AMSI
VACATION
RESEARCH
SCHOLARSHIPS
2018-2019

Schemes and their functors of points

Alexander Lai De Oliveira
Supervised by Professor Finnur Lárusson
University of Adelaide

Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Australian Government
Department of Education and Training

Abstract

Schemes are fundamental objects in algebraic geometry combining parts of algebra, topology and category theory. It is often useful to view schemes as contravariant functors from the category of schemes to the category of sets via the Yoneda embedding. A natural problem is then determining which functors arise from schemes. Our work addresses this problem, culminating in a representation theorem providing necessary and sufficient conditions for when a functor is representable by a scheme.

1 Introduction

Schemes are fundamental objects in modern algebraic geometry. A basic result in classical algebraic geometry is the bijection between an affine variety, solution sets of polynomials over an algebraically closed field, and its coordinate ring, a finitely generated, nilpotent-free ring over an algebraically closed field. Grothendieck sought to generalise this relationship by introducing affine schemes. If we relax our assumptions to include any commutative ring with identity, an affine scheme is the corresponding geometric object. Just as algebraic varieties are obtained by gluing affine varieties, a scheme is obtained by gluing affine schemes.

Relaxing the restrictions to include all commutative rings with identity was at first a radical generalisation. For example, unintuitive notions such as nonclosed points, or nonzero 'functions' which vanish everywhere, arise as a consequence. However, Grothendieck's generalisation is quite powerful and unifying. In particular, the ability to work over rings rather than algebraically closed fields links algebraic geometry with number theory.

Despite the importance of schemes, it is sometimes difficult to work with schemes directly. Many set-theoretic constructions such as products or intersections are not easily defined on schemes. However, a scheme can be reduced to a set by considering the functor of points of a scheme, which is the system of sets of scheme morphisms into the scheme from every other scheme. In this manner we obtain an embedding of schemes into the larger category of contravariant functors from the category of schemes to the category of sets. Many constructions are easier in this setting. The problem then reduces to determining which functors arise from schemes. The main result of this paper is a representation theorem characterising these functors, which we state below.

Theorem 1. *A functor* $F : \mathbf{Ring} \to \mathbf{Set}$ *is representable by a scheme if and only if*

(1) F *is a sheaf in the Zariski topology, and*

(2) F *has an open cover by affine schemes.*

2 Preliminaries

We cover preliminary results. As a matter of notation, by ‘ring’, we shall always refer to a commutative ring with identity. Morphisms of rings must map identity to identity.

The Yoneda lemma is a basic yet indispensable result in category theory. It essentially asserts that there is no information lost between passing from an object X in some category $\mathscr{C}$ to its functor of points $\mathrm{Mor}(-,X) : \mathscr{C}^o \to \mathbf{Set}$, where $\mathrm{Mor}(Y,X)$ is the collection of morphisms $Y \to X$. The statement given below is from Eisenbud and Harris [1]. Our proof of the lemma is based from Eisenbud and Harris [1], but given in more detail.

Lemma 2 (Yoneda lemma)**.** *Let $\mathscr{C}$ be a category and let X, X' be objects of $\mathscr{C}$.*

(a) If F is any contravariant functor from $\mathscr{C}$ to the category of sets, the natural transformations from $\mathrm{Mor}(-,X)$ to F are in natural correspondence with the elements of $F(X)$.

(b) If the functors $\mathrm{Mor}(-,X)$ and $\mathrm{Mor}(-,X')$ from $\mathscr{C}$ to the category of sets are isomorphic, then X is isomorphic to X'. More generally, the natural transformations from $\mathrm{Mor}(-,X)$ to $\mathrm{Mor}(-,X')$ are the same as maps from X to X'; that is, the functor $Mor : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^o, \mathbf{Set})$ sending X to $\mathrm{Mor}(-,X)$ is an equivalence of $\mathscr{C}$ with a full subcategory of the category of functors.

Proof. For part (a), let $f : Y \to X$ be an arbitrary morphism and consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Mor}(X,X) & \xrightarrow{\alpha_X} & F(X) & \quad & \mathbf{1}_X & \longmapsto & \alpha_X \mathbf{1}_X \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle F(f)} & & \downarrow & & \downarrow \\ \mathrm{Mor}(Y,X) & \xrightarrow{\alpha_Y} & F(Y) & & f & \longmapsto & \alpha_Y(f) = F(f)(\alpha_X \mathbf{1}_X). \end{array}$$

Given $\alpha : \mathrm{Mor}(-,X) \to F$, the association $\alpha \mapsto \alpha_X \mathbf{1}_X$ admits an inverse association $p \mapsto \beta : \mathrm{Mor}(-,X) \to F$ defined by $\beta_Y(f) = F(f)(p)$ for arbitrary $f : Y \to X$ and any given $p \in F(X)$. Explicitly, we have $\alpha_X \mathbf{1}_X \mapsto \beta : \mathrm{Mor}(-,X) \to F$ such that $\beta_Y : \mathrm{Mor}(Y,X) \to F(Y)$ is defined $f \mapsto F(f)(\alpha_X \mathbf{1}_X) = \alpha_Y(f)$. So $\beta_Y(f) = \alpha_Y(f)$ implying that $\beta = \alpha$. Conversely suppose $p \in F(X)$ is given. Apply the association $p \mapsto \beta : \mathrm{Mor}(-,X) \to F$ sending p to the natural transformation with components $\beta_Y : \mathrm{Mor}(Y,X) \to F(Y)$ defined by $f \mapsto F(f)(p)$ for $f : Y \to X$. When we then associate $\beta \mapsto \beta_X \mathbf{1}_X$, we induce by definition $\beta_X \mathbf{1}_X = F(\mathbf{1}_X)(p) = p$. Hence the two associations are mutual inverses.

For part (b), applying part (a) to $F = \mathrm{Mor}(-,Y)$ shows that natural transformations $\mathrm{Mor}(-,X) \to \mathrm{Mor}(-,Y)$ are in natural one-to-one correspondence with elements of $\mathrm{Mor}(X,Y)$, which are morphisms $f : X \to Y$. □

Remark 3. When we restrict the functor $\mathrm{Mor}(-, X)$ to the category of affine schemes, we write $h_X = \mathrm{Mor}(-, X)|\mathbf{Aff.Sch}$. For an affine scheme $\operatorname{Spec} R$, we occasionally abbreviate $h_R = h_{\operatorname{Spec} R}$. If $\operatorname{Spec} S$ is another affine scheme, we similarly may write $h_R(S) = h_{\operatorname{Spec} R}(\operatorname{Spec} S)$.

Lemma 4 (Extended Yoneda lemma)**.** *Scheme morphisms $f : X \to Y$ are in one-to-one correspondence with natural transformations $\varphi : h_X \to h_Y$; i.e., natural transformations of the functors* $\mathrm{Mor}(-, X)|\mathbf{Aff.Sch}$ *and* $\mathrm{Mor}(-, Y)|\mathbf{Aff.Sch}$.

Proof. The claim follows from the construction used in the proof of Proposition VI-2 of Eisenbud and Harris [1]. □

The Yoneda lemma and its extension will typically be used to interchange natural transformations between morphism sets $h_X \to h_Y$ with morphisms $X \to Y$. Another use is interchanging natural transformations $h_S \to F$ between functors from **Ring** to **Set** with elements of $F(S)$.

For the purposes of our representation theorem, we wish to extend geometric notions from the geometry of schemes to functors. In particular, we define the notions of open subfunctor, open covering and sheaf in the context of functors following the definitions of Eisenbud and Harris [1].

Definition 1. A subfunctor $\alpha : G \to F$ in $\mathrm{Fun}(\mathbf{Ring}, \mathbf{Set})$ is an *open subfunctor* if, for each map $\psi : h_{\operatorname{Spec} R} \to F$ from the functor represented by an affine scheme $\operatorname{Spec} R$ (that is, each $\psi \in F(R)$ by the Yoneda lemma), the fibered product

$$\begin{array}{ccc} G_\psi & \longrightarrow & h_{\operatorname{Spec} R} \\ \big\downarrow & & \big\downarrow \psi \\ G & \xrightarrow{\ \alpha\ } & F \end{array}$$

of functors yields a map $G_\psi \to h_{\operatorname{Spec} R}$ isomorphic to the injection from the functor represented by some open subscheme of $\operatorname{Spec} R$.

Definition 2. Let $F : \mathbf{Scheme} \to \mathbf{Set}$ be a functor. A collection of subfunctors $\{G_i \to F\}_{i \in I}$ of open subfunctors of F form an *open covering* if, for each map $h_X \to F$ from a representable functor h_X to F, the open subschemes $U_i \subset X$ induced by the fiber product $h_X \times_F G_i \cong h_{U_i}$ cover X.

Note that if $\{\alpha_i : G_i \to F\}$ is an open covering of F, it is not necessarily the case that $F(T) = \bigcup \alpha_i(T)(G_i(T))$ for all schemes T. The following lemma, which appears as Exercise VI-11 in Eisenbud and Harris [1], clarifies this relation. It provides an alternative characterisation of an open covering of a functor $F : \mathbf{Scheme} \to \mathbf{Set}$ by subfunctors $\alpha_i : G_i \to F$ which relates the sets $F(\operatorname{Spec} K)$ and $\alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$, where K is a field.

Lemma 5. *Let $\{\alpha_i : G_i \to F\}$ be a collection of open subfunctors of a functor F :* **Scheme** $\to$ **Set***. This collection of open subfunctors is an open cover of F if and only if*

$$F(\operatorname{Spec} K) = \bigcup \alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$$

for all fields K.

Proof. ($\Longrightarrow$) The inclusion $F(\operatorname{Spec} K) \supset \bigcup \alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$ always holds, so we need only show $F(\operatorname{Spec} K) \subset \bigcup \alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$. Suppose that $\alpha_i : G_i \to F$ is an open covering and let $\beta \in F(\operatorname{Spec} K)$. By the Yoneda lemma, we may view β as a natural transformation $\beta : h_{\operatorname{Spec} K} \to F$. Since $\operatorname{Spec} K$ consists of a single element, there is at least one G_i such that we have the commutative square

$$\begin{array}{ccc} h_{\operatorname{Spec} K} \times_F G_i \cong h_{\operatorname{Spec} K} & \xrightarrow{\text{id}} & h_{\operatorname{Spec} K} \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \beta} \\ G_i & \xrightarrow{\alpha_i} & F. \end{array}$$

Again by the Yoneda lemma, we may view β_i as an element in $G_i(\operatorname{Spec} K)$. But $\alpha_i(\operatorname{Spec} K)(\beta_i) = \beta$, so $\beta \in \alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$. Hence $F(\operatorname{Spec} K) \subset \bigcup \alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$.

($\Longleftarrow$) Suppose that $F(\operatorname{Spec} K) = \bigcup \alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$ for all fields K. Let X be an arbitrary scheme. For each G_i, we get a pullback square

$$\begin{array}{ccc} h_X \times_F G_i \cong h_{U_i} & \hookrightarrow & h_X \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \beta} \\ G_i & \xrightarrow{\alpha_i} & F. \end{array}$$

We need to show that the open subschemes $U_i \subset X$ cover X. It is sufficient to show that $X \subset \bigcup U_i$. Let $x \in X$. There exists some local affine scheme $\operatorname{Spec} R \subset X$ containing x, and in this affine scheme x corresponds to a prime ideal $\mathfrak{p}$ of R. We obtain the residue field $K = \kappa(x)$ as the quotient field of the integral domain $R/\mathfrak{p}$. The composition of maps $R \to \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathfrak{m}_{X,x} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = K$ induces a morphism $\operatorname{Spec} K \to \operatorname{Spec} R$, which extends to $h_X(\operatorname{Spec} K) \ni \varphi : \operatorname{Spec} K \to X$ by inclusion. But since $F(\operatorname{Spec} K) = \bigcup \alpha_i(\operatorname{Spec} K)(G_i(\operatorname{Spec} K))$, there exists $g_i \in G_i(\operatorname{Spec} K)$ in at least one of the sets $G_i(\operatorname{Spec} K)$ such that $\alpha_i(\operatorname{Spec} K)(g_i) = \beta(\operatorname{Spec} K)(\varphi)$. This is the precise requirement for (φ, g_i) to be in the fibered product $(h_X \times_F G_i)(\operatorname{Spec} K)$. Since this fibered product is isomorphic to $h_{U_i}(\operatorname{Spec} K)$, the pair (φ, g_i) corresponds to some $\varphi_i : \operatorname{Spec} K \to U_i$ such that φ_i extends to φ by inclusion of U_i into X. Hence φ factors through U_i, which implies U_i contains x. Therefore $X \subset \bigcup U_i$. □

Remark 6. Since the category of affine schemes is coequivalent to the category of rings, Lemma 5 implies that the same result holds viewing F and G_i as functors from **Ring** to **Set**.

Definition 3. A functor $F : \mathbf{Ring} \to \mathbf{Set}$ is a *sheaf in the Zariski topology* if for each ring R and each open covering of $X = \operatorname{Spec} R$ by distinguished open affines $\operatorname{Spec} R_{f_i}$, and every collection of elements $\alpha_i \in F(R_{f_i})$ such that α_i and α_j map to the same element in $F(R_{f_i f_j})$, there is a unique element $\alpha \in F(R)$ mapping to each of the α_i.

The gluability of compatible scheme morphisms is a rudimentary result. A statement of the result without proof is given in Theorem 3.3 of Hartshorne [2]. We reformulate Hartshorne's statement of the result and provide the proof. This result will often be invoked in the proof of Theorem 1.

Lemma 7. *Let X and Y be schemes. If X is covered by open subschemes U_i and we have scheme morphisms $f_i : U_i \to Y$ such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$, then there exists a unique scheme morphism $f : X \to Y$ such that $f|U_i = f_i$.*

Proof. We can glue together each f_i to get a map of underlying topological spaces $f : |X| \to |Y|$ since continuity is a local property. We need to check that f induces a sheaf morphism $f^{\#} : \mathscr{O}_Y \to f_* \mathscr{O}_X$. Let $V \subset Y$ be an open set and let $s \in \mathscr{O}_Y(V)$. From the sheaf morphisms $f_i^{\#}$, we have a collection of sections $f_i^{\#}(V)(s) \in \mathscr{O}_{U_i}(f_i^{-1}V) = \mathscr{O}_X(f^{-1}V \cap U_i)$. Observe that the open sets $f^{-1}V \cap U_i$ form an open cover of $f^{-1}V$. Furthermore, restricting $f_i^{\#}(V)(s)$ to $(f^{-1}V \cap U_i) \cap (f^{-1}V \cap U_j) = f^{-1}V \cap U_i \cap U_j$ is the same as restricting the sheaf morphism $f_i^{\#}$ to $U_i \cap U_j$. So $f_i^{\#}(V)(s)$ and $f_j^{\#}(V)(s)$ agree on overlap because $f_i|U_i \cap U_j = f_j|U_i \cap U_j$. Hence the sections $f_i^{\#}(V)(s) \in \mathscr{O}_X(f^{-1}V \cap U_i)$ glue to a section of $\mathscr{O}_X$ over $f^{-1}V$. Define this to be $f^{\#}(V)(s)$. Since each f_i is a scheme morphism, the induced map of local rings $\mathscr{O}_{Y,y} \to \mathscr{O}_{U_i,x}$ is a local homomorphism for every $x \in U_i$. All of the f_i agree on overlap, and so the induced map $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is local for every $x \in X$. Hence $f : X \to Y$ is a scheme morphism which, by construction, is such that $f|U_i = f_i$. □

3 Representation theorem

We are now in a position to prove the main result of this paper, Theorem 1. This appears as Theorem VI-14 in Eisenbud and Harris [1], stated without proof. Here condition (2) is instead given as the alternative characterisation in Lemma 5.

Theorem 1. *A functor $F : \mathbf{Ring} \to \mathbf{Set}$ is of the form h_Y for some scheme Y if and only if*

(1) F is a sheaf in the Zariski topology, and

(2) there exists a family of rings $\{R_i\}_{i \in I}$ such that the open subfunctors $\alpha_i : h_{R_i} \to F$ form an open covering of F.

In the introduction, we presented condition (2) as having an open cover by affine schemes. We state this more precisely as F having an open cover by subfunctors representable by affine schemes $\operatorname{Spec} R_i$. The set I which indexes the family of rings $\{R_i\}$ can be infinite.

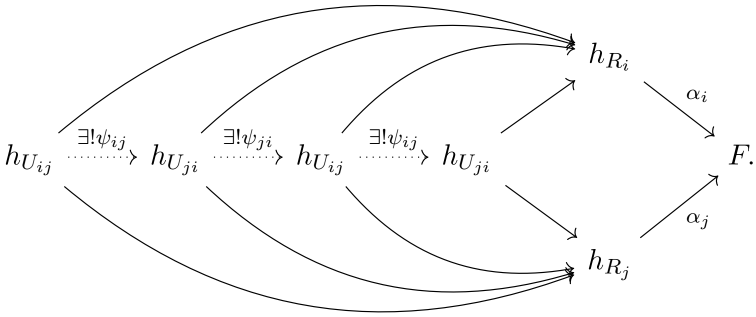
Let us first give an outline of the proof. The 'only if' direction of this proof is relatively straightforward and requires no preamble. The converse is a considerably longer proof, which we summarise as follows. By condition (2), we are given a family of rings $\{R_i\}_{i\in I}$. We show that the corresponding affine schemes $\operatorname{Spec} R_i$ are all compatible, and glue to a scheme Y. We then construct an isomorphism $\alpha : h_Y \to F$ which is induced locally by the subfunctors $\alpha_i : h_{R_i} \to F$. This construction is done in three parts: (i) for any ring S, we first define $\alpha(S) : h_Y(S) \to F(S)$ and its candidate inverse as set-theoretic maps; (ii) we then show that these two maps are in fact mutual inverse; and (iii), we conclude the proof by establishing that the family of components $\{\alpha(S) : h_Y(S) \to F(S)\}_{S\in\mathbf{Ring}}$ defines a natural transformation $\alpha : h_Y \to F$.

Proof of Theorem 1. For the 'only if' direction, suppose that $F \cong h_Y$. The fact that h_Y is a sheaf follows immediately from the gluability of compatible scheme morphisms (Lemma 7). More precisely, suppose that for the affine scheme $\operatorname{Spec} R$, we are given a distinguished open cover $\{\operatorname{Spec} R_{f_i}\}$ with elements $\alpha_i \in h_Y(R_{f_i})$ such that $\alpha_i|\operatorname{Spec} R_{f_if_j} = \alpha_j|\operatorname{Spec} R_{f_if_j}$. By definition of $h_Y(R_{f_i})$, each α_i is a morphism $\operatorname{Spec} R_{f_i} \to Y$. These morphisms are all compatible and cover $\operatorname{Spec} R$, so they glue to a morphism $h_Y(R) \ni \alpha : \operatorname{Spec} R \to Y$ restricting to each α_i. For the second condition, take an affine open cover $\{\operatorname{Spec} R_i\}$ of Y. We have a natural collection of open subfunctors via the inclusion $\iota_i : h_{R_i} \to h_Y$. Now let K be a field and $\varphi : \operatorname{Spec} K \to Y$ a scheme morphism. The affine scheme $\operatorname{Spec} K$ consists of a single point, so the image of $\operatorname{Spec} K$ under φ has an affine neighbourhood $\operatorname{Spec} R_i \subset Y$. Hence φ factors through $\operatorname{Spec} R_i$ such that there exists $\varphi_i : \operatorname{Spec} K \to \operatorname{Spec} R_i$ satisfying $\varphi = \iota_i \circ \varphi_i = \iota_i(K)(\varphi_i)$. So $h_Y(K) = \bigcup \iota_i(K)(h_{R_i}(K))$, and h_Y satisfies the second condition by Lemma 5.

For the 'if' direction, we need to first construct a suitable scheme Y and then show that $F \cong h_Y$. We are given by condition (2) an open cover $\{\alpha_i : h_{R_i} \to F\}$. This implies the following commutative diagram,

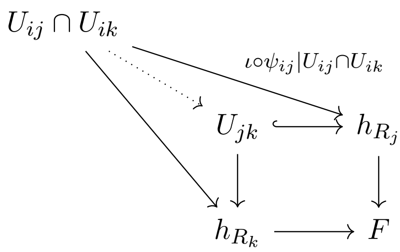
$$\begin{array}{ccc} h_{U_{ij}} \cong h_{R_i} \times_F h_{R_j} & \longrightarrow & h_{R_i} \\ \big\downarrow & & \big\downarrow{\scriptstyle \alpha_i} \\ h_{R_j} & \xrightarrow[\alpha_j]{} & F \end{array}$$

where $U_{ij} \subset \operatorname{Spec} R_i$ is an open subscheme. By symmetry we can obtain the open subscheme U_{ji} representing $h_{U_{ji}} \cong h_{R_j} \times_F h_{R_i}$. We obtain an isomorphism between $h_{U_{ij}}$ and $h_{U_{ji}}$ via the universal property of fiber products

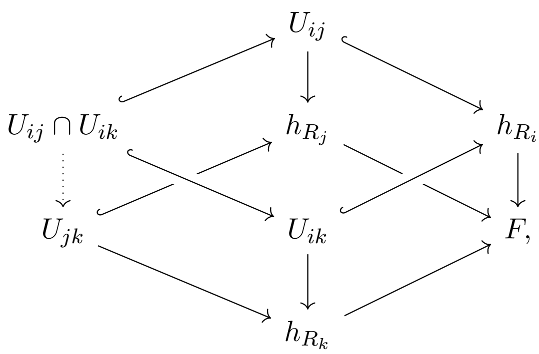


By the Yoneda lemma, there is a canonical identification of the map $\psi_{ij} : h_{U_{ij}} \to h_{U_{ji}}$ with the map $U_{ij} \to U_{ji}$; we denote both with the same symbol ψ_{ij}, and freely interchange between h_{U_i} and U_i where notation is convenient. Moreover, the universal property implies $\psi_{ij} \circ \psi_{ji} = \mathrm{id}_{U_{ij}}$ and vice versa, so we obtain our first gluing condition $\psi_{ji} = \psi_{ij}^{-1}$.

The following diagram



commutes by factoring through h_{R_i} as shown



and so we have $\psi_{ij}(U_{ij} \cap U_{ik}) \subset U_{ji} \cap U_{jk}$ for distinct i, j, k. Reversing the roles of i and j implies that

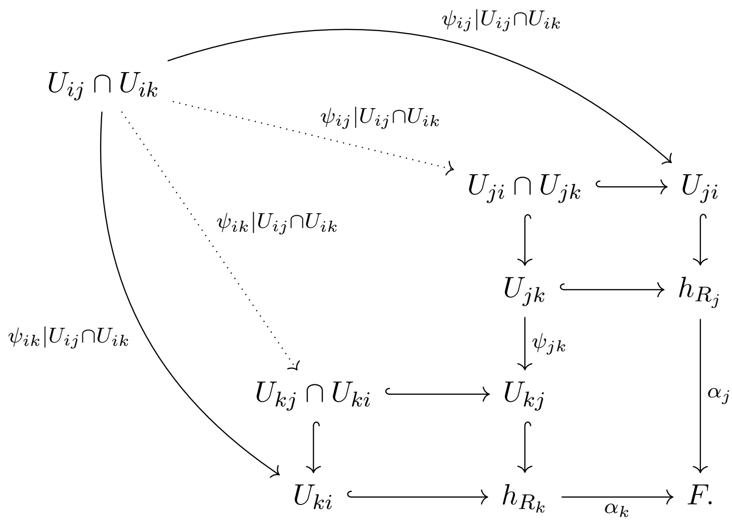
$$\psi_{ji}(U_{ji} \cap U_{jk}) \subset U_{ij} \cap U_{ik}.$$

Recalling that $\psi_{ij} = \psi_{ji}^{-1}$, we have

$$\psi_{ij}^{-1}(U_{ji} \cap U_{jk}) \subset U_{ij} \cap U_{ik}$$
$$U_{ji} \cap U_{jk} \subset \psi_{ij}(U_{ij} \cap U_{ik}),$$

and hence the second condition $\psi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ is satisfied.

The compatibility condition $\psi_{jk} \circ \psi_{ik}|U_{ij} \cap U_{ik} = \psi_{ik}|U_{ij} \cap U_{ik}$ follows from the commutative diagram



Our construction yields a scheme Y by gluing the affine schemes $\{\operatorname{Spec} R_i\}$. Under this construction, if we let $U_i \cong \operatorname{Spec} R_i$ such that $Y = \bigcup U_i$, the isomorphisms ψ_{ij} correspond to the identity maps on $U_i \cap U_j$.

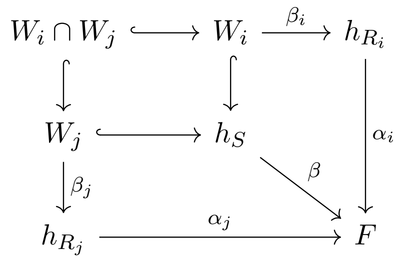
We show that $F \cong h_Y$. It suffices to find a natural transformation $\alpha : h_Y \to F$ such that $\alpha(S) : h_Y(S) \to F(S)$ is an isomorphism for all rings S. Let S be a commutative ring with identity. Let $h_Y(S) \ni \varphi : \operatorname{Spec} S \to Y$. Any map from an affine scheme to an arbitrary scheme $Y = \bigcup \operatorname{Spec} R_i$ may be defined as a family of compatible maps $h_{R_i}(S_{f_i}) \ni \varphi_i : \operatorname{Spec} S_{f_i} \to \operatorname{Spec} R_i$, where the collection of $\operatorname{Spec} S_{f_i}$ form an open cover of S by distinguished sets. Applying the natural transformations $\alpha_i(S_{f_i}) : h_{R_i}(S_{f_i}) \to F(S_{f_i})$, we obtain $\alpha_i(S_{f_i})(\varphi_i) \in F(S_{f_i})$ such that $\alpha_i(S_{f_i})(\varphi_i)$ and $\alpha_j(S_{f_j})(\varphi_j)$ map to the same element in $F(S_{f_i f_j})$. Hence there exists a unique element $\beta \in F(S)$ such that β maps to each $\alpha_i(S_{f_i})(\varphi_i)$. Define $\alpha : h_Y \to F$ by $\alpha(S)(\varphi) = \beta$.

Conversely, suppose we are given an element $\beta \in F(S)$. Regarding β as a natural transformation $\beta : h_S \to F$ by the Yoneda lemma, for each ring R_i, we obtain a pullback square

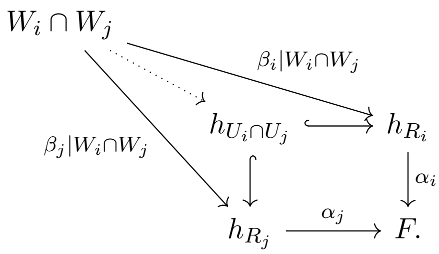
$$\begin{array}{ccc} h_S \times_F h_{R_i} \cong h_{W_i} & \hookrightarrow & h_S \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \beta} \\ h_{R_i} & \xrightarrow{\alpha_i} & F \end{array}$$

where $W_i \subset \operatorname{Spec} S$ is an open subscheme. Since the subfunctors h_{R_i} cover F by condition (2), we have an open cover of $\operatorname{Spec} S$ by the collection of open subschemes W_i. By the extended Yoneda lemma, we can regard β_i as a morphism $\beta_i : W_i \to \operatorname{Spec} R_i$.

Since the diagram



commutes, the intersection $W_i \cap W_j$ factors through $U_{ij} \cong U_i \cap U_j$ as shown in the commutative diagram



Note that we get an inclusion from $h_{U_i \cap U_j}$ to both h_{R_i} and h_{R_j} since the isomorphisms ψ_{ij} and ψ_{ji} are identified with the identity map on $U_i \cap U_j$ in the scheme Y. Commutativity implies that $\beta_i|W_i \cap W_j = \beta_j|W_i \cap W_j$ since we can cover $W_i \cap W_j$ with distinguished open subsets and compose with inclusions. Hence by Lemma 7 we can glue the maps together to obtain a scheme morphism $\varphi : \operatorname{Spec} S \to Y$ such that $\varphi|W_i = \beta_i$.

We now show that this construction is the inverse of $\alpha(S) : h_Y(S) \to F(S)$. We start with a scheme morphism $\varphi : \operatorname{Spec} S \to Y$.

1. Given a scheme morphism $h_Y(S) \ni \varphi : \operatorname{Spec} S \to Y$, break φ into components $h_{R_i}(S_{f_i}) \ni \varphi_i : \operatorname{Spec} S_{f_i} \to \operatorname{Spec} R_i$, compatible in the sense that $\varphi_i|V_i \cap V_j = \varphi_j|V_i \cap V_j$, where $V_i = \operatorname{Spec} S_{f_i}$.

2. Apply the natural transformation $\alpha_i(S_{f_i}) : h_{R_i}(S_{f_i}) \to F(S_{f_i})$ to obtain $\alpha_i(S_{f_i})(\varphi_i) \in F(S_{f_i})$.

   The compatibility condition implies that $\varphi_i \circ \psi_{ij} = \varphi_i|V_i \cap V_j$ maps into the intersection $U_i \cap U_j$. Since $V_i \cap V_j = \operatorname{Spec} S_{f_i f_j}$ is a distinguished open set, the commutative diagram

   $$\begin{array}{ccc} h_{U_i \cap U_j}(S_{f_i f_j}) & \hookrightarrow & h_{R_i}(S_{f_i f_j}) \\ \downarrow & & \downarrow \alpha_i(S_{f_i f_j}) \\ h_{R_j}(S_{f_i f_j}) & \xrightarrow{\alpha_j(S_{f_i f_j})} & F(S_{f_i f_j}). \end{array}$$

   shows that $\alpha_i(S_{f_i f_j})(\varphi_i|V_i \cap V_j) = \alpha_j(S_{f_i f_j})(\varphi_i|V_i \cap V_j)$.

3. Glue the components together, since F is a sheaf by condition (1), to obtain $\alpha(S)(\varphi) \in F(S)$ such that $\alpha(S)(\varphi)|V_i = \alpha_i(S_{f_i})(\varphi_i) \in F(S_{f_i})$.

4. Apply $\alpha(S)(\varphi)$ to the pullback square

$$\begin{array}{ccc} h_S \times_F h_{R_i} \cong h_{W_i} & \hookrightarrow & h_S \\ \downarrow \beta_i & & \downarrow \alpha(S)(\varphi) \\ h_{R_i} & \xrightarrow{\alpha_i} & F. \end{array}$$

   In this case, we cannot guarantee that W_i is an affine scheme. However, we can cover W_i by affine schemes W_{ij}, and then examine the pullback square on any distinguished affine scheme $W_{ij\lambda}$ induced by the open set $D(f_{ij\lambda})$ distinguished in both W_{ij} and V_i. If we start with the natural inclusion $\iota : W_{ij\lambda} \to W_i$, we find that $\alpha(S)(\varphi)|W_{ij\lambda} = \alpha_i(S_{f_{ij\lambda}})(\beta_i|W_{ij\lambda})$. But $\alpha(S)(\varphi)|V_i = \alpha_i(S_{f_i})(\varphi_i)$, and this can be further restricted to $W_{ij\lambda}$. Hence $\alpha_i(S_{f_{ij\lambda}})(\varphi_i|W_{ij\lambda}) = \alpha_i(S_{f_{ij\lambda}})(\beta_i|W_{ij\lambda})$. Since α_i is a subfunctor, every component is injective. Hence $\beta_i|W_{ij\lambda} = \varphi_i|W_{ij\lambda}$.

5. Glue together the components β_i to obtain a scheme morphism $\beta : \operatorname{Spec} S \to Y$ such that $\beta|W_i = \beta_i$. The previous discussion implies that $\beta = \varphi$ since $\operatorname{Spec} S$ can be covered by the distinguished open affines $W_{ij\lambda}$.

For the other direction, consider an arbitrary element $\beta \in F(S)$.

1. Viewing β as a natural transformation $\beta : h_S \to F$, apply β to the pullback square

$$\begin{array}{ccc} h_S \times_F h_{R_i} \cong h_{W_i} & \hookrightarrow & h_S \\ \downarrow \beta_i & & \downarrow \beta \\ h_{R_i} & \xrightarrow{\alpha_i} & F. \end{array}$$

2. Glue together components β_i to get a scheme morphism $\varphi : \operatorname{Spec} S \to Y$ satisfying $\varphi|W_i = \beta_i$.

3. Cover φ by compatible components $\varphi_{ij\lambda} : W_{ij\lambda} \to \operatorname{Spec} R_i$, where the collection $\{W_{ij}\}$ forms an affine cover of W_i, and $\{W_{ij\lambda}\}$ is a cover of W_{ij} by distinguished open affines induced by the open set $D(f_{ij\lambda})$.

   Since $\varphi|W_i = \beta_i$, we have $(\varphi|W_i)|W_{ij\lambda} = \beta_i|W_{ij\lambda}$. But $(\varphi|W_i)|W_{ij\lambda} = \varphi|W_{ij\lambda} = \varphi_{ij\lambda}$ by definition. Hence $\varphi_{ij\lambda} = \beta_i|W_{ij\lambda}$.

4. Glue together components $\alpha_i(S_{f_{ij\lambda}})(\varphi_{ij\lambda})$ to obtain $\alpha(S)(\varphi)$ with the property that this is the unique map such that $\alpha(S)(\varphi)|W_{ij\lambda} = \alpha_i(S_{f_{ij\lambda}})(\varphi_{ij\lambda})$ for all $i \in I, j \in J_i, \lambda \in \Lambda_{ij}$. But $\beta|W_{ij\lambda} = \alpha_i(S_{f_{ij\lambda}})(\beta_i|W_{ij\lambda}) = \alpha_i(S_{f_{ij\lambda}})(\varphi_{ij\lambda})$ for all $i \in I, j \in J_i, \lambda \in \Lambda_{ij}$. By uniqueness we conclude that $\beta = \alpha(S)(\varphi)$. Hence $\alpha(S) : h_Y(S) \to F(S)$ is an isomorphism for all rings S.

Finally, we must verify that defining $\alpha : h_Y \to F$ in this manner induces a natural transformation. This comes down to showing that, for morphisms $f : \operatorname{Spec} T \to \operatorname{Spec} S$ and $\varphi : \operatorname{Spec} S \to Y$ the square

$$\begin{array}{ccc} h_Y(S) & \xrightarrow{\alpha(S)} & F(S) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ h_Y(T) & \xrightarrow{\alpha(T)} & F(T) \end{array}$$

commutes, which in turn is a matter of proving that $\alpha(T)(\varphi \circ f) = F(f)(\alpha(S)(\varphi))$. We first break φ into compatible components $\varphi_i : \operatorname{Spec} S_{s_i} \to \operatorname{Spec} R_i$. This gives a cover of $\operatorname{Spec} S$ in terms of affine open subschemes $\operatorname{Spec} S_{s_i}$. Hence the scheme morphism $f : \operatorname{Spec} T \to \operatorname{Spec} S$ can be given in terms of compatible components $f_i : \operatorname{Spec} T_{t_i} \to \operatorname{Spec} S_{s_i}$. The composition of components $\varphi_i \circ f_i$ induces a cover of $\varphi \circ f$ into compatible components $(\varphi \circ f)_i = \varphi_i \circ f_i : \operatorname{Spec} T_{t_i} \to \operatorname{Spec} R_i$. By gluing, since F is a sheaf by condition (1), we have $\alpha(T)(\varphi \circ f)|_{\operatorname{Spec} T_{t_i}} = \alpha_i(T_{t_i})(\varphi_i \circ f_i)$. Each α_i is a natural transformation, so $\alpha_i(T_{t_i})(\varphi_i \circ f_i) = F(f_i)(\alpha_i(S_{s_i})(\varphi_i))$. For an arbitrary ring R and localisation R_{r_i}, let $\psi_{R_i} : \operatorname{Spec} R_{r_i} \to \operatorname{Spec} R$ denote the natural map. First note that the diagram

$$\begin{array}{ccc} F(S) & \xrightarrow{F(f)} & F(T) \\ {\scriptstyle F(\psi_{S_i})}\downarrow & & \downarrow{\scriptstyle F(\psi_{T_i})} \\ F(S_{s_i}) & \xrightarrow{F(f_i)} & F(T_{t_i}) \end{array}$$

commutes because $f \circ \psi_{T_i} = \psi_{S_i} \circ f_i$. So for every $p \in F(S)$ such that $p|\operatorname{Spec} S_{s_i} = p_i \in F(S_{s_i})$, we have $F(f)(p)|\operatorname{Spec} T_{t_i} = F(f_i)(p_i)$. Setting $p = \alpha(S)(\varphi)$ and $p_i = \alpha(S_i)(\varphi_i)$ establishes the required equality $F(f)(\alpha(S)(\varphi))|\operatorname{Spec} T_{t_i} = F(f_i)(\alpha_i(S_{s_i})(\varphi_i))$. By uniqueness of the element restricting to $F(f_i)(\alpha_i(S_{s_i})(\varphi_i)) = \alpha_i(T_{t_i})(\varphi_i \circ f_i)$ for each $\operatorname{Spec} T_{t_i}$, we have $F(f)(\alpha(S)(\varphi)) = \alpha(T)(\varphi \circ f)$. Hence $\alpha : h_Y \to F$ is a natural isomorphism, and F is of the form h_Y. □

References

[1] D. Eisenbud and J. Harris, *The Geometry of Schemes.* Springer-Verlag New York, 1999.

[2] R. Hartshorne, *Algebraic Geometry.* Springer-Verlag New York, 1977.