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# Powers of Maximal Monotone Operators on Hilbert Spaces

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**Abstract**

The problem of taking higher powers of single-valued operators on suitable spaces is well understood and can be solved via the method of composition. Here, we will examine operators that are possibly multi-valued. We will compare the method of composition with a newer approach to taking powers of operators based upon a method discovered by Alaarabiou and Benilan in 1996 for squared powers [1]. We will attempt to generalise the results of Alaarabiou and Benilan to $k \geq 2$, specifically in showing that $A \subseteq (A_{\frac{1}{k}})^k$ for maximal monotone operators $A$.

# 1. Introduction

The question of taking higher powers for abstract operators is a rather natural question to ask. As alluded to briefly in the abstract, this problem varies from trivial to complex rather quickly. This report will explore the problem at a surface level past Brezis and Barbu up to the current research being done.

Consider the following Dirichlet problem in the multi-valued setting on the half-space $H_+$

$$\begin{cases} u_{tt}(t) + Bu(t) \ni 0 \text{ on } H_+ = H \times (0, +\infty) \\ u(0) = \varphi \text{ on } \partial H_+ = H \end{cases} \tag{1}$$

Brezis and Barbu were able to show that the "square root" of an operator $B$ denoted as $B_{\frac{1}{2}}$ coincided with the infinitesimal generator of $\{T_t\}_{t\geq 0}$ of flows $T_t\varphi = u(t)$. Due to The flows satisfy the following multiplication rule: $T_{t+s}\varphi = T_tT_s\varphi$. They form a strongly continuous semi-group. To be precise, Brezis and Barbu found that for maximal monotone operators $B$, we have that

$$B_{\frac{1}{2}} = \left\{ (\varphi, v) \in H \times H \middle| \lim_{h\to 0} \frac{T_h\varphi - u(0)}{h} = -v \right\}$$

To the keen eye, this coincides with the Dirichlet to Neumann map (DtN) for the Dirichlet problem above, that is, we have that

$$B_{\frac{1}{2}} = \Lambda : \varphi \to -u'(0)$$

Given this new interpretation for the square root of $B$, one naively expects or at least hopes for a result of the following nature

$$(B_{\frac{1}{2}})^2 = B \tag{2}$$

That is, the composition of the square root of an operator with itself is the original operator. In linear cases of $B$, (2) holds true. However, if $B$ is a non-linear operator, then we have that (2) fails. An elementary example of this would be the maximal monotone operator $B : \mathbb{R} \to \mathbb{R}$ such that $Bt = t^3$. To verify that (2) does indeed breakdown, we must solve the following system

$$\begin{cases} u_{tt}(t) + t^3 u(t) = 0 \text{ on } \mathbb{R}_+ = \mathbb{R} \times (0, +\infty) \\ u(0) = \varphi \text{ on } \partial\mathbb{R}_+ = \mathbb{R} \end{cases} \tag{3}$$

Leaving out the gory details, one will obtain that $B_{\frac{1}{2}} t = \frac{|t|t}{2}$. From here, we compose the DtN map of (3) with itself to obtain $(B_{\frac{1}{2}})^2 t = \frac{|t|t^3}{8}$, which clearly is not equal to $Bt = t^3$.

From the above work of Brezis and Barbu, mathematicians looked for a new procedure for taking higher powers of operators seeing as the composition rule did not yield favourable results (such as the one above). Alaarabiou and Benilan, redefined the square power of $B$ as an approximation denoted by $B_2$ in the following manner

$$B_2 := \liminf_{\lambda \to 0+} \frac{B - B_\lambda}{\lambda} \tag{4}$$

Where $B$ is some maximal monotone operator and $B_\lambda$ is the Yosida approximation (13). If $B$ is linear, then there is a rather simple heuristic as to why this is a suitable "approximation" of composition

$$\frac{B - B_\lambda}{\lambda} = \frac{B - BJ_\lambda^B}{\lambda} = B\left(\frac{I - J_\lambda^B}{\lambda}\right) = BB_\lambda$$

From this equation, we can take the lim inf of both sides as $\lambda \to 0+$ to obtain something i.e. $\liminf_{\lambda \to 0+} BB_\lambda$ resembling the square of an operator. Here we have that $J_\lambda^B$ is the resolvant of $B$ for a given $\lambda > 0$ (12).

Within the same paper, Alaarabiou and Benilan were able to establish the following theorem

**Theorem 1.** *If $\Phi : H \to [-\infty, \infty)$ is a convex, proper, lower semi-continuous functional and $B = \partial\Phi$. Then we have that*

$$B \subseteq (B_{\frac{1}{2}})_2$$

In addition, Alaarabiou and Benilan successfully established the reverse inclusion in the above theorem for when $H = \mathbb{R}$.

Within this research project, we attempted to extend the above result to all integer powers $k \geq 2$, that is

**Conjecture 1.** *If $\Phi : H \to [-\infty, \infty)$ is a convex, proper, lower semi-continuous functional and $B = \partial\Phi$. Then we have that*

$$B \subseteq (B_{\frac{1}{k}})_k$$

From here, it is clear that one needs a sensible definition for $B_k$ or $k$-th powers of $B$ as well as the notion of fractional powers $B_{\frac{1}{k}}$. However, both of these can be defined in a vaguely similar manner to $B_2$ and $B_{\frac{1}{2}}$. Finally, one can then bootstrap Theorem 2 to hold for all maximal monotone operators $B$ via an auxiliary theorem of Rockafellar [4].

## 2. The Linear Case

We begin by attempting to define a "3rd power" of $B$ by following the heuristic of Alaarabiou and Benilan

$$\frac{B_2 - BB_\lambda}{\lambda} = \frac{B^2 - BB_\lambda}{\lambda} = B\left(\frac{B - B_\lambda}{\lambda}\right) = B^2 B_\lambda$$

From here, the pattern seems to imply that under suitable conditions on $B$, perhaps a logical definition for $B_k$ is

$$B_k = \liminf_{\lambda \to 0+} \frac{B_{k-1} - B_{k-2}B_\lambda}{\lambda} \tag{5}$$

We compare the operation of composition with its approximation (5). Essentially this involved looking for when the sets $B^k$ and $B_k$ where included in one another (essentially when equality occurs and when it doesn't). Formally we have that

**Theorem 2.** *Let $B \in \mathscr{M}(X)$ be linear and $k \geq 2$ be an integer. Then* *(i)* $B^k$ *is linear.*

*(ii)* $B^k \subseteq B_k$.

*Proof.* (i) The case for $B^2$ was done by Alaarabiou and Benilan [1], one can adapt their proof to fit for induction. The spirit of the proof however is essentially identical.

(ii) We proceed by Induction over $k$ with our base case as $k = 2$. Suppose $(x, z) \in B^2$. Then by the definition of $B^2$, we have that there exists $y \in D(B)$ such that $y \in Bx$ and $z \in By$. We now set

$$x_\lambda := x + \lambda y \to x \text{ for every } 0 < \lambda < \lambda(B)$$

as $\lambda \to 0$. Now we have that $y = B_\lambda x_\lambda$ if and only if $y \in B(x_\lambda - \lambda y) = Bx$. Thus we have the following

$$\begin{aligned} B_\lambda x_\lambda + \lambda z = y + \lambda z \in B(x + \lambda y) \;\; &\text{(By the Linearity in B)} \\ = Bx_\lambda \end{aligned}$$

So we end up with

$$z \in \left(\frac{B - B_\lambda}{\lambda}\right) x_\lambda \tag{6}$$

Thus we have that $(x, z) \in B_2$ and therefore, $B^2 \subseteq B_2$. From here, we use strong induction by assuming that for all $2 \leq j \leq k-1$, the statement $B^j \subseteq B_j$ holds. So, we suppose $(x, z) \in B^k$. We have that $B^k = B^{k-2} \circ B^2$. So we have that there exists $y \in B^2x$ such that $z \in B^{k-2}y$. From (6), we have that there exists a sequence $(x_\lambda)$ in $X$ such that $x_\lambda \to x$ as $\lambda \to 0+$ and

$$y \in \left(\frac{B - B_\lambda}{\lambda}\right) x_\lambda \text{ for all } 0 < \lambda < \lambda(B)$$

We then have that

$$z \in B^{k-2}y \subseteq B^{k-2}\left(\frac{B - B_\lambda}{\lambda} x_\lambda\right) = \frac{B^{k-1} - B^{k-2}B_\lambda}{\lambda} x_\lambda \subseteq \frac{B_{k-1} - B_{k-2}B_\lambda}{\lambda} x_\lambda$$

Which implies that $(x, z) \in B_k$ and thus $B^k \subseteq B_k$.

□

Ideally, we would like $B_k \subseteq B^k$ in general. This however is wishful thinking and holds for certain operators. We however can break the problem of the above inclusion down into determining whether $B$ is a closed operator via the following

**Theorem 3.** *Suppose $B \in \mathscr{M}(X)$ is linear and $\liminf_{\lambda\to 0+}\|J_\lambda^B\| < \infty$. Then $B_k \subseteq B^k$ for all integers $k \geq 2$ if $B^k$ is closed for all integers $k \geq 2$.*

*Proof.* ($\Longleftarrow$) We proceed by induction over $k \geq 2$. The base case for induction comes from Alaarabiou and Benilan [1], which states that under the conditions, $B_2 = B^2$. Next, assume that $B_j = B^j$ for all integers $2 \leq j \leq k-1$. We aim to show that $B_k = B^k$. To do so, let $(x, z) \in B_k$. Then there exist sequences $(\lambda_n) \subseteq (0, \lambda(B))$ and $((x_{\lambda_n}, z_{\lambda_n})) \subseteq X \times X$ such that

$$\begin{aligned}
&\lambda_n \to 0+ \ \text{ as } n \to \infty, \\
&(x_{\lambda_n}, z_{\lambda_n}) \in \frac{B_{k-1} - B_{k-2}B_{\lambda_n}}{\lambda_n} \ \text{ for all } n, \\
\text{and } &(x_{\lambda_n}, z_{\lambda_n}) \to (x, z) \text{ in } X \times X.
\end{aligned}$$

Let $u_{\lambda_n} := J_{\lambda_n}^B x_{\lambda_n}$ and $y_{\lambda_n} := B_{\lambda_n} x_{\lambda_n} \in BJ_{\lambda_n}^B x_{\lambda_n} = Bu_{\lambda_n}$. Then, by the linearity of $B$, and since by the induction hypothesis one has that $B_{k-1} = B^{k-1}$ and $B_{k-2} = B^{k-2}$, it follows that

$$\begin{aligned}
z_{\lambda_n} &\in \frac{B_{k-1}x_{\lambda_n} - B_{k-2}B_{\lambda_n}x_{\lambda_n}}{\lambda_n} = \frac{B^{k-1}x_{\lambda_n} - B^{k-2}B_{\lambda_n}x_{\lambda_n}}{\lambda_n} \\
&\subseteq \frac{B^{k-1}x_{\lambda_n} - B^{k-1}u_{\lambda_n}}{\lambda_n} = B^{k-1}\left(\frac{x_{\lambda_n} - u_{\lambda_n}}{\lambda_n}\right) = B^{k-1}(y_{\lambda_n})
\end{aligned}$$

so $z_{\lambda_n} \in B^{k-1}(y_{\lambda_n})$ and $y_{\lambda_n} \in B(u_{\lambda_n})$. Hence $z_{\lambda_n} \in B^k(u_{\lambda_n})$. Now, note that $u_{\lambda_n} \to x$ as $n \to \infty$ since

$$\begin{aligned}
\|u_{\lambda_n} - x\| = \|J_{\lambda_n}^B x_{\lambda_n} - x\| &= \|J_{\lambda_n}^B x_{\lambda_n} - x_{\lambda_n} + x_{\lambda_n} - x\| \\
&\leq \|J_{\lambda_n}^B x_{\lambda_n} - x_{\lambda_n}\| + \|x_{\lambda_n} - x\| \to 0 + 0
\end{aligned}$$

due to Proposition 3 (v) in Appendix D. Since we also have that $z_{\lambda_n} \to z$, and that $B^k$ is closed by assumption, we can conclude that $z \in B^k x$. Hence $(x, z) \in B^k$, and so $B_k \subseteq B^k$. □

Once the above equivalence was established, we looked into special classes of operators that can be shown are closed and hence obey $B_k = B^k$. The first class of operators considered were the bounded operators. Showing such operators are closed is rather easy

**Proposition 1.** *If $B \in \mathcal{L}(X)$ ($B$ is linear and bounded). Then $B^k$ is closed for all integers $k \geq 2$.*

*Proof.* Since $B \in \mathscr{L}(X)$, we have that $B^k$ is also in $\mathscr{L}(X)$, for all integers $k \geq 2$. Since $B^k$ is linear and $B$ is bounded, we have that $B^k$ is also bounded since

$$\|B^k x\| = \|B(B^{k-1}x)\| \leq \|B\|\|B^{k-1}x\| \leq \|B\|^2\|B^{k-2}x\| \leq \cdots \leq \|B\|^k\|x\| \text{ for all } x \in X.$$

Since $B^k \in \mathscr{L}(X)$, we have that $B^k$ is single valued and continuous. This means that for any sequence $(x_n) \subseteq X$ converging to $x$, the sequence $(B^k x_n)$ converges to $B^k x$. Hence $B^k$ is closed. □

We then considered coercive operators (see Appendix E) and were able to show the following

**Proposition 2.** *Let $V$ be a reflexive Banach space and $H$ a Hilbert space defined as in Appendix E. If $B\colon V \to V^*$ be linear and coercive, and $B_H \in \mathscr{M}(H)$ with $J_\lambda^{B_H} \in \mathscr{L}(H)$ for all $0 < \lambda < \lambda(B_H)$. Then $(B_H)^k$ is closed for all integers $k \geq 2$.*

*Proof.* Let $((x_n^{(0)}, x_n^{(k)})) \subseteq (B_H)^k$ and $(x_n^{(0)}, x_n^{(k)}) \to (x_0, x_k) \in H \times H$. We want to show that $(x_0, x_k) \in (B_H)^k$. Since $(x_n^{(0)}, x_n^{(k)}) \in (B_H)^k$, there exist $x_n^{(i)}$ for $i = 1, \dots, k-1$ such that $(x_n^{(i)}, x_n^{(i+1)}) \in B_H$ for all $i = 0, \dots, k-1$. Let

$$u_{\lambda,n} := x_n^{(0)} - (-\lambda)^k x_n^{(k)} \to x_0 - (-\lambda)^k x_k =: u_\lambda \text{ as } n \to \infty, \text{ for all } \lambda \in \Lambda.$$

Now

$$\begin{aligned} u_{\lambda,n} &= x_n^{(0)} - (-\lambda)^k x_n^{(k)} \\ &= x_n^{(0)} + \lambda(x_n^{(1)} - x_n^{(1)}) + \lambda^2(x_n^{(2)} - x_n^{(2)}) + \cdots + \lambda^{k-1}(x_n^{(k-1)} - x_n^{(k-1)}) - (-\lambda)^k x_n^{(k)} \\ &= \left[x_n^{(0)} - \lambda x_n^{(1)} + \lambda^2 x_n^{(2)} + \cdots + (-\lambda)^{k-1} x_n^{(k-1)}\right] + \lambda\left[x_n^{(1)} - \lambda x_n^{(2)} + \lambda^2 x_n^{(3)} + \cdots + (-\lambda)^{k-1} x_n^{(k)}\right], \end{aligned}$$

and since $B_H$ is linear by Lemma 2 (Appendix E), we have that

$$u_{\lambda,n} = \left[x_n^{(0)} - \lambda x_n^{(1)} + \lambda^2 x_n^{(2)} + \cdots + (-\lambda)^{k-1} x_n^{(k-1)}\right] + \lambda B_H\left[x_n^{(0)} - \lambda x_n^{(1)} + \lambda^2 x_n^{(2)} + \cdots + (-\lambda)^{k-1} x_n^{(k-1)}\right].$$

Hence

$$J_\lambda^{B_H} u_{\lambda,n} = x_n^{(0)} - \lambda x_n^{(1)} + \lambda^2 x_n^{(2)} + \cdots + (-\lambda)^{k-1} x_n^{(k-1)}. \tag{7}$$

Note that the left hand side converges to $J_\lambda^{B_H} u_\lambda$ as $n \to \infty$ since $J_\lambda^{B_H} \in \mathscr{L}(H)$. Now, since $(x_n^{(k-1)}, x_n^{(k)}) \in B_H$ and using the coercivity of $B$, we have that

$$\eta \|x_n^{(k-1)}\|_V^2 \leq \langle x_n^{(k)}, x_n^{(k-1)} \rangle_H \leq \|x_n^{(k)}\|_H \|x_n^{(k-1)}\|_H. \tag{8}$$

Young's inequality yields that

$$\|x_n^{(k)}\|_H \|x_n^{(k-1)}\|_H \leq \frac{\|x_n^{(k)}\|_H^2}{2} + \frac{\|x_n^{(k-1)}\|_H^2}{2}.$$

We can obtain tighter control over the second term by sacrificing control over the first term since for every $\varepsilon > 0$, we have that

$$\|x_n^{(k)}\|_H \|x_n^{(k-1)}\|_H \leq \frac{\|x_n^{(k)}\|_H^2}{4\varepsilon} + \varepsilon \|x_n^{(k-1)}\|_H^2.$$

Plugging the above inequality onto the right hand side of (8) gives us

$$\begin{aligned} \eta \|x_n^{(k-1)}\|_V^2 &\leq \frac{\|x_n^{(k)}\|_H^2}{4\varepsilon} + \varepsilon \|x_n^{(k-1)}\|_H^2 \\ &= \frac{\|x_n^{(k)}\|_H^2}{4\varepsilon} + \varepsilon C_p^2 \|x_n^{(k-1)}\|_V^2, \end{aligned}$$

Since the above inequality holds for all $\varepsilon > 0$, we select $\varepsilon_p = \frac{\eta}{2C_p^2}$ so that we have

$$\eta \|x_n^{(k-1)}\|_V^2 \leq \frac{\|x_n^{(k)}\|_H^2}{4\varepsilon_p} + \frac{\eta}{2} \|x_n^{(k-1)}\|_V^2,$$

and hence

$$\frac{\eta}{2} \|x_n^{(k-1)}\|_V^2 \leq \frac{1}{4\varepsilon_p} \|x_n^{(k)}\|_H^2. \tag{9}$$

Since $(x_n^{(k)})$ converges in $H$ by assumption, it must also be bounded. This in turn implies that $(x_n^{(k-1)})$ is bounded in $V$. Since $V$ is reflexive, we have that there exist $x_{k-1} \in V$ and a subsequence of $(x_n^{(k-1)})$ which converges weakly to $x_{k-1}$ in $V$. Then, since $V \hookrightarrow H$, and passing to the appropriate subsequence, we get that $x_n^{(k-1)} \rightharpoonup x_{k-1}$ also in $H$. Repeating this argument with $(x_n^{(k-2)}, x_n^{(k-1)})$ up to $(x_n^{(1)}, x_n^{(2)})$, and noting that a weakly convergent sequence is also bounded, we get that there exist $x_i \in H$ for all $i = 1, \ldots, k-1$ such that a subsequence of $(x_n^{(i)})$ converges weakly to $x_i$ for all $i$. Hence, by passing to the appropriate subsequences and taking the limit of (7) as $n \to \infty$, we get by uniqueness of limits that

$$J_\lambda^{B_H} u_\lambda = x_0 - \lambda x_1 + \lambda^2 x_2 - \cdots + (-\lambda)^{k-1} x_{k-1}.$$

This means that

$$\begin{aligned} u_\lambda &= \left[x_0 - \lambda x_1 + \lambda^2 x_2 - \cdots + (-\lambda)^{k-1} x_{k-1}\right] + \lambda B_H \left[x_0 - \lambda x_1 + \lambda^2 x_2 - \cdots + (-\lambda)^{k-1} x_{k-1}\right] \\ &= x_0 - (-\lambda)^k x_k. \end{aligned}$$

Subtracting $x_0$ from both sides and using the linearity of $B_H$, we get that

$$\left[-\lambda x_1 + \lambda^2 x_2 - \cdots + (-\lambda)^{k-1} x_{k-1}\right] + \left[\lambda B_H x_0 - \lambda^2 B_H x_1 + \lambda^3 B_H x_2 - \cdots - (-\lambda)^k B_H x_{k-1}\right] = -(-\lambda)^k x_k,$$

so

$$-\lambda(x_1 - B_H x_0) + \lambda^2(x_2 - B_H x_1) - \cdots + (-\lambda)^{k-1}(x_{k-1} - B_H x_{k-2}) + (-\lambda)^k (x_k - B_H x_{k-1}) = 0 \text{ for all } \lambda \in \Lambda.$$

Viewing the left hand side as a polynomial in $-\lambda$, and recalling that a polynomial of degree $k$ is uniquely determined by $k+1$ points, we get from equating coefficients that

$$x_i - B_H x_{i-1} = 0,$$

and so $x_i = B_H x_{i-1}$ for all $i = 1, \dots, k$. This implies that $x_k = (B_H)^k x_0$, proving that $(B_H)^k$ is closed. □

**Remark 1.** *Where $V$ and $H$ are as in the definition of coercivity. For certain linear elliptic operators $B \subseteq V \times H$ (where $\omega < 2\eta$), we can also conclude using a similar technique as above that $B^k$ is closed for all integers $k \geq 2$ and hence $B^k = B_k$.*

# 3. The Non-Linear Case

This section is very short, seeing as it is currently a work in progress. However, the assumption that $B$ is non-linear implies that we cannot expect any results that are of a similar nature to the linear section. That is, it would be wishful thinking to have reasonable assumptions leading to $B^k = B_k$. This is clear if we consider the case $B : x \in \mathbb{R} \mapsto x^3 \in \mathbb{R}$. Here $B$ is a non-linear, maximal monotone operator. As was seen in the introduction, we had that using Brezis and Barbu's definition for $B_{\frac{1}{2}}$, $B_{\frac{1}{2}} x = \frac{|x|x}{2}$. The thing to note though is that $B_{\frac{1}{2}}$ is non-linear. It is clear that $(B_{\frac{1}{2}})^2 \neq (B^{\frac{1}{2}})_2$. In fact, it is also clear that the only point that both graphs share is at the origin. So we have no set inclusion whatsoever, thus Theorem 3 and 4 can not be generalised for non-linear operators. The best comparison one can perhaps obtain is based upon the results of [1], specifically the strengthening of Theorem 1 to Conjecture 1. We plan to attempt to establish Conjecture 1 over the next few weeks, however this requires us to understand the properties of $(B_{\frac{1}{3}})_3$ since we have that

$$(B_{\frac{1}{3}})_3 = \liminf_{\lambda \to 0+} \frac{(B_{\frac{1}{3}})_2 - B_{\frac{1}{3}}(B_{\frac{1}{3}})_\lambda}{\lambda}$$

Essentially what seems to happen is, as $k$ increases, we get nested $\liminf$'s due to the recursive nature of the definition for $k$. This is concerning. However there appears to be a paper that might contain techniques as to how to deal with this issue. One avenue of attack is to utilise the assumption that $B = \partial\Phi$, and to look for a relationship between how "raising" $B$ to certain powers would change $\Phi$.

# A. The Multi-Valued Setting

We begin this section, by introducing multi-valued operators.

**Definition 1.** *A relation $B : X \to Y$ is not a function if there exists $x \in X$, such that $Bx = y_1$ and $Bx = y_2$ where $y_1 \neq y_2$. Such a relation, is known as a multi-valued operator.*

**Remark 2.** *Such an operator can be thought of as a single-valued, set-valued map from $X$ to $2^Y$. Hence, the multi-valued setting perfectly encapsulates the single-valued setting.*

Such operators are identified with their *graphs*. Thus we say that $B = \{(x, y) \in X \times Y | y \in Bx\}$. Examples of multi-valued operators include the following:

$$Bx = \begin{cases} 0 & \text{if } \ x \neq 0 \\ \mathbb{R} & \text{if } \ x = 0 \end{cases} \quad \& \quad Bx = \begin{cases} -1 & \text{if } \ x < 0 \\ [-1, 1] & \text{if } \ x = 0 \\ 1 & \text{if } \ x > 0 \end{cases}$$

Additionally, we can construct more exotic multi-valued operators such as the $n$-th root of a complex number $z \in \mathbb{C}$ or the inverse trigonometric functions with domain and co-domain without restriction, that is $\mathbb{R} \to \mathbb{R}$.

Since we are dealing with a more general class of operators with whom we know little about. It seems that the first logical step would be to build up some algebraic structure on the class of multi-valued operators. Suppose we have two multi-valued operators $A_1, A_2 \subseteq X \times X$. Then we have

$$A_1 + A_2 = \{(x, y + z) \in X \times X | y \in A_1 x \ \ \& \ \ z \in A_2 x\}$$

One can also define additional algebraic operations on this class of operators, however we are really only interested in the following

$$A_1^{-1} := \{(y, x) \in X \times X | y \in Ax\}$$

Additionally, we will define a composition rule as follows

$$A_1 \circ A_2 := \{(x, y) \in X \times X | A_2 x \cap A_1^{-1} y \neq \emptyset\}$$

It isn't clear that this is a logical definition for composition. However, it is hoped that an example may shed light as to why composition was chosen like this. Consider $Bx = x^3 : \mathbb{R} \to \mathbb{R}$. We will compute $B \circ B$. So we have two equations

$$Bx = x^3 \ \ \& \ \ B^{-1}y = y^{\frac{1}{3}}$$

For the non-empty intersection to be satisfied, we need $y = x^9$. Thus, we have that $B^2 := B \circ B = \{(x, x^9) \in \mathbb{R} \times \mathbb{R} | x \in \mathbb{R}\}$.

We next introduce the reader to arguably the most important aspect of the preliminary theory. This is the theory of *Maximal Monotone Operators*.

## B. Maximal Monotone Operators

These operators can be thought of as the abstract generalisation of monotone increasing/decreasing operators with some kind of surjectivity condition. For example, $Bx = x^3$ is a maximal monotone operator since it is monotone increasing and is maximal since it is surjective. Keep in mind, that maximality should never be identified with the surjective property. We begin by defining these terms

**Definition 2.** *An operator $B \subseteq H \times H$ is called Monotone if we have that*

$$\langle x_1 - x_2, y_1 - y_2 \rangle_H \geq 0 \ \text{ for all } \ (x_1, y_1), (x_2, y_2) \in B$$

In addition, we say that such an operator $B$ is *Maximal* if for every monotone operator $\widetilde{B} \subseteq H \times H$, we have that $B \subseteq \widetilde{B}$ implies that $B = \widetilde{B}$.

**Remark 3.** *These operators are also known as accretive or m-accretive operators (depending on maximality). This is because of the Riesz-Frechet theorem which allows for one to identify a Hilbert space $H$ with its dual space. Essentially, one can use the terms accretive and m-accretive synonymously with monotonicity and maximal monotonicity if one is working within a reflexive Banach space.*

Monotonicity is a property that is not unique to Hilbert spaces, one can define monotonicity via the duality pairing. This means that monotone operators are defined within a *variational framework*. Since, we are however working a priori within a Hilbert space $H$, we can tie the notion of monotonicity to metric geometry via *Kato's Lemma*.

**Lemma 1.** *An operator $B$ is monotone if and only if the following inequality holds*

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| \ \text{ for all } \ \lambda > 0, \quad (x_i, y_i) \in B \tag{10}$$

*where $i = 1, 2$*

We omit the proof for this lemma. See [2] for a more proper treatment of monotone/accretive operators in Hilbert/Banach spaces respectively.

## C. Convex Functionals on Hilbert Spaces

**Definition 3.** *A functional $\Phi : H \to (-\infty, +\infty]$ is convex provided*

$$\Phi[\mu u + (1 - \mu)v] \leq \mu\Phi[u] + (1 - \mu)\Phi[v]$$

*for all $u, v \in H$ and each $\mu \in [0, 1]$.*

Notice, we allow $\Phi$ to take on the value $+\infty$ but not $-\infty$. We say that $\Phi$ is *proper* if $\Phi$ isn't identically equal to $+\infty$. The *domain* of $\Phi$ is

$$D(\Phi) := \{u \in H | \Phi[u] < +\infty\}$$

**Definition 4.** $\Phi : H \to (-\infty, +\infty]$ *is said to be lower semi-continuous (l.s.c for short) if*

$$\begin{cases} u_k \to u \ \ in \ \ H \ \ implies \\ \Phi[u] \leq \liminf_{k\to\infty} \Phi[u_k] \end{cases}$$

From here, we can define the notion of a Subdifferential. This can be naively thought of as a generalisation of differentiation to Hilbert spaces as well as the "differentiation of corners".

**Definition 5.** *Let* $\Phi : H \to (-\infty, +\infty]$ *be convex and proper.*

*(i) For each* $u \in H$*, we write*

$$\partial\Phi[u] := \{v \in H | \Phi[w] \geq \Phi[u] + (v, w-u) \ \ for\ all \ \ w \in H\} \tag{11}$$

*Such a mapping* $\partial\Phi : H \to 2^H$ *is called the Subdifferential of* $\Phi$*.*

*(ii) We say* $u \in D(\partial\Phi)$*, provided* $\partial\Phi[u] \neq \emptyset$*.*

As was alluded to earlier, one geometric interpretation of the Subdifferential (11) would be to consider $H = \mathbb{R}$ and $\Phi[x] = |x|$. Here, we would have that $\partial\Phi[x]$ would take on values of $-1$ and $+1$ for $x < 0$ and $x > 0$ respectively. This is as expected and coincides with the ordinary derivative. However, we can also compute $\partial\Phi[x]$ at the corner, where $x = 0$. This yields an uncountable set and thus points to why the Subdifferential is in general multi-valued. To summarise, for our concrete example, we would obtain

$$\partial\Phi[x] = \begin{cases} -1 & \text{if} \ \ x < 0 \\ [-1, +1] & \text{if} \ \ x = 0 \\ +1 & \text{if} \ \ x > 0 \end{cases}$$

**Remark 4.** *One thing to note is that the Subdifferential of any convex, proper functional is a maximal monotone operator. Similarly, see [3] for a more complete treatment.*

## D. The Resolvant and The Yosida Approximation

We begin by defining $J_\lambda^B$ as follows:

$$J_\lambda^B x := (I + \lambda B)^{-1} x \ \ \text{for} \ \ x \in R(I + \lambda B) \tag{12}$$

This is known as the *Resolvant of B* for a given $\lambda > 0$. Additionally, we define $B_\lambda$ as follows:

$$B_\lambda x := \frac{1}{\lambda}\left(I - J_\lambda^B\right) x \quad \text{for} \quad x \in R(I + \lambda B) \tag{13}$$

This operator forms a family of operators indexed by $\lambda$ which approximate $B$. This family of operators is known as the *Yosida Approximation.* We begin by listing out some elementary properties of both $J_\lambda^B$ and $B_\lambda$.

**Proposition 3.** *Let $B \subseteq H \times H$ be monotone. Then we have the following:*

*(i)* $\|J_\lambda^B x - J_\lambda^B y\| \leq \|x - y\|$ *for all* $x, y \in R(1 + \lambda B)$

*(ii)* $B_\lambda$ *is monotone and Lipschitz continuous*

*(iii)* $B_\lambda x \in B J_\lambda^B x$ *for all* $x \in R(1 + \lambda B)$

*(iv)* $\|B_\lambda x\| \leq |B^0 x| := \inf_{y \in Bx} \|y\|$

*(v)* $\lim_{\lambda \to 0+} J_\lambda^B x = x$ *for all* $x \in \overline{D(B)} \cap_{\lambda > 0} R(1 + \lambda B)$

See [2] for the proof of the above proposition.

**Remark 5.** *We let $\mathscr{M}(X)$ denote the set of all closed operators $B \colon X \to 2^X$ having a family of resolvents $(J_\lambda^B)_{0<\lambda<\lambda(B)}$ in $X$ which are Lipschitz continuous.*

# E. Coercive and Elliptic Operators

Let $V$ be a Banach space and $H$ a Hilbert space, and suppose that there exists an embedding $V \overset{d}{\hookrightarrow} H$, that is, there exists an injection $i \in \mathscr{L}(V; H)$ such that $\overline{i(V)}^H = H$. For the rest of this section, we let $B$ be a mapping from $V$ to $V^*$.

**Definition 6.** *The part $B_H$ of $B$ in $H$ is defined by*

$$B_H := \{(u, h) \in H \times H \mid \langle Bu, v\rangle_{V^*, V} = \langle h, i(v)\rangle_H \quad \textit{for all } v \in V\}.$$

**Lemma 2.** *$B_H \colon D(B_H) \to H$ is a well-defined map, where*

$$D(B_H) := \{u \in V \mid \textit{there exists } h \in H \textit{ such that } \langle Bu, v\rangle_{V^*, V} = \langle h, i(v)\rangle_H \textit{ for all } v \in V\}$$

*Moreover, $B_H$ is linear if $B$ is linear.*

*Proof.* Fix $u \in D(B_H)$ arbitrary, and let $h_1, h_2 \in B_H u$, that is,

$$\langle Bu, v\rangle_{V^*,V} = \langle h_1, i(v)\rangle_H = \langle h_2, i(v)\rangle_H \quad \text{for all } v \in V.$$

Then $\langle h_1 - h_2, i(v)\rangle_H$ for all $v \in V$. But since $i(V)$ is dense in $H$, this implies that $\langle h_1 - h_2, \xi\rangle_H$ for all $\xi \in H$, and so $h_1 = h_2$. Hence $B_H$ is well-defined.

Now suppose $B$ is linear, and let $(u_1, h_1), (u_2, h_2) \in B_H$ and $\lambda, \mu \in \mathbb{R}$. We claim that $(\lambda u_1 + \mu u_1, \lambda h_1 + \mu h_2) \in B_H$. We have that

$$\langle Bu_1, v\rangle_{V^*,V} = \langle h_1, i(v)\rangle_H \quad \text{and} \quad \langle Bu_2, v\rangle_{V^*,V} = \langle h_2, i(v)\rangle_H \quad \text{for all } v \in V.$$

Then since $B$ is linear, and $\langle \cdot, \cdot\rangle_{V^*,V}$ is linear in its first argument, we get that

$$\langle B(\lambda u_1 + \mu u_2), v\rangle = \langle \lambda Bu_1 + \mu Bu_2, v\rangle_{V^*,V} = \langle \lambda h_1 + \mu h_2, i(v)\rangle \quad \text{for all } v \in V.$$

Thus, we have that $B_H$ is linear. □

**Definition 7.** *An operator $B\colon V \to V^*$ is* coercive *if there exists $\eta > 0$ such that*

$$\langle Bx, x\rangle_{V^*,V} \geq \eta \|x\|_V^2 \quad \textit{for all} \quad x \in D(B).$$

**Definition 8.** *An operator $B\colon V \to V^*$ is said to be* elliptic *if there exist $\eta > 0$ and $\omega \in \mathbb{R}$ such that*

$$\eta \|x\|_V^2 \leq \langle Bx, x\rangle_{V^*,V} + \frac{\omega}{2}\|x\|_H^2 \quad \textit{for all } x \in D(B),$$

# F. Limit Inferior and Limit Superior of Sets

Given a net of subsets $(A_\lambda)_{\lambda>0}$ of $H \times H$.

**Definition 9.** *The Limit Inferior of the net $(A_\lambda)_{\lambda>0}$ is defined to be*

$$\liminf_{\lambda\to 0+} A_\lambda = \left\{ (x, y) \in H \times H \,\middle|\, \textit{there exists a sequence } ((x_\lambda, y_\lambda))_{\lambda>0} \subseteq A_\lambda \ \textit{s.t.} \lim_{\lambda\to 0+} (x_\lambda, y_\lambda) = (x, y) \right\}$$

One interpretation of this set, is that it contains the limit points of the net $(A_\lambda)_{\lambda>0}$.

**Definition 10.** *The Limit Superior of the net* $(A_\lambda)_{\lambda>0}$ *is defined as*

$$\limsup_{\lambda\to 0+} A_\lambda = \bigcap_{\lambda>0}\overline{\bigcup_{\mu\leq\lambda} A_\mu}$$

To make these definitions slightly more concrete or understandable, we consider the following sequence as an example.

$$B_n = \begin{cases} [0, 1-\frac{1}{n}] & \text{if } n \text{ even} \\ [0, 2-\frac{1}{n}] & \text{if } n \text{ odd} \end{cases}$$

Here, we would have that $\liminf_{n\to\infty} B_n = [0,1]$ and $\limsup_{n\to\infty} B_n = [0,2]$

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## References

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