \to \text{Cliff}(E)
\]
defined by extending the map \( \alpha \) in Example 3.1, then the eigenbundles for \( \pm 1 \) gives a corresponding decomposition
\[
\text{Cliff}(E) = \text{Cliff}^0(E) \oplus \text{Cliff}^1(E).
\]

### 3.3 Spinor Bundles

**Definition 3.9.** The spin groups are defined, for \( n \in \mathbb{N} \), as
\[
\text{Spin}(n) = \{ v_1 \cdots v_k : v_i \in \mathbb{R}^n, \| v_i \| = 1 \text{ for } 1 \leq i \leq k \text{ and } k \text{ is even} \}.
\]
Note that for a unit vector \( v \in \mathbb{R}^n \) we have \( v \cdot v = -1 \) and \( v \cdot (-v) = 1 \) we see that \( \{ 1, -1 \} \subset \text{Spin}(n) \). We also see that by definition \( \text{Spin}(n) \subseteq \text{Cliff}_0(\mathbb{R}^n) \).

For \( n \geq 3 \) there exists a universal covering homomorphism \( \xi_0 : \text{Spin}(n) \to \text{SO}(n) \). The kernel of this homomorphism is \( \{ -1, 1 \} \).

**Definition 3.10.** Suppose \( n \geq 3 \). Then a spin structure on \( E \) is a principal \( \text{Spin}(n) \) bundle \( P_{\text{Spin}}(E) \) together with a 2-sheeted covering
\[
\xi : P_{\text{Spin}}(E) \to P_{\text{SO}}(E)
\]
such that \( \xi(pg) = \xi(p)\xi_0(g) \) for all \( p \in P_{\text{Spin}}(E) \) and all \( g \in \text{Spin}(n) \).

When \( n = 2 \), the definition remains the same except that \( \xi_0 : \text{SO}_2 \to \text{SO}_2 \) is now the connected double cover. When \( n = 1 \), \( P_{\text{SO}}(E) \cong M \) and a spin structure is a double cover of \( M \).

**Definition 3.11.** Let \( E \to M \) be an oriented Riemannian vector bundle with a spin structure \( \xi : P_{\text{Spin}}(E) \to P_{\text{SO}}(E) \). A real spinor bundle of \( E \) is a bundle of the form
\[
S(E) = P_{\text{Spin}}(E) \times_{\mu} L,
\]
where \( L \) is a left module for \( \text{Cliff}(\mathbb{R}^n) \) and \( \mu : \text{Spin}_n \to \text{SO}(L) \) is the representation given by left multiplication by elements of \( \text{Spin}_n \subset \text{Cliff}_0(\mathbb{R}^n) \).

Likewise, a complex spinor bundle of \( E \) is a bundle of the form
\[
S_\mathbb{C}(E) = P_{\text{Spin}}(E) \times_{\mu} L_\mathbb{C},
\]
where \( L_\mathbb{C} \) is a complex left module for \( \text{Cliff}(\mathbb{R}^n) \otimes \mathbb{C} \). If \( L \) (or \( L_\mathbb{C} \)) is \( \mathbb{Z}_2 \)-graded, we say the corresponding spinor bundle is \( \mathbb{Z}_2 \)-graded.

**Example 3.12.** Consider \( \text{Cliff}(\mathbb{R}^n) \) as a module acting on itself by left multiplication \( \mu \). This gives a 'principal \( \text{Cliff}(\mathbb{R}^n) \) bundle' with a free right action \( \alpha : \text{Cliff}(\mathbb{R}^n) \times S(\text{Cliff}(\mathbb{R}^n)) \to S(\text{Cliff}(\mathbb{R}^n)) \).
3.4 Connections

Definition 3.13. Let $M$ be a Riemannian manifold and let $E$ be a smooth ($\mathbb{R}$ or $\mathbb{C}$) vector bundle over $M$ of dimension $n$. A connection on $E$ is a linear map $\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ such that

$$\nabla(fu) = df \otimes u + f \nabla u$$

for all $f \in C^\infty(M)$ and $u \in C^\infty(M, E)$. Given a smooth vector field $X : M \to TM$ we thus have a map $\nabla_X : C^\infty(M, E) \to C^\infty(M, E)$, called the covariant derivative with respect to $X$. Note that this covariant derivative satisfies

$$\nabla_X(fu) = Xu + f \nabla_X u.$$

Definition 3.14. Let $(v_1, \ldots, v_n)$ be a local orthonormal basis for $E_x$ and $(e_1, \ldots, e_m)$ a basis for $T_xM$. Then we define the connection coefficients by

$$\nabla_{e_i} v_j = \Gamma^k_{ij} v_k.$$

Definition 3.15. A connection $\nabla$ on a Riemannian bundle $E$ is called Riemannian if it satisfies the inner product rule

$$X(u, u') = (\nabla_X u, u') + (u, \nabla_X u')$$

for all $X \in C^\infty(M, TM)$ and $u, u' \in C^\infty(M, S)$.

4 Dirac Operators

We give a motivation for the development of Dirac operators as presented in [7, Chapter 2] The study of Dirac operators begins historically with physicist Paul Dirac, who was searching for a first order differential operator that was Lorentz invariant and squared to the Klein-Gordon operator,

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + \frac{m^2 c^2}{\hbar^2}.$$

Thus he was essentially looking for an operator $D$ satisfying $D^2 = \Box$ (the d’Alembertian) compatible with special relativity, and guessed the form

$$D = \sum_{k=1}^{n} \psi_k \frac{\partial}{\partial x^k}.$$

Dirac realised that the $\psi_k$ must be matrices and satisfy what we now recognise as the generating relations for a representation of $\text{Cliff}(\mathbb{R}^n)$. Generalisations of these operators will be the differential operators of interest to us.

In modern physics we are often working in 3D-spacetime, called Minkowski space, rather than Euclidean space. Minkowski space is defined as the inner product space $(\mathbb{R}^{n+1}, q)$ where for $x = (x_0, x_1, \ldots, x_n)$ and
\( y = (y_0, y_1, \ldots, y_n) \) in \( \mathbb{R}^n \) we have

\[ q(x, y) = -x_0 y_0 + \sum_{k=1}^{n} x_k y_k. \]

This is the framework used for the study of special relativity. Just as Euclidean space can be generalised to Riemannian manifolds, Minkowski space can be generalised to an object called a Lorentzian manifold. A Lorentzian manifold is defined in the same manner as a Riemannian manifold however the metric is no longer required to be positive definite, but is required to have signature \((1, n)\) where \((n + 1)\) is the dimension of the manifold. For an in depth discussion on pseudo-Riemannian and Lorentzian geometry, see [8] and [1, Appendix 2].

4.1 Definition

**Definition 4.1.** Let \( M \) be a Riemannian manifold of dimension \( n \) and \( E \) a smooth complex vector bundle over \( M \). Write left Clifford multiplication as \( c : C^\infty(M, TM \otimes E) \to C^\infty(M, E) \) and let \( \nabla : C^\infty(M, E) \to C^\infty(M, T^* M \otimes E) \) be a connection. Let \( \phi : C^\infty(M, T^* M \otimes E) \to C^\infty(M, TM \otimes E) \) denote the isomorphism between vector and covector fields. We define the Dirac operator associated to the bundle \( E \) and connection \( \nabla \),

\[ D : C^\infty(M, E) \to C^\infty(M, E), \]

as \( D = c \circ \phi \circ \nabla \). Note that this also depends on the representation chosen.

Thus for an orthonormal frame \((e_k)\) of \((TM)_x\), with dual frame \((e^k)\) of \((T^* M)_x\) we have that

\[
Du(x) = c \circ \phi \left( \sum_{k=1}^{n} e^k \otimes \nabla e_k u|_x \right) \\
= c \left( \sum_{k=1}^{n} e^k \otimes \nabla e_k u|_x \right) \\
= \sum_{k=1}^{n} e^k \cdot \nabla e_k u|_x.
\]

**Example 4.2.** Choosing normal coordinates at \( x \in M \) so that \( \frac{\partial}{\partial x^j} (x) = e_j(x) \), we may compute the Dirac operator.
Laplacian $D^2$ as

$$D^2 u(x) = \sum_{k=1}^{n} c_k(x) \cdot \nabla e_k(x) (Du)$$

$$= \sum_{k=1}^{n} c_k(x) \cdot \nabla e_k(x) \left( \sum_{j=1}^{n} e_j(x) \cdot \nabla e_j(x) u \right)$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ e_k(x) \cdot e_j(x) \cdot \nabla e_k(x) \nabla e_k(x) u + e_k(x) \nabla e_k(x) e_j(x) \cdot \nabla e_j(x) u \right]$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ -g^{jk} \cdot \nabla e_k(x) \nabla e_j(x) u + e_k(x) \nabla e_k(x) e_j(x) \cdot \nabla e_j(x) u \right]$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ -g^{jk} \cdot \nabla \frac{\partial u}{\partial x_k} \nabla \frac{\partial u}{\partial x_j} f_l + e_k(x) \nabla e_k(x) e_j(x) \cdot \nabla \frac{\partial u}{\partial x_j} u \right]$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ -g^{jk} \cdot \nabla \frac{\partial u}{\partial x_k} \left( \frac{\partial u}{\partial x_l} (x) f_l \right) + u^l \nabla \frac{\partial u}{\partial x_j} f_l \right] + \frac{\partial}{\partial x_j} (x) u^l f_l, x \nabla W e_k(x) \nabla W e_j(x) u$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ -g^{jk} \cdot \nabla \frac{\partial u}{\partial x_k} \nabla \frac{\partial u}{\partial x_j} f_l + e_k(x) \nabla e_k(x) e_j(x) \cdot \nabla \frac{\partial u}{\partial x_j} u \right]$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ -g^{jk} \cdot \nabla \frac{\partial u}{\partial x_k} \left( \frac{\partial u}{\partial x_l} (x) f_l \right) + u^l \nabla \frac{\partial u}{\partial x_j} f_l \right] + \nabla \frac{\partial u}{\partial x_j} v_l + u^l \nabla \frac{\partial u}{\partial x_j} v_l$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ -g^{jk} \cdot \nabla \frac{\partial u}{\partial x_k} \left( \frac{\partial u}{\partial x_l} (x) f_l \right) + u^l \nabla \frac{\partial u}{\partial x_j} f_l \right] + \nabla \frac{\partial u}{\partial x_j} v_l + u^l \nabla \frac{\partial u}{\partial x_j} v_l$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left[ -g^{jk} \cdot \nabla \frac{\partial u}{\partial x_k} \left( \frac{\partial u}{\partial x_l} (x) f_l \right) + u^l \nabla \frac{\partial u}{\partial x_j} f_l \right] + \nabla \frac{\partial u}{\partial x_j} v_l + u^l \nabla \frac{\partial u}{\partial x_j} v_l, x \nabla W e_k(x) \nabla W e_j(x) u$$

Note that everything in this computation is evaluated at $x \in M$. From this computation we see that

$$\sigma_{D^2}(x, \xi) = -\sum_{j=1}^{n} \sum_{k=1}^{n} g^{jk} \xi_j \xi_k$$

$$= -\|\xi\|^2.$$ 

Thus $D^2$ is a Laplace type operator, as expected.

### 4.2 Examples

We will now examine some Euclidean examples of Dirac operators. Consider the simple manifold $M = \mathbb{R}^n$ with the standard inner product and let $\text{Cliff}(\mathbb{R}^n)$ be its Clifford algebra. Let $W$ be a complex vector space of dimension $m$ and $\varphi : \text{Cliff}(\mathbb{R}^n) \rightarrow \text{End}_\mathbb{C}(W)$ be a representation of $\text{Cliff}(\mathbb{R}^n)$. Let $E = M \times W$ have the trivial Levi-Civita connection. We consider the Dirac operator $D : C^\infty(M, E) \rightarrow C^\infty(M, E)$. Then from Definition 4.1 we have for $u \in C^\infty(M, E)$ and $\{e_1, \ldots, e_n\}$ the standard basis vectors for $\mathbb{R}^n$ that

$$Du = \sum_{k=1}^{n} c_k \cdot \nabla e_k u$$

$$= \sum_{k=1}^{n} \varphi(e_k) \frac{\partial u}{\partial x^k}$$

$$= \sum_{k=1}^{n} \varphi_k \frac{\partial u}{\partial x^k}$$

where $\varphi(e_k) := \varphi_k : W \rightarrow W$ are linear maps satisfying

$$\varphi_k \circ \varphi_j + \varphi_j \circ \varphi_k = -2\delta_{kj} \text{Id}_W.$$  

(4.2)
So identifying \( \text{End}_C(W, W) \cong M_n(C) \) and applying Equation 4.2 with \( j = k \) shows that \( \varphi_k \in \text{GL}_m(C) \) for all \( 1 \leq k \leq n \). Note also that applying 4.1 and the fact that \( g^{jk} = \delta_{jk} \) and \( \Gamma^k_{ij} = 0 \) for all \( i, j, k \) gives

\[
D^2 u = -\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x^k} \text{Id}_W.
\]

**Example 4.3.** Let \( n = 1 \), so \( \text{Cliff}(\mathbb{R}) = \mathbb{C} \) and let \( W = \mathbb{C} \). Then we have a single linear \( \varphi : \mathbb{C} \to \mathbb{C} \) satisfying \( \varphi \circ \varphi = -1 \). So we have two choices, \( \varphi(x) = \pm ix \). The usual convention is \( \varphi(x) = -ix \) for the Dirac operator to be self adjoint. Thus the Dirac operator on \( \mathbb{R} \) is

\[
D = \frac{1}{i} \frac{d}{dx}.
\]

**Example 4.4.** Let \( n = 2 \) and \( W = \mathbb{C} \oplus \mathbb{C} \), so that \( \text{Cliff}(\mathbb{R}^2) = \mathbb{H} \cong \mathbb{C} \oplus \mathbb{C} \), as \( \mathbb{R} \)-linear spaces (not as algebras). Note that the decomposition of \( \mathbb{H} \) into \( \mathbb{C} \oplus \mathbb{C} \) corresponds to the \( \mathbb{Z}_2 \) grading \( \text{Cliff}(\mathbb{R}^2) = \text{Cliff}^0(\mathbb{R}^2) \oplus \text{Cliff}^1(\mathbb{R}^2) \).

To see this, define \( \alpha_1 : \text{Cliff}^0(\mathbb{R}^2) \to \mathbb{C} \) and \( \alpha_2 : \text{Cliff}^1(\mathbb{R}^2) \) by

\[
\alpha_0(x + ye_1e_2) = x + iy = \alpha_2(xe_1 + ye_2).
\]

Define \( \varphi_0 = \varphi(1) = \text{Id} \). To find \( \varphi_1 \) and \( \varphi_2 \), we note that we need two \( 2 \times 2 \) matrices who square to \(-\text{Id}\) and anticommute.

Hence we recall two of the three Pauli matrices as

\[
\varphi_1 = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

and

\[
\varphi_2 = \begin{pmatrix}
0 & i \\
i & 0
\end{pmatrix}.
\]

Note that the third Pauli matrix is generated by the relation

\[
\varphi(e_1e_2) = \varphi(e_1)\varphi(e_2) = \begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix}.
\]

Then we have for \( u \in C^\infty(\mathbb{R}^2, S) \) that

\[
Du = \varphi_1 \frac{\partial u}{\partial x^1} + \varphi_2 \frac{\partial u}{\partial x^2} = \begin{pmatrix}
0 & \frac{\partial u}{\partial x^1} - i \frac{\partial u}{\partial x^2} \\
\frac{\partial u}{\partial x^2} + i \frac{\partial u}{\partial x^1} & 0
\end{pmatrix}.
\]

Thus we see that on \( \mathbb{R}^2 \), \( D \) is the Cauchy-Riemann operator.

A more interesting example to consider is the Dirac operator on a cylinder.

**Example 4.5.** Let \( M = \mathbb{R} \times S^1 \) be the infinite cylinder with the product metric \( g = dt^2 + d\theta^2 \). Let \( C = [t_1, t_2] \times S^1 \subset M \) be the finite cylinder with metric \( \tilde{g} = g|_C \) and \( S = C \times \mathbb{C}^2 \). The Levi-Civita connection on \( C \) is
Note that Cliff\((T^*M, g) = \mathbb{H}\) and we aim to find a representation for this algebra. So we aim to find an \(\mathbb{R}\) algebra homomorphism \(\varphi: \mathbb{H} \to \text{End}_\mathbb{C}(\mathbb{C}^2)\). Define \(\varphi: \mathbb{H} \to \text{End}_\mathbb{C}(\mathbb{C}^2)\) by
\[
\varphi(\alpha + i\beta + j\gamma + k\delta) = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \delta \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

Note that a basis for Cliff\((T^*M)\) is \(\{1, dt, d\theta, dt d\theta\}\) and so we see that we are identifying \(\varphi(dt) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and \(\varphi(d\theta) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\).

So applying Definition 4.1 we have for a local basis \(\{v_1, v_2\}\) of \(C\) that
\[
Du = \sum_{j=1}^{2} \sum_{l=1}^{2} \frac{\partial}{\partial x^j} \cdot \nabla_{\frac{\partial}{\partial x^j}} (u^l v_l)
\]
\[
= \sum_{j=1}^{2} \sum_{l=1}^{2} \nabla_{\frac{\partial}{\partial x^j}} \left( \frac{\partial u^l}{\partial x^j} v_l \right)
\]
\[
= \sum_{j=1}^{2} \sum_{l=1}^{2} dx^j \left( \frac{\partial u^l}{\partial x^j} v_l \right)
\]
\[
= \sum_{l=1}^{2} \left[ dt \left( \frac{\partial u^l}{\partial t} v_l \right) + d\theta \left( \frac{\partial u^l}{\partial \theta} v_l \right) \right]
\]
\[
= \begin{pmatrix} 0 & -\partial_t u^2 \\ \partial_t u^1 & 0 \end{pmatrix} + i\partial_\theta u^1 \begin{pmatrix} 0 & i\partial_\theta u^2 \\ i\partial_\theta u^1 & 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} 0 & -\partial_t + i\partial_\theta \\ \partial_t + i\partial_\theta & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}.
\]

Note that this has the same form as the Dirac operator on \(\mathbb{R}^2\) in Example 4.2.

### 5 A Dirac Operator on the Lorentzian Cylinder \(\mathbb{R} \times S^1\)

In physical problems, we are often working with a Lorentzian manifold. We will now consider the Dirac operator on a cylinder with a Lorentzian metric. We will impose APS boundary conditions as found in [2] and solve the eigenvalue problem for \(D\). The aim of this section is to apply the results of [2] to a simple situation and compute some traces with the goal of combining this work with that presented by Connes’ in [3], generalising to the setting of Lorentzian rather than Riemannian manifolds.

#### 5.1 Constructing the Dirac Operator

Let \(M = \mathbb{R} \times S^1\) be the infinite cylinder with the Lorentzian metric \(g = -dt^2 + d\theta^2\). Let \(C = [t_1, t_2] \times S^1 \subset M\) be the finite cylinder with metric \(\tilde{g} = g|_C\) and \(S = C \times \mathbb{C}^2\) be a spinor bundle over \(C\). The Levi-Civita connection on \(C\) is flat. Note that since the metric is Lorentzian, \(D\) will no longer be an elliptic operator, rather a generalisation of a hyperbolic operator. We use the following definition from [1].

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Definition 5.1. Let $M$ be a Lorentzian manifold and let $E \to M$ be a real or complex vector bundle. A second order linear differential operator $D : C^\infty(M,E) \to C^\infty(M,E)$ is called normally hyperbolic if its principal symbol is given by the metric,

$$\sigma_D(x,\xi) = -\|\xi\|^2 \cdot \text{Id}_{E_x},$$

for all $(x,\xi) \in T^*M$.

Example 5.2. The computation in Example 4.2 works for a metric of any signature, hence for a Dirac operator $D$ on a Lorentzian manifold we have that $D^2$ is a normally hyperbolic operator rather than an elliptic operator.

Note also that $	ext{Cliff}(T^*M,g) = M_2(\mathbb{R})$ and we aim to find a representation for this algebra. So we aim to find an $\mathbb{R}$-algebra homomorphism $\varphi : M_2(\mathbb{R}) \to \text{End}_\mathbb{C}(\mathbb{C}^2)$.

Construct a basis of $M_2(\mathbb{R})$ as

$$\begin{cases}
\text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e_1e_2 = e_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{cases}.$$  

Note that this basis satisfies

$$\text{Id}^2 = e_1^2 = -e_2^2 = e_3^2 = 1.$$

So we define $\varphi : M_2(\mathbb{R}) \to \text{End}_\mathbb{C}(\mathbb{C}^2)$ by inclusion, which is clearly an $\mathbb{R}$-algebra homomorphism. So we identify $dt = e_1$ and $d\theta = e_2$ as in the previous example.

So applying Definition 4.1 we have for a local basis $\{v_1, v_2\}$ of $C$ that

$$D\psi = \sum_{j=1}^{2} \sum_{l=1}^{2} \frac{\partial}{\partial x^j} \cdot \nabla_{\frac{\partial}{\partial x^j}} (\psi^l v_l)$$

$$= \sum_{j=1}^{2} \sum_{l=1}^{2} \nabla_{\frac{\partial}{\partial x^j}} \left( \frac{\partial \psi^l}{\partial x^j} v_l \right)$$

$$= \sum_{j=1}^{2} \sum_{l=1}^{2} dx^j \left( \frac{\partial \psi^l}{\partial x^j} v_l \right)$$

$$= \sum_{j=1}^{2} dx^j \frac{\partial \psi}{\partial x^j}$$

$$= \begin{pmatrix} 0 & \frac{\partial \psi}{\partial t} \\ \frac{\partial \psi}{\partial t} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\partial \psi}{\partial \theta} \\ \frac{\partial \psi}{\partial \theta} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \theta} \\ \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \theta} & 0 \end{pmatrix}.$$  

5.2 APS Boundary Conditions

As is usual with physical problems, we would like to impose boundary conditions on our operator $D$. The boundary conditions we will work with are Atiyah-Patodi-Singer (APS) boundary conditions, as described in
PDE, we apply the method of separation of variables. First, suppose \( \psi \in C^\infty(\mathbb{R}^n) : P_{20}(t_1)\psi = 0 = P_{20}(t_2)\psi \).

These definitions were adapted from those given in [2, Section 1.3]. Here we have \( P_{20}(t) = \chi_{[0,\infty)}(\partial_0 |\partial_0|) \) and \( P_{20}(t) = \chi_{(-\infty,0]}(\partial_0 |\partial_0|) \), spectral projections onto the non-negative and non-positive eigenspaces for the operator \( \partial_0 \). Physically, \( C^\infty_{\text{APS}}(\mathbb{R}^n) \) represents particles whose energy is non-positive at time \( t_1 \) and is non-negative at time \( t_2 \). Likewise for \( C^\infty_{\text{APS}}(\mathbb{R}^n) \). To see what this means explicitly, first note that we can decompose any function \( \psi : C \rightarrow \mathbb{C} \) in a Fourier basis as

\[
\psi(t,\theta) = \sum_{n \in \mathbb{Z}} \psi_n(t,\theta).
\]

A particle has entirely non-positive energy at \( t_1 \) if \( \psi_n(t_1,\theta) = 0 \) for all \( n \geq 0 \) and non-negative energy at time \( t_2 \) if \( \psi_n(t_2,\theta) = 0 \) for all \( n \leq 0 \). Hence we find that

\[
D = \begin{pmatrix}
0 & D^-
D^+ & 0
\end{pmatrix},
\]

where \( A = \frac{1}{i} \partial_0, D_+ = \partial_t + iA \) and \( D_- = \partial_t - iA \). Note also that \( \text{Dom}(D^+) = C^\infty_{\text{APS}}(\mathbb{R}^n) \) and \( \text{Dom}(D^-) = C^\infty_{\text{APS}}(\mathbb{R}^n) \).

We consider the Dirac operator acting on spinors \( \psi \) which take the form \( \psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \), where \( \psi^+ \) represents a particle and \( \psi^- \) represents an antiparticle.

### 5.3 Determining the Eigenvalues

As is standard when working with Dirac operators, we first find the eigenvalues of the associated Laplacian. Compute that

\[
D^2 = \begin{pmatrix}
D^- D^+ & 0 \\
0 & D^- D^+
\end{pmatrix} = \begin{pmatrix}
\partial_t^2 - \partial_0^2 & 0 \\
0 & \partial_t^2 - \partial_0^2
\end{pmatrix}.
\]

Note that although \( D^- D^+ \) and \( D^- D^+ \) have the same expression, they are in fact different operators, since \( \text{Dom}(D^- D^+) = \{ \psi \in \text{Dom}(D^+) : D^- \psi \in \text{Dom}(D^+) \} \) and \( \text{Dom}(D^- D^+) = \{ \psi \in \text{Dom}(D^-) : D^+ \psi \in \text{Dom}(D^-) \} \).

As mentioned, a solution \( \psi \) will be particle-antiparticle pair of the form \( \psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \). As usual with a linear PDE, we apply the method of separation of variables. First, suppose \( \psi^+(t,\theta) = T(t) \Theta(\theta) \). Applying this to the
eigenvalue problem we see that
\[ \frac{T''(t)}{T(t)} - \frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2. \]

So we have that \( \Theta''(\theta) = -k\Theta(\theta) \) and \( T''(t) = (-k + \lambda^2)T(t) \) for some constant \( k \in \mathbb{C} \). Since \( \Theta(\theta) \in S^1 \) we require that \( \Theta(0) = \Theta(2m\pi) \) for all \( m \in \mathbb{Z} \) and so this has solution if and only if \( \sqrt{k} \in \mathbb{Z} \). Hence for each \( n \in \mathbb{Z} \) we have the solution \( \Theta_n(\theta) = e^{in\theta} \). Next, we see that for each \( n \in \mathbb{Z} \) solutions \( T_n(t) \) will be of the form
\[ T_n(t) = \alpha_n e^{i\sqrt{n^2-\lambda^2}(t-t_1)} + \beta_n e^{-i\sqrt{n^2-\lambda^2}(t-t_1)} \]
for some \( \alpha_n, \beta_n \in \mathbb{C} \) when \( \lambda^2 \neq n^2 \) and
\[ T_n(t) = \alpha_0(t-t_1) + \beta_n \]
when \( \lambda^2 = n^2 \). Suppose for now that \( \lambda^2 \in \mathbb{R} \).

We now apply our APS boundary conditions to determine possible values of \( \lambda \). The APS boundary conditions tell us that \( T_n(t_1) = 0 \) for \( n \geq 0 \) and \( T_n(t_2) = 0 \) for \( n \leq 0 \). So for \( n \geq 0 \) and \( n^2 \neq \lambda^2 \) we have \( \alpha_n = -\beta_n \) and for \( n \geq 0 \) and \( n^2 = \lambda^2 \) we have \( \beta_n = 0 \). For \( n \leq 0 \) we have \( T_n(t_2) = 0 \). This forces \( \alpha_n = -e^{-2i\sqrt{n^2-\lambda^2}(t_2-t_1)}\beta_n \) when \( n^2 \neq \lambda^2 \) and \( \alpha_n = 0 \) when \( n^2 = \lambda^2 \). Hence we have eigenvectors
\[ \psi_{n,\lambda}^+(t, \theta) = \begin{cases} 
(e^{i\sqrt{n^2-\lambda^2}(t-t_1)} - e^{-i\sqrt{n^2-\lambda^2}(t-t_1)}) e^{in\theta}, & \text{if } n \geq 0 \text{ and } \lambda^2 \neq n^2, \\
0, & \text{if } n \leq 0 \text{ and } \lambda^2 \neq n^2.
\end{cases} \]

We find similarly for antiparticles that
\[ \psi_{n,\lambda}^-(t, \theta) = \begin{cases} 
(e^{i\sqrt{n^2-\lambda^2}(t-t_1)} - e^{-i\sqrt{n^2-\lambda^2}(t-t_1)}) e^{in\theta}, & \text{if } n \leq 0 \text{ and } \lambda^2 \neq n^2, \\
0, & \text{if } n \geq 0 \text{ and } \lambda^2 \neq n^2.
\end{cases} \]

Now we attempt to solve the eigenvalue problem
\[ (\partial_t + \partial_\theta) \psi_{n,\lambda}^+ = \lambda \psi_{n,\lambda}^- \]

With \( n \geq 0 \) and \( n^2 \neq \lambda^2 \) this leads to
\[ 0 = \left( (i\sqrt{n^2-\lambda^2} + in - \lambda) e^{i\sqrt{n^2-\lambda^2}(t-t_1)} + (i\sqrt{n^2-\lambda^2} - in + \lambda)e^{2i\sqrt{n^2-\lambda^2}(t_2-t_1)} \right) e^{in\theta}. \]
Comparing coefficients and using linear independence we see that
\[ i\sqrt{n^2-\lambda^2} + in = \lambda \]
and
\[ i\sqrt{n^2 - \lambda^2} - in = -\lambda e^{2i\sqrt{n^2 - \lambda^2}(t_2 - t_1)}. \]

Combining these gives
\[ \frac{n + \sqrt{n^2 - \lambda^2}}{n - \sqrt{n^2 - \lambda^2}} = -e^{2i\sqrt{n^2 - \lambda^2}(t_2 - t_1)}. \] (5.1)

If \( n^2 > \lambda^2 \) then the left hand side is real, forcing
\[ 2\sqrt{n^2 - \lambda^2}(t_2 - t_1) = m\pi \]
for some \( m \in \mathbb{Z} \). Rearranging we see that the eigenvalues satisfy
\[ \lambda^2 = n^2 - \frac{m^2\pi^2}{4(t_2 - t_1)^2}. \]

If \( n^2 < \lambda^2 \), let \( \sqrt{n^2 - \lambda^2} = i\rho_n \) where \( \rho_n \in \mathbb{R} \) and so
\[ \frac{n + i\rho_n}{n - i\rho_n} = e^{2\rho_n(t_2 - t_1)}. \]

The left hand side is real and so we must have \( \rho_n = 0 \), a contradiction. A similar analysis for \( n \leq 0 \) gives the same result. Hence we have eigenvalues
\[ \lambda_{n,m} = \sqrt{n^2 - \frac{m^2\pi^2}{4(t_2 - t_1)^2}}, \]
where \( n, m \in \mathbb{Z} \) and \( n^2 > \frac{m^2\pi^2}{4(t_2 - t_1)^2} \). From this expression we see that finding an analogue of Weyl’s theorem in this case will be difficult, as the eigenvalues are indexed by two variables.

Next, consider the case \( \lambda^2 \notin \mathbb{R} \). Then we must have \( \lambda = \alpha + i\beta \) for some \( 0 \neq \alpha, \beta \in \mathbb{R} \). Note that all of the analysis up to and including Equation 5.1 still applies.

### 5.4 Action Principles

Many physical problems require us to find and solve equations of motion. To do this, a common technique is to develop an action principle, which usually involves finding minima of functions called actions. For an excellent overview of material regarding action principles in physics, refer to [11].

**Example 5.3** (Fermat’s Principle). In optics, Fermat’s principle states that light will take the path of shortest time. Mathematically, this says that light will minimise the integral
\[ \int_{t_1}^{t_2} \frac{ds}{v}. \]

**Example 5.4** (Hamilton’s Principle). In classical mechanics, Hamilton’s principle states that the actual path taken by a particle will minimise the action of the Lagrangian, that is it will minimise the integral
\[ S(x, \dot{x}) = \int_{t_1}^{t_2} \mathcal{L}(t, x(t), \dot{x}(t)) \, dt, \]
where \( \mathcal{L} = T - U \) is the Lagrangian, \( T \) is the kinetic energy and \( U \) is the potential energy.
Physicist Richard Feynman attempted to extend this to quantum mechanics with his path integral formalisation, however this is often poorly defined and difficult to work with, so another approach is needed.

5.5 Connes’ Spectral Action Principle

In the 1990’s, Connes’ attempted to develop an action principle for compact Riemannian manifolds defined in terms of a trace of a differential operator, found in [3].

Let \( \mathcal{H} \) be a Hilbert space, \( A \) an algebra of bounded operators on \( \mathcal{H} \) and \( D \) a self-adjoint operator on \( \mathcal{H} \). Consider an operator on \( \mathcal{H} \) of the form \( D_A = D + A + JAJ^* \), where \( J \) is an anti-linear isometry. Connes’ spectral action principle is defined in terms of the trace

\[
\text{Trace} \left( \chi \left( \frac{D_A^2}{\Lambda^2} \right) \right),
\]

where \( \chi : \sigma(D_A) \to \mathbb{R} \) is measurable and satisfies \( \chi(\lambda) = 1 \) for \( \lambda \leq 1 \), and \( \Lambda \geq 0 \) is a spectral cutoff. For large values of \( \Lambda \) this can be computed using a heat kernel expansion, with the coefficients containing information regarding the geometry.

However spacetime is neither compact nor Riemannian, so more work is required for this to apply to a non-idealistic physics problem. Compactness can be overcome by multiplying by a compactly supported function, however it is not so simple to move to the Lorentzian setting. To do this, we would like to be able to define a trace on such Dirac operators \( D \).

5.6 Existence and Analysis of Trace Functionals

Before being able to determine any useful geometric or physical information from these traces, we must first do some analysis to determine that the integrals involved converge. In this section we present some candidates for a trace and some calculations involving them.

A simple trace we would like to compute is

\[
\text{Trace} \left( M_f \left( 1 + D^2_{APS,\varepsilon,R} \right)^{-\frac{1}{2}} \right),
\]

where we have imposed an energy cutoff \( R \) and a lower limit \( \varepsilon > 0 \) for our operator. To compute this, we will require some technical results.

**Lemma 5.5.** Let \( f : \mathbb{R}^n \to \mathbb{C} \) be rotationally and reflectionally symmetric. Then for any \( v \in \mathbb{R}^n \) with \( \| v \| = 1 \) we have that

\[
\int_{\mathbb{R}^n} f(x) \, dx = \text{Vol}(S^{n-1}) \int_0^\infty f(rv) r^{n-1} \, dr.
\]

**Proof.** First, recall the form of the Beta function \( \beta(x,y) = 2 \int_0^\frac{\pi}{2} \sin^{2x-1}(\theta) \cos^{2y-1}(\theta) \, d\theta \). Then we compute...
that
\[
\int_{\mathbb{R}^n} f(x) \, d^nx = \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^{\infty} f(rv) r^{n-1} \sin^{n-2} (\phi_1) \sin^{n-3} (\phi_2) \cdots \sin (\phi_{n-2}) \, dr \, d\phi_1 \, d\phi_2 \cdots d\phi_{n-1}
= \left( \int_0^{\pi} d\phi_1 \right) \left( \int_0^{\pi} \sin^{n-2} (\phi_2) \, d\phi_2 \right) \cdots \left( \int_0^{\pi} \sin (\phi_{n-2}) \, d\phi_{n-2} \right) \int_0^{\infty} f(rv) r^{n-1} \, dr
= 2\pi \left( \int_0^{\pi} \sin^{n-2} (\phi_2) \, d\phi_2 \right) \cdots \left( \int_0^{\pi} \sin (\phi_{n-2}) \, d\phi_{n-2} \right) \int_0^{\infty} f(rv) r^{n-1} \, dr
= 2\pi \beta \left( \frac{n-1}{2}, \frac{1}{2} \right) \cdots \beta \left( \frac{n-2}{2}, \frac{1}{2} \right) \int_0^{\infty} f(rv) r^{n-1} \, dr
= 2\pi \Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n-2}{2} \right) \cdots \Gamma \left( \frac{1}{2} \right) \int_0^{\infty} f(rv) r^{n-1} \, dr
= 2\pi \frac{n-1}{2} \int_0^{\infty} f(rv) r^{n-1} \, dr
= \frac{\pi}{n} \int_0^{\infty} f(rv) r^{n-1} \, dr
= \text{Vol}(S^{n-1}) \int_0^{\infty} f(rv) r^{n-1} \, dr.
\]

\[\square\]

**Lemma 5.6.** For any \(r \in \mathbb{Z}\), define \(f_r = u^r e^{-u}\). Then for \(k \in \mathbb{N}\) we have
\[
\frac{\partial^k}{\partial u^k} f_r = \sum_{j=0}^k (-1)^{j+k} \frac{\Gamma(r+1)}{\Gamma(r-j+1)} \left( \begin{array}{c} l \\ j \end{array} \right) \frac{\partial^{k-l}}{\partial u^{k-l}} f_{r-j}.
\]

**Proof.** We proceed by induction. The case \(l = 0\) is clearly true. For \(0 < l < k\), we have that
\[
\sum_{j=0}^l (-1)^{j+l} \frac{\Gamma(r+1)}{\Gamma(r-j+1)} \left( \begin{array}{c} l \\ j \end{array} \right) \frac{\partial^{k-l}}{\partial u^{k-l}} f_{r-j} = \sum_{j=0}^l (-1)^{j+l} \frac{\Gamma(r+1)}{\Gamma(r-j+1)} \left( \begin{array}{c} l \\ j \end{array} \right) \left( \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-j-1} - \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-j} \right)
= \sum_{j=0}^l (-1)^{j+l} \frac{\Gamma(r+1)}{\Gamma(r-j)} \left( \begin{array}{c} l \\ j \end{array} \right) \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-j-1}
- \sum_{j=0}^l (-1)^{j+l} \frac{\Gamma(r+1)}{\Gamma(r-j+1)} \left( \begin{array}{c} l \\ j \end{array} \right) \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-j}.
\]

We now make the change of indices \(n = j + 1\) on the first sum and relabel the second sum. This gives
\[
\sum_{j=0}^l (-1)^{j+l} \frac{\Gamma(r+1)}{\Gamma(r-j+1)} \left( \begin{array}{c} l \\ j \end{array} \right) \frac{\partial^{k-l}}{\partial u^{k-l}} f_{r-j} = \sum_{n=1}^{l+1} (-1)^{n+l+1} \frac{\Gamma(r+1)}{\Gamma(r-n+1)} \left( \begin{array}{c} l \\ n-1 \end{array} \right) \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-n}
- \sum_{n=0}^l (-1)^{n+l} \frac{\Gamma(r+1)}{\Gamma(r-n+1)} \left( \begin{array}{c} l \\ n \end{array} \right) \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-n}.
\]
We now combine these into a single sum so that we may use the identity \[
\binom{l+1}{n} = \binom{l}{n-1} + \binom{l}{n}.
\] This gives
\[
\sum_{j=0}^{l} (-1)^{j} \frac{\Gamma(r+1)}{\Gamma(r-j+1)} \binom{l}{j} \partial^{k-l} f_{r-j} = \sum_{n=1}^{l} (-1)^{n+l+1} \frac{\Gamma(r+1)}{\Gamma(r-n+1)} \left( \binom{l}{n-1} + \binom{l}{n} \right) \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-n} \\
+ \frac{\Gamma(r+1)}{\Gamma(r-l)} \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-l} + \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r} \\
= \sum_{n=0}^{l} (-1)^{n+l+1} \frac{\Gamma(r+1)}{\Gamma(r-n+1)} \left( \binom{l}{n-1} + \binom{l}{n} \right) \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-n} \\
= \sum_{n=0}^{l} (-1)^{n+l+1} \frac{\Gamma(r+1)}{\Gamma(r-n+1)} \left( \binom{l+1}{n} \right) \frac{\partial^{k-l-1}}{\partial u^{k-l-1}} f_{r-n}.
\]

Note that in the last equality, we have used the fact that for the terms \(n = 0\) and \(n = l + 1\) there are only a single term in the sum.

**Example 5.7.** We now attempt to compute

\[
\text{Trace} \left( M_{f} \left( 1 + D^{2}_{\kappa S_{\xi,R}} \right)^{-\frac{1}{2}} \right).
\]

We will be computing the trace over the domain of the eigenvalues, so we will integrate over the region displayed in Figure 1.

Note that the two lines have gradient \(c\) and \(-c\) respectively, where \(c = \frac{\pi}{2(t_2 - t_1)}\). Since there are no zero eigenvalues, we introduce a lower limit \(\varepsilon\) for \(\lambda^{2}\) and a spatial cutoff \(R\). The kernel for this operator is

\[
K_{\epsilon}(x, y) = 2^{\left[\frac{d+1}{2}\right]}(2\pi)^{\frac{d-1}{2}} f(x)\hat{g}_{\epsilon}(x - y),
\]
where $g$ is defined as

$$g_c(\xi) = \left( 1 + \|\xi\|_d^2 - c^2\xi^2 \right)^{-\frac{1}{2}} H \left( \|\xi\|_d^2 - c^2\xi^2 - \varepsilon^2 \right).$$

Let $c_d = 2(\frac{d+1}{2})(2\pi)^{-\frac{d+1}{2}}$, $A = \{(\xi_t, \xi_1, \ldots, \xi_d) \in \mathbb{R}^{d+1} : \|\xi\|_d^2 - c^2\xi_t^2 > \varepsilon^2 \}$ and $B = \{(r, p) \in \mathbb{R}^2 : r^2 - c^2p^2 > \varepsilon^2 \}$. Also let $\xi = (\xi_t, \xi_1, \ldots, \xi_d)$, $x = (x_t, x_1, \ldots, x_d)$ and $\|\xi\|_d^2 = \sum_{k=1}^d \xi_k^2$. Then we compute that

$$\text{Trace} \left( M_f (1 + D^2_{\text{APS}, x, R})^{-\frac{1}{2}} \right) = \int_{\mathbb{R}^{d+1}} K_s(x, x) \, dx$$

$$= c_d \int_{\mathbb{R}^{d+1}} M_f(\xi) \, dx$$

$$= c_d \int_{\mathbb{R}^{d+1}} M_f(2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R}^{d+1}} \left( 1 + \|\xi\|_d^2 - c^2\xi^2 t \right)^{-\frac{1}{2}} \, d\xi \, dx$$

$$= \frac{2(\frac{d+1}{2})}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} f(x) \, dx \int_{A} \left( 1 + \|\xi\|_d^2 - c^2\xi^2 t \right)^{-\frac{1}{2}} \, d\xi.$$

We now carefully calculate the rightmost integral, using an analogue of Lemma 5.5, as

$$\int_{A} \left( 1 + \|\xi\|_d^2 - c^2\xi^2 t \right)^{-\frac{1}{2}} \, d\xi = \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} u^{-\frac{1}{2}} \int_{A} e^{-u(1 + \|\xi\|_d^2 - c^2\xi^2 t)} \, d\xi \, du$$

$$= \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} u^{-\frac{1}{2}} \int_{B} \text{Vol} \left( S^{d-1} \right) r^{d-1} e^{-u(1 + r^2 - c^2p^2)} \, dr \, dp \, du.$$

We will need a change of coordinates. Let $v = r - cp$ and $w = r + cp$, so that $v + w = 2r$ and $w - v = 2cp$. Calculate the Jacobian as

$$J(r, p) = \det \begin{pmatrix} \frac{\partial r}{\partial v} & \frac{\partial r}{\partial w} \\ \frac{\partial p}{\partial v} & \frac{\partial p}{\partial w} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix}$$

$$= 1 + c^2.$$

Figure 5.7 demonstrates the coordinate transformation given above. Thus we have that

$$\int_{B} \text{Vol} \left( S^{d-1} \right) r^{d-1} e^{-u(1 + r^2 - c^2p^2)} \, dr \, dp = \frac{1}{2(1 + c^2)} \int_{v}^{w} \int_{e}^{2R} \left( \frac{1}{2} (v + w) \right)^{d-1} e^{-uvw} \, dv \, dw$$

$$+ \frac{1}{2(1 + c^2)} \int_{v}^{w} \int_{e}^{2R} \left( \frac{1}{2} (v + w) \right)^{d-1} e^{-uvw} \, dv \, dw$$

$$= \frac{1}{2^d (1 + c^2)} \int_{v}^{w} \int_{e}^{2R} \sum_{k=0}^{d-1} k e^{-kuvw} \, dv \, dw \quad (5.2)$$

$$= \frac{1}{2^d (1 + c^2)} \int_{v}^{w} \int_{e}^{2R} \sum_{k=0}^{d-1} k e^{-kuvw} \, dv \, dw. \quad (5.3)$$

We may now note that $v^* e^{-uvw} = \left( -\frac{1}{w} \right)^k \frac{\partial^k}{\partial u^k} e^{-uvw}$ and apply Lemma 5.6 to further compute the first integral

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5.2 as (for $2k \neq d - 1$)

\[
\int_{\epsilon}^{R} \int_{x}^{w} u^{d-1-k} e^{-uw} du \, dw = \int_{\epsilon}^{R} \int_{x}^{w} (-1)^k w^{d-1-2k} \frac{\partial}{\partial u^k} e^{-uw} du \, dw \\
= (-1)^k \frac{\partial}{\partial u^k} \int_{\epsilon}^{R} w^{d-2-2k} \left( \frac{1}{u} \left( e^{-u^2} - e^{-uw^2} \right) \right) du \\
= (-1)^k \frac{\partial}{\partial u^k} \left( \frac{R^{d-1-2k}}{d-1-2k} - \frac{e^{d-1-2k}}{d-1-2k} \right) \\
- (-1)^k \frac{\partial}{\partial u^k} \int_{\epsilon}^{R} w^{d-2-2k} e^{-uw^2} du.
\]

To work with these integrals, we recall that they were inside an integral over $u$. Then for $s$ large and using integration by parts we find that

\[
\int_{0}^{\infty} u^{\frac{s}{2}-1} e^{-u} \int_{\epsilon}^{R} \int_{x}^{w} u^{d-1-k} e^{-uw} du \, dw \, du = \int_{0}^{\infty} (-1)^k u^{\frac{s}{2}-1} e^{-u} \frac{\partial}{\partial u^k} \left( \frac{e^{-u^2}}{u} \right) du \left( \frac{R^{d-1-2k}}{d-1-2k} - \frac{e^{d-1-2k}}{d-1-2k} \right) \\
- \int_{0}^{\infty} (-1)^k u^{\frac{s}{2}-1} e^{-u} \frac{\partial}{\partial u^k} \int_{\epsilon}^{R} u^{d-2-2k} e^{-uw^2} du \, dw \\
= \sum_{j=0}^{k} c_{j,k,s} \int_{0}^{\infty} u^{\frac{s}{2}-j-2} e^{-u} e^{-u^2} \left( \frac{R^{d-1-2k}}{d-1-2k} - \frac{e^{d-1-2k}}{d-1-2k} \right) du \\
- \sum_{j=0}^{k} c_{j,k,s} \int_{\epsilon}^{R} \int_{0}^{\infty} u^{\frac{s}{2}-j-2} e^{-u} e^{-u^2} w^{d-2-2k} du \, dw \\
= \sum_{j=0}^{k} c_{j,k,s} \left( 1 + \epsilon^2 \right)^{j+1} \left( \frac{R^{d-1-2k}}{d-1-2k} - \frac{e^{d-1-2k}}{d-1-2k} \right) \\
- \sum_{j=0}^{k} c_{j,k,s} \int_{\epsilon}^{R} \left( 1 + w^2 \right)^{j+1} w^{d-2-2k} dw
\]

where $c_{j,k,s} = (-1)^{j+k} \frac{\Gamma \left( \frac{s}{2} \right)}{\Gamma \left( \frac{s}{2} - j \right)} \binom{k}{j}$. Using integration by parts we see that this is much more computable when $d - 2 - 2k$ is odd than even.

If $2k = d - 1$ then the same calculation applies, however every term of the form $\left( \frac{R^{d-1-2k}}{d-1-2k} - \frac{e^{d-1-2k}}{d-1-2k} \right)$

![Figure 2: Coordinate transformation applied to the domain of integration](image-url)
becomes \( \ln \left( \frac{R}{\varepsilon} \right) \), so that

\[
\int_0^\infty u^{\frac{1}{2}-1} e^{-u} \int_\varepsilon^R \int_0^w u^{d-1-k} v^k e^{-uw} dv \ du \ dw = \sum_{j=0}^k c_{j,k,s} (1 + \varepsilon^2)^{-\frac{1}{2}+j+1} \ln \left( \frac{R}{\varepsilon} \right)
- \sum_{j=0}^k c_{j,k,s} \int_\varepsilon^R (1 + w^2)^{-\frac{1}{2}+j+1} w^{d-2-2k} \ dw
\]

We compute this last integral as

\[
\int_\varepsilon^R (1 + w^2)^{-\frac{1}{2}+j+1} w^{d-2-2k} \ dw = \int_\varepsilon^R \frac{1}{\Gamma \left( \frac{1}{2} - j - 1 \right)} \int_0^\infty u^{\frac{1}{2}-j-2} e^{-w(1+u^2)} w^{d-2-2k} \ du \ dw
\]

\[
= \frac{1}{\Gamma \left( \frac{1}{2} - j - 1 \right)} \int_0^\infty u^{\frac{1}{2}-j-2} e^{-u} \frac{\partial^d e^{-2k}}{\partial u^{2-2k}} \int_\varepsilon^R w^P(d-2-2k) e^{-w(1+u^2)} \ dw \ du
\]

\[
= \frac{1}{\Gamma \left( \frac{1}{2} - j - 1 \right)} \int_0^\infty \frac{\partial^d e^{-2k}}{\partial u^{2-2k}} (u^{\frac{1}{2}-j-2} e^{-u}) m_{d,k}(u) \ du
\]

\[
= \frac{1}{\Gamma \left( \frac{1}{2} - j - 1 \right)} \left[ \frac{\partial^d e^{-2k}}{\partial \left( u^{\frac{1}{2}-j-2} \right)} \right] \sum_{p=0}^{\frac{d-2-2k}{2}} (-1)^p \frac{\Gamma \left( \frac{1}{2} - j - 1 \right)}{\Gamma \left( \frac{1}{2} - j - 1 + p \right)} \left( \frac{d-2-2k}{2} \right) M_{d,k,s,p},
\]

where \( P \) is the parity, giving \( 1 \) if odd and \( 0 \) if even,

\[
m_{d,k}(u) = \begin{cases} \sqrt{\pi} \left( \text{erf} \left( \sqrt{\pi}R \right) - \text{erf} \left( \sqrt{\pi}u \right) \right) & \text{if } d - 2 - 2k \text{ is even}, \\ \frac{1}{2} \left( e^{-uR^2} - e^{-u^2} \right) & \text{if } d - 2 - 2k \text{ is odd}. \end{cases}
\]

and

\[
M_{d,k,s,p} = \int_0^\infty u^{\frac{1}{2}-j-1-p-1} e^{-u} m_{d,k}(u) \ du.
\]

We may now also compute in a similar manner the integral 5.3 as

\[
\int_0^{2R} \int_0^{2R-w} w^{d-1-k} v^k e^{-uw} dv \ dw = (-1)^k \int_0^{2R} \int_0^{2R-w} w^{d-1-2k} \frac{\partial^k}{\partial u^k} (e^{-uw}) \ dv \ dw
\]

\[
= (-1)^k \frac{\partial^k}{\partial u^k} \int_0^{2R} \int_0^{2R-w} w^{d-1-2k} \frac{e^{-uw^2}}{uw} \ dv \ dw
\]

\[
= (-1)^k \frac{\partial^k}{\partial u^k} \left( \frac{(2R)^{d-1-2k}}{d-1-2k} - \frac{R^{d-1-2k}}{d-1-2k} \right)
\]

\[
= (-1)^k \frac{\partial^k}{\partial u^k} \left( \frac{(2R)^{d-1-2k}}{d-1-2k} - \frac{R^{d-1-2k}}{d-1-2k} \right) - I_{d,k}.
\]

Note that the case \( 2k = d - 1 \) gives rise to a logarithm term, as in the previous integral. For further computation,
we again bring back the integral over $u$ and apply integration by parts. This gives
\[
\int_0^\infty u^{\frac{d}{2}-1}e^{-u} I_{d,k} \, du = (-1)^k \int_0^\infty u^{\frac{d}{2}-1}e^{-u} \frac{\partial^k}{\partial u^k} \int_R \frac{2R \, u^{d-2k} \, e^{-u(2R-2)}}{u} \, du \, dw \, du
\]
\[
= \int_0^\infty \frac{\partial^k}{\partial u^k} \left( u^{\frac{d}{2}-1}e^{-u} \right) \int_R \frac{2R \, u^{d-2k} \, e^{-u(2R-w)}}{u} \, du \, dw \, du
\]
\[
= \sum_{j=0}^k \left( -1 \right)^{j+k} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - j \right)} \left( \frac{k}{2} \right)_j \int_0^\infty u^{\frac{d}{2}-j}e^{-u(1+u(2R-w))} \int_R 2R \, u^{d-2k} \, du \, dw \, du
\]
\[
= \sum_{j=0}^k \left( -1 \right)^{j+k} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - j \right)} \left( \frac{k}{2} \right)_j \int_0^\infty u^{\frac{d}{2}-j}e^{-u(1+u(2R-w))} \int_R 2R \, u^{d-2k} \, du \, dw.
\]
Combining these calculations together, we see that
\[
\int_A (1 + \|\xi\|_d^2 - c^2 \xi_1^2)^{-\frac{d}{2}} \, d\xi = \sum_{k=0}^{d-1} \left( \frac{d-1}{k} \right) \sum_{j=0}^k \left( -1 \right)^{j+k} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - j \right)} \left( \frac{k}{2} \right)_j \left( 1 + \varepsilon^2 \right)^{-j+1} \left( \frac{R^{d-1-2k}}{d-1-2k} - \frac{\varepsilon^{d-1-2k}}{d-1-2k} \right)
\]
\[
= \sum_{k=0}^{d-1} \left( \frac{d-1}{k} \right) \sum_{j=0}^k \left( -1 \right)^{j+k} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - j \right)} \left( \frac{k}{2} \right)_j \left( 1 + \varepsilon^2 \right)^{-j+1} \left( \frac{(2R)^{d-1-2k}}{d-1-2k} - \frac{\varepsilon^{d-1-2k}}{d-1-2k} \right)
\]
\[
= \sum_{k=0}^{d-1} \left( \frac{d-1}{k} \right) \sum_{j=0}^k \left( -1 \right)^{j+k} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - j \right)} \left( \frac{k}{2} \right)_j \left( 1 + \varepsilon^2 \right)^{-j+1} \left( \frac{(2R)^{d-1-2k}}{d-1-2k} - \frac{\varepsilon^{d-1-2k}}{d-1-2k} \right)
\]
and so
\[
\text{Trace} \left( M_f (1 + D^2)^{-\frac{d}{2}} \right) = \frac{2^{d+1} \cdot d \cdot \text{Vol}(S^{d-1}) \cdot \int_{\mathbb{R}^{d+1}} f(x) \, dx \cdot \int_A (1 + \|\xi\|_d^2 - c^2 \xi_1^2)^{-\frac{d}{2}} \, d\xi}
\]

We will now compute this trace for some low dimensional examples from $d = 1$ to $d = 3$. First we consider $d$ odd since the integrals are simpler.

**Example 5.8** (Dimension $d + 1 = 2$). We will consider the expression for $I(s) = \int_A (1 + \|\xi\|_d^2 - c^2 \xi_1^2)^{-\frac{d}{2}} \, d\xi$ computed in example 5.7. Note that there is a simple pole at $s = 2$. We compute the residue at this pole to determine its behaviour.

\[
\text{Res}_{s=2}(I(s)) = \ln \left( \frac{2R}{\varepsilon} \right) - (R - \varepsilon) - (2R - R)
\]
\[
= \ln \left( \frac{2R}{\varepsilon} \right) - 2R + \varepsilon.
\]

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Example 5.9 (Dimension $d + 1 = 4$). We will consider the expression for $I(s) = \int_A \left(1 + \|\xi\|_d^2 - c^2 \xi^2\right)^{-\frac{1}{2}} d\xi$ computed in example 5.7. Note that there are simple poles at $s = 2, s = 4$ and $s = 6$. We compute the residue at these poles to determine their behaviour.

For the pole at $s = 2$, we only need to consider terms with $j = 0$. Hence we have

$$\text{Res}_{s=2}(I(s)) = -2R^2 + \frac{\varepsilon^2}{2} - \ln\left(\frac{2R}{\varepsilon}\right) + \left(\frac{2R}{\varepsilon}\right)^{-1} - \frac{\varepsilon^{-1}}{2} - (2R)^{-2} + \varepsilon^{-2}.$$ 

For the pole at $s = 4$, we only need to consider terms with $j = 1$. Thus we have

$$\text{Res}_{s=4}(I(s)) = \left(\begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \end{array}\right) (2)(1 + \varepsilon^2)^0 \ln\left(\frac{R}{\varepsilon}\right) + \left(\begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \end{array}\right) (-1)(2)(1 + \varepsilon^2)^0 \left(\frac{(2R)^{-2}}{-2} - \frac{\varepsilon^{-2}}{-2}\right)$$

$$- \left(\begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \end{array}\right) (2) \left(\ln\left(\frac{R}{\varepsilon}\right) + \ln\left(\frac{2R}{R}\right)\right) + 2 \left(\begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \end{array}\right) (2) \int_{\varepsilon}^{R} w^{-3} dw$$

$$= 4\ln\left(\frac{R}{\varepsilon}\right) - 4\ln\left(\frac{R}{\varepsilon}\right) - 4\ln(2) + 4\left(2R^{-2} - \varepsilon^{-2}\right) - \left(R^{-4} - \varepsilon^{-4}\right)$$

$$= -4\ln(2) + 8R^{-2} - 2R^{-4} - 4\varepsilon^{-2} + 2\varepsilon^{-4}.$$ 

For the pole at $s = 6$, we only need to consider terms with $j = 2$. We compute this as

$$\text{Res}_{s=6}(I(s)) = \left(\begin{array}{c} 2 \\ 2 \\ 2 \end{array}\right) (-1)(2)^2 \Gamma(4) \left(\frac{(2R)^{-1}}{-1} - \frac{\varepsilon^{-1}}{-1} - \int_{\varepsilon}^{R} w^{-2} dw - \int_{R}^{2R} w^{-2} dw\right)$$

$$= 6\left(-\frac{(2R)^{-1}}{-1} + \frac{\varepsilon^{-1}}{-1} + \frac{(2R)^{-1}}{-1}\right)$$

$$= 0.$$ 

We now compute an odd example with $d + 1 = 3$.

Example 5.10 (Dimension $d + 1 = 3$). We will consider the expression for $I(s) = \int_A \left(1 + \|\xi\|_d^2 - c^2 \xi^2\right)^{-\frac{1}{2}} d\xi$ computed in example 5.7. Note that there is a simple pole at $s = 2$ and at $s = 4$. We compute the residue at these poles to determine their behaviour.

For $s = 2$, we only need to consider terms with $j = 0$. Calculate the residue as

$$\text{Res}_{s=2}(I(s)) = \left(\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right) \Gamma(1) \left(\varepsilon^{-1} - (2R)^{-1} - \int_{\varepsilon}^{R} w^{-2} dw - \int_{R}^{2R} w^{-2} dw\right)$$

$$+ \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right) \left[2R - \varepsilon - \int_{\varepsilon}^{R} w dw - \int_{R}^{2R} w dw\right]$$

$$= \varepsilon^{-1} - (2R)^{-1} - \varepsilon^{-1} + (2R)^{-1} + 2R - \varepsilon - \frac{1}{2}(2R)^2 + \frac{1}{2} \varepsilon^2$$

$$= 2R - \varepsilon - \frac{1}{2}(2R)^2 + \frac{1}{2} \varepsilon^2.$$
For $s = 4$, we need only consider terms with $j = 1$. This residue is calculated as

$$\text{Res}_{s=4}(l(s)) = -\left(\frac{2}{2}\right)\left(\frac{2}{1}\right)\left[\frac{e^{-2} - (2R)^{-2}}{2} - \frac{\int_{\varepsilon}^{\infty} w^{-3} dw}{2} - \frac{\int_{R}^{2R} w^{-3} dw}{2}\right]$$

$$+ \left(\frac{2}{2}\right)\left(\frac{1}{1}\right)\left[\ln (\frac{2R}{\varepsilon}) - \frac{\int_{\varepsilon}^{\infty} w^{-1} dw}{2} - \frac{\int_{R}^{2R} w^{-1} dw}{2}\right]$$

$$= 0.$$

It should be noted that in all of the above examples, taking the residue made the integral computations significantly easier, removing the distinction between even and odd. We would also like to take the limit as $R \to \infty$, which is not expected to commute with the operation of taking a residue, hence these results can be expected to have lost information about the problem.

**Example 5.11.** Another possible candidate for our trace is $\text{Trace}\left(M_fD^2\left(1 + D^*D\right)^{-\frac{1}{2}}\right)$. Let $\xi = (\xi_t, \xi_1, \ldots, \xi_d)$, $x = (x_t, x_1, \ldots, x_d)$ and $y = (y_t, y_1, \ldots, y_d)$ then compute, using analogous results to those found in [9, Corollary 14], and Example 5.7 that

$$\text{Trace}\left(M_fD^2\left(1 + D^*D\right)^{-\frac{1}{2}}\right) = 2\left[\frac{d+1}{d}\right](2\pi)^{-(d+1)} \int_{\mathbb{R}^{d+1}} \int_{A} f(x) \left(-\frac{\partial^2}{\partial x_t^2} + \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}\right)[e^{-((x-y)\cdot \xi)}(1 + \|\xi\|^2)^{-\frac{1}{2}}]_{x=y} dx$$

$$= c_d \int_{\mathbb{R}^{d+1}} \int_{A} \left(f(x) \left(\xi_t^2 - \|\xi_d\|^2\right)\left(1 + \xi_t^2 + \|\xi_d\|^2\right)^{-\frac{1}{2}} e^{-((x-y)\cdot \xi)}\right)_{x=y} dx$$

$$= c_d \int_{A} \left((1 + \xi_t^2 + \|\xi_d\|^2)^{-\frac{1}{2}} \left(2\xi_t^2 + 1\right) - \left(1 + \xi_t^2 + \|\xi_d\|^2\right)^{-\frac{1}{2}}\right) dx \int_{\mathbb{R}^{d+1}} f(x) dx,$$

where $c_d = 2\left[\frac{d+1}{d}\right](2\pi)^{-(d+1)}$ and $A = \{(\xi_t, \xi_1, \ldots, \xi_d) \in \mathbb{R}^{d+1} : \|\xi_d\|^2 - c^2 \xi_t^2 > \varepsilon^2\}$. We will calculate these two integrals separately. Let $B = \{(r, p) \in \mathbb{R}^2 : r^2 - c^2 p^2 > \varepsilon^2\}$ and compute that

$$\int_{A} \left(1 + \xi_t^2 + \|\xi_d\|^2\right)^{-\frac{1}{2}} \left(2\xi_t^2 + 1\right) dx = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\infty} \int_{A} u^{-\frac{1}{2}} \left(2\xi_t^2 + 1\right) e^{-u(1 + \xi_t^2 + |\xi_d|^2)} dx$$

$$= \text{Vol}(S^{d-1}) \int_{0}^{\infty} \int_{B} u^{-\frac{1}{2}} \left(2\xi_t^2 + 1\right) e^{-u(1 + \xi_t^2 + p^2)} dp du.$$

We will evaluate the integrals separately and recombine. First, consider the $r$ integral and note that the region of integration is $D = (-\infty, -\sqrt{\varepsilon^2 + c^2 p^2}) \cup (\sqrt{\varepsilon^2 + c^2 p^2}, \infty)$. Then we have that

$$I_d = \int_{\mathbb{R}} r^{d-1} e^{-ur^2} dr$$

$$= \int_{\mathbb{R}} -\frac{d}{2u} r^{d-2} e^{-ur^2} dr$$

$$= \left(-\frac{d}{2u}\right)^{d-2} e^{-ur^2} dr$$

$$= e^{-u(\varepsilon^2 + c^2 p^2)} \left(\left(\sqrt{\varepsilon^2 + c^2 p^2}\right)^{d-2} - \left(-\sqrt{\varepsilon^2 + c^2 p^2}\right)^{d-2}\right) + \frac{d}{2u} I_{d-3}.$$
So we calculate $I_2$ as

$$I_2 = \int_D r e^{-ur^2} \, dr$$
$$= \frac{1}{2} (e^{-ur^2}) |_{\partial D}$$
$$= 0$$

and $I_1$ as

$$I_1 = \int_D e^{-ur^2} \, dr$$
$$= \sqrt{\frac{\pi}{u}} \text{erfc} \left( \sqrt{\varepsilon^2 + c^2 p^2} \right).$$

This integral appears quite difficult to evaluate and so we will not attempt to continue the computation.

6 Conclusion

In this paper we developed the theory of differential operators on manifolds and clifford algebras in order to define and work with Dirac operators, of particular interest in physics. The main result of this report is that a trace can be computed for the Dirac operator on a Lorentzian cylinder. Further work could include analysing the trace in the limit as $R \to \infty$ before taking the residues and continuing with the integral computations found in Example 5.11. A further extension would be to consider a compact manifold with more interesting topology, such as a torus.
References


