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Precession of the Perihelia in a Schwarzschild Spacetime

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1 Abstract

The Newtonian theory of gravity, while a useful approximation, is unable to account for the entirety of the precession of Mercury's perihelia. The precession of Mercury's orbit is observed to be approximately $575.3''$/century, however the Newtonian theory predicts a precession of only $532.3''$/century. This project calculated the missing $43''$/century precession by finding the precession of Mercury's orbit in a Schwarzschild spacetime, using Einstein's theory of relativity. A ballpark figure was first calculated by analytically solving the geodesic equation to a first-order approximation, and a more accurate value was then calculated from a numerical solution. The precession returned by the numerical calculation matches the observed value to an accuracy of 0.01%.

2 Introduction

Since Newcomb (1882), it has been known that Mercury's orbit differed from that predicted by Newtonian gravity. When accounting for the different forces exerted upon Mercury from the sun, other planets in the solar system, and even smaller effects like the oblateness of the sun, the theory predicted a precession rate of $532.3301''$/century. Observations at the time showed the precession rate to be significantly larger, with an extra $43''$/century not accounted for by the theory of the time. Although the exact value of this missing precession has fluctuated, it has remained a constant that the Newtonian theory of gravity is unable to accurately predict this phenomena. Indeed, the ability of the following theory of gravity - Einstein's theory of General Relativity - to accurately predict this motion was an important test in determining it's validity.

General Relativity is a theory in which the force of gravity is replaced by the curvature of spacetime. Instead of bodies being attracted to each other, they follow the geodesics of this curved space. To calculate the path of an orbit in a given curved spacetime, it is sufficient to calculate the geodesic from the initial conditions of the body.

The added precession under a Schwarzschild metric, when compared to the equivalent Newtonian theory, accounts for the 43″/century precession discrepancy. In this project, a numerical simulation of Mercury's orbit under a Schwarzschild spacetime is performed, to calculate this precession. An analytical analysis of geodesics is also performed to a first order approximation, to provide an idea of the ballpark figures expected from the simulation.

3 Background

3.1 Precession

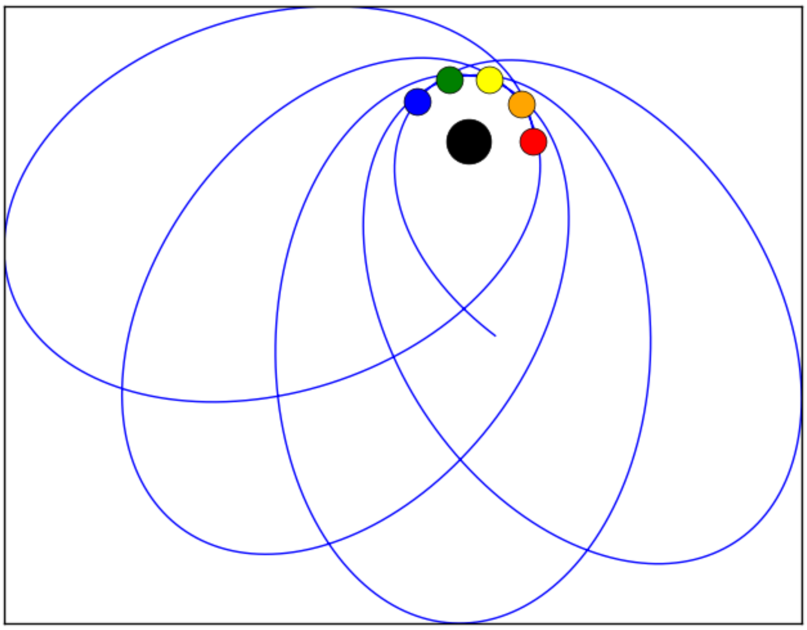

Figure 1: Exaggerated example path of a planet orbiting around a star (shown in black), with the perihelia represented by the coloured dots.

The precession of an orbit looks at the motion of the *perihelia* over time. The perihelia of an orbit is the point closest to the host body. In Figure 1 the perihelia is initially at the red dot, then after completing an orbit has shifted slightly to the orange dot, continuing to move after each completed orbit. The discussion of the precession of the perihelia of an orbit looks at the motion of those dots over time.

3.2 Geodesics

A geodesic is the generalisation of straight lines to curved surfaces. There are two major ways of thinking about geodesics outlined in this project, each with different benefits.

3.2.1 Extremising Distances

On a curved surface, a geodesic is the path between two points that extremises distance. There will always exist a geodesic between any two points, but there is not guaranteed to be only one geodesic.

3.2.2 Locally straight

Another method of locally finding geodesics is to take small, straight 'steps' along the surface. It may be easier to think of this as an ant on a curved surface walking in what it believes is a straight line. As long as the ant continues walking straight, it will map out a geodesic on the surface.

3.3 Metrics

A metric is a structure added to a manifold to enable the calculation of distances. The simplest example of a metric is the flat, 2-dimensional metric, expressed as

$$ds^2 = dx^2 + dy^2 \tag{1}$$

Another example is the metric for a sphere of constant radius R,

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{2}$$

Finally, the most simple example of a spacetime metric, known as the Minkowski metric, is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3}$$

Minkowski space is the spacetime equivalent of flat space, with no curvature. The spatial co-ordinates of this metric are the same as in the 2-dimensional flat space example, but the time co-ordinate has a negative coefficient. This introduction of a negative sign means that the distance, squared, can be a negative number. To avoid dealing with complex numbers, it is easier to instead use a different variable, τ, where $d\tau^2 = -ds^2$. This ensures that it is possible to work exclusively with real numbers.

3.4 Schwarzschild

A Schwarzschild spacetime is a spacetime where the only curvature comes from a single, static mass. This introduces two parameters into the metric; mass of the object (m), and distance from the mass (r). The Schwarzschild Metric is

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \tag{4}$$

The Schwarzschild spacetime will not account for the sun's spin when calculating Mercury's orbit. To include the spin of the sun, it is necessary to replace the Schwarzschild metric with the Kerr metric. In this project, the Schwarzschild approximation was used, but for a discussion of the effects of the Kerr metric on a body's orbit, see Appendix A.

4 2-dimensional Euler-Lagrange Equation

The Euler-Lagrange (E-L) equation can be used to calculate a geodesic from the metric of a curved space.

4.1 Lagrangian

Consider a 2-dimensional space, described using coordinates (x, y). One way to define a path through that space would be to take y as a function of x, or, put simply, $y = y(x)$. Consider a path between points P and Q, with x coordinates a and b, respectively. Computing the length L along the path between those two points requires the equation

$$L = \int_a^b \mathcal{L} dx \tag{5}$$

where

$$\mathcal{L} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{6}$$

In this case, $\mathcal{L}$ is known as the Lagrangian, and is dependant on the metric used to describe the space. Equation (5) can be generalised to hold for spaces with other metrics by simply using the appropriate Lagrangian.

4.2 Deriving the E-L equation - Variational Calculus

The aim of this section is to derive the E-L equation. This is an equation that describes the path extremising the integral L from equation (5). To begin, assume that the path $y = y(x)$ satisfies this condition. Small variations on that path are expressed as

$$\tilde{y} = y + \varepsilon\eta \tag{7}$$

Here, ε is some small number and η is any arbitrary function that satisfies $\eta(a) = \eta(b) = 0$, and does not approach infinity between a and b. The Lagrangian, and as such the length, is now a function of ε. This means that the length of the curve $\tilde{y}$ between a and b will be extremised at $\varepsilon = 0$. This results in

$$\left.\frac{dL}{d\varepsilon}\right|_{\varepsilon=0} = 0 \tag{8}$$

Returning to equation (5), the value $dL/d\varepsilon$ can be calculated by moving the derivative inside of the integral.

$$\frac{dL}{d\varepsilon} = \int_a^b \frac{d\mathcal{L}}{d\varepsilon} dx \tag{9}$$

Using Integration by Parts on equation (9) results in the following:

$$0 = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) \tag{10}$$

Equation (10) is the Euler-Lagrange equation, and describes how to calculate the geodesic $y = y(x)$ from the Lagrangian of a space.

It is also possible to instead represent the geodesic parametrically, as $(x(\lambda), y(\lambda))$. When using this representation, the E-L equation is

$$0 = \frac{\partial \mathcal{L}}{\partial z^a} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}^a}\right) \tag{11}$$

where

$$\dot{z}^\alpha = \frac{dz^\alpha}{d\lambda} \tag{12}$$

Equation 11 is actually two equations; a is an index, not a power, so in this case there will be one equation for $a = 1$, where $z^1 = x$, and another for $a = 2$, where $z^2 = y$. Solving those two equations would produce a parametric solution to the geodesic.

5 4-dimensional Euler-Lagrange equation

5.1 Einstein Summation Notation

Einstein Summation Notation (ESN) is a method of keeping equations shorter when dealing with General Relativity. It works by using indices, where having a repeated index indicates a sum over all possible values, such as

$$x_\alpha y^\alpha = \sum_{i=0}^{3} x_i y^i \tag{13}$$

5.2 Calculating the 4-dimensional E-L equations

To calculate the geodesic for the Schwarzschild metric, some small changes need to be made. Spacetime consists of four dimensions, so the parametric version of the E-L equations is used. The difference between a 4-dimensional and 2-dimensional solution is that a in equation (11) is replaced with α, which goes from 0 to 3. This makes the 4-dimensional E-L equation

$$0 = \frac{\partial \mathcal{L}}{\partial z^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}^\alpha}\right) \tag{14}$$

First, it is necessary to rewrite the metric using ESN. By deriving both sides with respect to parametrisation variable λ, any metric can be written using ESN as

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \tag{15}$$

where $g_{\alpha\beta}$ is a matrix representation of the coefficients. As an example, instead of writing the Minkowski metric as in equation (3), it can instead be written as equation (15) with

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

The Lagrangian of a metric expressed with equation (15) is

$$\mathcal{L} = \left(g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)^{1/2} d\lambda \tag{17}$$

with $\dot{x}^\alpha$ as defined in equation (12). Using the E-L equation with respect to variable x^α results in

$$0 = \frac{1}{2} g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu - g_{\alpha\mu,\nu}\dot{x}^\nu \dot{x}^\mu - g_{\alpha\mu}\ddot{x}^\mu \tag{18}$$

where

$$g_{ab,c} = \frac{\partial g_{ab}}{\partial x^c}$$

Then, rearranging the equation and performing some manipulations, the final equation is

$$0 = \ddot{x}^\nu + \Gamma^\nu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \tag{19}$$

with

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{2} g^{\nu\rho}\left(g_{\rho\beta,\alpha} + g_{\alpha\rho,\beta} - g_{\alpha\beta,\rho}\right)$$

5.3 Schwarzschild

The Schwarszschild metric is defined by

$$g_{\mu\nu} = \begin{bmatrix} -(1-\frac{2m}{r}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2m}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix} \tag{20}$$

Using equation (19) to find the parametric differential equation that describes geodesics on a Schwarzschild spacetime results in the four following equations:

$$\ddot{t} = -\frac{2m}{r(r-2m)}\dot{r}\dot{t} \tag{21}$$

$$\ddot{r} = -\frac{m(r-2m)}{r^3}\dot{t}^2 + \frac{m}{r(r-2m)}\dot{r}^2 + (r-2m)\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right) \tag{22}$$

$$\ddot{\phi} = -2\cot(\theta)\dot{\theta}\dot{\phi} - \frac{2}{r}\dot{r}\dot{\phi} \tag{23}$$

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin(\theta)\cos(\theta)\dot{\phi}^2 \tag{24}$$

6 Analytical Approximation

While these equations cannot be solved analytically, it is possible to find a first-order approximation, to get an idea of what the precession should be. Simplifying equations (21), (23) and

(24) results in:

$$\left(1-\frac{2m}{r}\right)\dot{t}=k \tag{25}$$

$$r^2\dot{\phi}=h \tag{26}$$

$$\theta=\pi/2 \tag{27}$$

where h and k are the constants of integration, whose precise value can be found from the specifics of Mercury's orbit. Finally, equation (22) can be replaced, using special relativistic principles and the knowledge that Mercury is a massive object. This results in the final equation being

$$-1=-\left(1-\frac{2m}{r}\right)\dot{t}^2+\frac{1}{1-\frac{2m}{r}}\dot{r}^2+r^2\dot{\phi}^2 \tag{28}$$

Combining equations (25), (26), (27) and (28) pproduces

$$0=-k^2+\left(1+\frac{h^2}{r^2(\tau)}\right)\left(1-\frac{2m}{r(\tau)}\right)+\left(\dot{r}(\tau)\right)^2 \tag{29}$$

Instead of finding the radius as a function of τ, it makes more sense to have it as a function of ϕ. Using the chain rule, it is clear that

$$\begin{aligned}\dot{r}&=\frac{dr}{d\phi}\frac{d\phi}{d\tau}\\&=r'\frac{h}{r^2}\end{aligned} \tag{30}$$

Making the simple substitution $u(\phi)=\frac{1}{r(\phi)}$ will also make the equation easier to solve. Using this substitution and equation (30), equation (29) becomes

$$0=-k^2+\left(h^2u(\phi)^2+1\right)+\left(1-2mu(\phi)\right)+h^2u(\phi)'^2 \tag{31}$$

Taking the derivative of equation (31) with respect to ϕ gives

$$0=2u'(\phi)\left(h^2u''(\phi)+h^2u(\phi)-3h^2u^2(\phi)m-m\right) \tag{32}$$

The solution $u'(\phi)=0$ applies only to circular orbits, so can be rejected. This leaves the equation

$$0=u''(\phi)+u(\phi)-3mu(\phi)^2-\frac{m}{h^2} \tag{33}$$

The differential equation from equation (33) cannot be solved fully in terms of elementary functions. However, it is possible to instead look only at a first order approximation of $u(\phi)$.

Equation (33) differs from the Newtonian version only in the inclusion of the $3mu(\phi)^2$ term. This value, as a fraction of the Newtonian approximation (neglecting the 3) is defined to be ϵ.

$$\epsilon = \frac{mu_c^2}{u_c} \tag{34}$$

where u_c is a circular approximation of Mercury's orbit. Substituting the average value of Mercury's orbital radius produces

$$\epsilon = 2.5600 \times 10^{-8} \tag{35}$$

As $\epsilon << 1$, it is reasonable to take a perturbation of $u(\phi)$ as

$$u(\phi) = u_0(\phi) + \epsilon v_0(\phi) \tag{36}$$

where u_0 and v_0 are functions that satisfy equation (33). Substituting equation (36) into equation (33), expanding, and discarding higher order terms of ϵ results in two second-order linear differential equations.

$$u_0'' + u_0 = \frac{m}{h^2} \tag{37}$$

$$v_0'' + v_0 = 3u_0^2\frac{h^2}{m} \tag{38}$$

Setting the perihelia to be at $\phi = 0$, solving equations (37) and (38), and substituting the solutions for u_0 and v_0 into equation (36) provides solution:

$$u(\phi) = (A + B\epsilon)\cos(\phi) + \frac{3\epsilon m}{h^2} + \frac{m}{h^2} + \frac{A^2\epsilon h^2}{m}\sin^2(\phi) + 3A\epsilon\phi\sin(\phi) \tag{39}$$

This solution is only accurate to a first order in ϵ, so without loss of accuracy, $3A\epsilon\phi\sin(\phi)$ can be replaced with $3(A + \epsilon B)\epsilon\phi\sin(\phi)$, resulting in

$$u(\phi) = (A + B\epsilon)\left(\cos(\phi) + 3\epsilon\phi\sin(\phi)\right) + \frac{3\epsilon m}{h^2} + \frac{m}{h^2} + \frac{A^2\epsilon h^2}{m}\sin^2(\phi) \tag{40}$$

Using trigonometric formulae and a Taylor expansion, it is clear that

$$\begin{aligned}\cos(\phi - 3\epsilon\phi) &= \cos(\phi)\cos(-3\epsilon\phi) - \sin(\phi)\sin(-3\epsilon\phi)\\ &= \cos(\phi) + 3\epsilon\phi\sin(\phi) + O(\epsilon^2)\end{aligned} \tag{41}$$

This presents the final solution to equation (33) as

$$u(\phi) = C\cos(\phi - 3\epsilon\phi) + \frac{m}{h^2} + 3\epsilon\left(\frac{m}{h^2} + \frac{A^2h^2}{2m} - \frac{A^2h^2}{6m}\cos(2\phi)\right) \tag{42}$$

with $C = A + B\epsilon$. The only part of this solution that is not periodic in 2π is the first cos term. As the precession is the result of the orbit not being periodic in 2π, the precession must result from this term, and so can be calculated from it. Take the precession as $\delta\phi$. This then gives

$$u(0) = u(2\pi + \delta\phi) \tag{43}$$

The perihelia first occurs at ϕ=0, as was defined earlier. This means that the second perihelia will occur when the cos term generating the precession has gone through a full 2π.

$$2\pi = (2\pi + \delta\phi)(1 - 3\epsilon) \tag{44}$$

Solving equation (44) presents the value

$$\delta\phi = 6\pi\epsilon$$

Substituting the value of ϵ found earlier results in an estimated precession of

$$\delta\phi \approx 4.8254 \times 10^{-7} \tag{45}$$

7 Computational Simulation

To get a more accurate precession value, the solution to the differential equations from the E-L equation was numerically calculated. This was done using a fourth order Runge-Kutta method.

7.1 Visual Results

The initial conditions for Mercury's orbit were; $m = 1477.015$, $r = 46 \times 10^9$, $\theta = \pi/2$, and $\dot{\phi} = 2.478 \times 10^{-15}$, with all values in *geometrised units*. The geometrised unit system sets the speed of light and the gravitational constant to 1. Running the simulation for approximately 400 orbits, and plotting the result provides Figure 2. From this image alone, it is not possible to tell if the precession is accurate. Although 400 orbits have been plotted, there is no immediately visible precession. This image is useful as a confirmation that the precession is not extreme enough to easily be seen, however, to get a more accurate result it is important to use a numerical method better able to calculate small precessions.

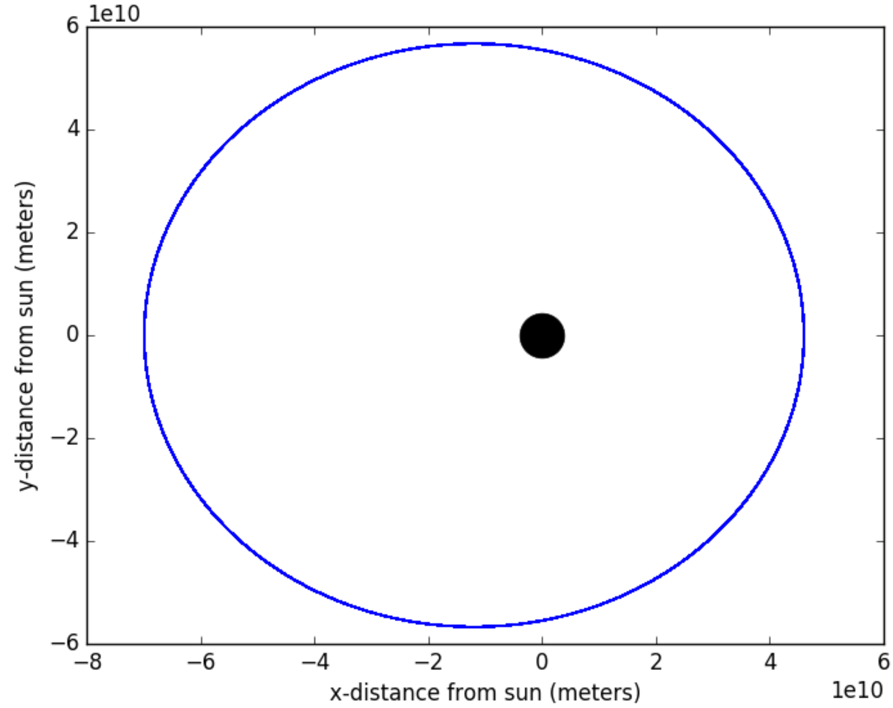

Figure 2: Plot of Mercury's Path Over 400 Orbits, using a numerical solution to the geodesic equation.

7.2 Numerical Results

As the precession of Mercury's orbit caused by the Schwarzschild metric is so small, it is important to calculate a numeric value of the precession, based on the simulation. Due to the fact that Geodesics in a Schwarzschild spacetime are precessing ellipses, a simple method of calculating this precession is to find the values for a precessing ellipse that most closely match the simulated version, and calculate the precession from there. The formula for a precessing ellipse is

$$r = \frac{A}{1 + B\cos(C\theta)} \tag{46}$$

with $C \neq 1$. In this case, as the precession is small, and $C \approx 1$, we can follow the method outlined in Brewin (1993), and approximate the formula for a precessing ellipse as

$$\frac{1}{r} = A' + B'\cos(\theta) + C'\theta\sin(\theta) \tag{47}$$

This is a reasonable approximation, as the precession is already known to be extremely small. Equation (47) is linear in A', B', and C'. This means that a least-squares solution can be found to the set of linear equations by using the simulated values of r and θ. This method provides values of

$$A' = 1.8 \times 10^{-11} \qquad B' = 3.7 \times 10^{-12} \qquad C' = 3.0 \times 10^{-19}$$

Calculating the precession from this gives a value of 5.0183×10^{-7}rad/orbit

8 Discussion

The precession of Mercury's orbit due to General Relativistic effects was most recently calculated by Park et al. (2017) as $42.9799''$/century. This converts to 5.0186×10^{-7} rad/orbit.

The first order approximation to the analytical solution had an inaccuracy of $< 5\%$. This method was valuable in providing a ballpark figure to confirm that the simulation was working as expected, and that looking at a Schwarzschild spacetime was the correct direction to investigate, but was never going to provide enough accuracy in the predicted value itself. This is due in part to the use of a first order approximation, but as the value for ϵ was of order 10^{-8}, this lack of accuracy more likely comes from the approximation of Mercury's orbit as circular when initially calculating ϵ. Mercury's orbit has an eccentricity of 0.21, so this approximation would have decreased the accuracy.

When comparing this value to the simulated value of 5.0183×10^{-7}rad/orbit, the simulated value is shown to have an inaccuracy of $< 0.01\%$. This level of accuracy suggests that the vast majority of Mercury's missing precession is due to the relativistic effects of the Schwarzschild metric. Although the precession predicted does not exactly match the observed value, it is important to remember that these values were obtained through the use of many approximations. The use of a simulation to calculate the orbit is an approximation in itself, and in this case it is an approximation that could have significant effect on the final values. This is due to the natural instability of orbital mechanics. A slight error in the distance between the planet and the host star would compound over time, and as over 900,000 timesteps were taken, it is not unrealistic that an error of 0.01% could have appeared from the simulation itself.

These results can be compared to a similar experiment conducted by Brewin (1993), in which a precession value with an error less than 0.2% was obtained. The difference in accuracy is likely due to rounding errors; this paper has kept 5 significant figures, while the other only kept 3. There is also likely some difference in computational error as different units of measurement were used.

Through these experiments, it has been shown that that using a Schwarzschild spacetime to calculate Mercury's orbit can accurately account for the precession discrepancy in the Newtonian approximation.

References

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A Kerr Metric

A.1 Background

Instead of looking at the Kerr metric specifically with regard to Mercury's orbit, the motion of a planet under the effects of a Kerr spacetime was explored in a much more qualitative manner. This is because the difference in precession of Mercury between a Schwarzschild and Kerr spacetime would have been negligible when compared to errors accumulated by the simulation, and so would not provide a more accurate value for the precession of Mercury.

A Kerr spacetime is similar to a Schwarzschild spacetime, with the singular difference that the mass creating all of the curvature is now allowed to spin. This introduces a new parameter, a, into the metric. The Kerr metric is

$$ds^2 = -\Big(1 - \frac{2m}{r^2 + a^2\cos^2(\theta)}\Big)dt^2 - \frac{4mra\sin^2(\theta)}{r^2 + a^2\cos^2(\theta)}dtd\phi + \frac{r^2 + a^2\cos^2(\theta)}{r^2 - 2mr + a^2}dr^2 \qquad (48)$$
$$+ (r^2 + a^2\cos^2(\theta)d\theta^2 + \Big(r^2 + a^2 + \frac{2mra^2\sin^2(\theta)}{r^2 + a^2\cos^2(\theta)}\Big)\sin^2(\theta)d\phi^2$$

The sign of the parameter a represents the direction the mass is spinning in. To test the effect that this spin would have on an orbit, three simulations were run, keeping all parameters except the spin constant. This ensured that any variation between the orbits would be solely due to the spin of the mass.

A.2 Results

The spin of the mass did indeed affect the precession of the orbiting body. When the mass was spinning in the same direction as the precession, as in Figure 3, the precession of the orbit increased, as compared to the case with no spin in Figure 4. Similarly, when the mass spun in the opposite direction to the precession, as in Figure 5, the precession decreased.

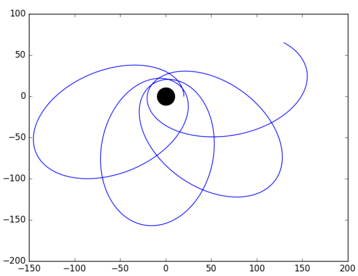

Figure 3: Kerr with $a < 0$

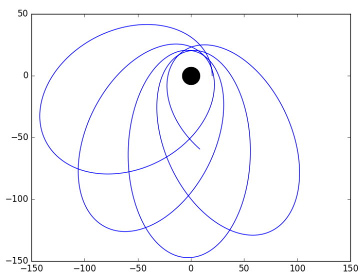

Figure 4: Kerr with a=0

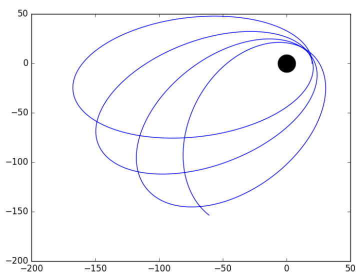

Figure 5: Kerr with $a > 0$

This can be thought of as the mass 'pulling' the orbital body along with it as it spins. For example, a mass spinning clockwise would 'pull' the orbiting planet clockwise to follow it. If the planet's precession was already clockwise, this would increase precession. However, if the precession was counterclockwise, this would decrease the precession of the body.