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EXPLORE THE MATHEMATICAL SCIENCES THIS SUMMER

Fourier Analysis on Graphs

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1 Abstract

In this report we look at the fundamental concepts of Fourier analysis, which is conventionally used to investigate functions on the real line, and see how an analogous form of these concepts can be used on functions defined on graphs. We will start by revisiting important ideas about classical Fourier analysis of functions defined on the real line, as well as basic concepts of graphs. Using the concepts of Fourier analysis we explored a way of representing ideas of Fourier analysis on graphs.

2 Introduction

Fourier analysis is conventionally used to study functions that are defined upon the real line, and fundamentally looks at studying these functions by decomposing them into basis elements and understanding the functions in terms of these basis elements. Some of the concepts of Fourier analysis include Fourier series, in which we look at how some function can be expressed as a linear combination of the basis elements which are of the same space as the function, Fourier transform and the Shannon sampling theorem. The Shannon sampling theorem in particular gives a way of recovering certain types of functions given that we only know the value of the function at certain uniform points. Functions can be described on graphs as a mapping from the vertex set to the real or complex field, and having tools working as closely as possible to the ones in classical Fourier Analysis can help us investigate and further deepen our knowledge of these functions on graphs.

Statement of Authorship

I studied various concepts of classical Fourier analysis and then explored how these concepts could be defined on graphs under the direction of my academic supervisor. All definitions and theorems about Fourier analysis and sampling on graphs are respectfully sourced from the authors referenced. To interpret my findings I then implemented these concepts through specifically created examples.

3 Classical Fourier Analysis

We begin by defining what a Cauchy sequence is, then a Hilbert space.

Definition 3.1 (Cauchy Sequence) : Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in a normed linear space. Then $\{x_k\}_{k=1}^{\infty}$ is called a Cauchy sequence if, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n > N \Rightarrow ||x_m - x_n\ || < \varepsilon.$$

Now that we have established what a Cauchy sequence is, we can now move onto defining a Hilbert space.

Definition 3.2 (Hilbert Space) :

A Hilbert space is a complete inner product space, and by complete we mean that every Cauchy sequence is a convergent sequence.

In general, Classical Fourier Analysis looks at studying vectors that are an element of some Hilbert space in terms of a set of orthonormal basis elements of that Hilbert space. This way of expressing functions as a sum of the orthonormal basis elements in a Hilbert space is called a Fourier series.

Definition 3.3 (Orthonormal basis) :

Let $\{e_k\}_{k=1}^{\infty} \in \mathcal{H}$ be a set of elements from a Hilbert space $\mathcal{H}$. We say that $\{e_k\}$ is an orthonormal basis if, every element of $\mathcal{H}$ can be represented as a linear combination of the elements in $\{e_k\}$ and if $< e_i, e_j >= 1$ if $i = j$ and $< e_i, e_j >= 0$ if $i \neq j$, where $e_i, e_j \in \{e_k\}$.

Definition 3.4 (Fourier Series) :

Let $B = \{e_k\}_{k=1}^{\infty} \in \mathcal{H}$, be an orthonormal basis for Hilbert space $\mathcal{H}$. Then if $f \in \mathcal{H}$,

$$f = \sum_{k=1}^{\infty} < f, e_k > e_k\ .$$

We will now define a Lebesgue space and then use that to define a Fourier transform.

Definition 3.5 (Lebesgue spaces) : For each $1 \leq p < \infty$, let

$L^p(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C}; f$ is measurable and $||f||_p = (\int |f|^p)^{1/p} < \infty\}$.

Now that we have the definition of a Lebesgue space we will define what a Fourier transform is, in particular the Fourier transform in $L^1(\mathbb{R})$

Definition 3.6 (Fourier transform in $L^1(\mathbb{R})$) : Let $f \in L^1(\mathbb{R})$. Then the Fourier transform $\hat{f}$ of f is given by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-2\pi itx}dt.$$

Definition 3.7 (Schwarz Space) : The Schwarz space $S(\mathbb{R})$ is defined as,

$$S(\mathbb{R}) = \{\ f \in C^{\infty}(\mathbb{R}); \sup_{t\in\mathbb{R}} |t^p D^q f(t)| < \infty \text{ for all } p, q \in \mathbb{Z}^+\}.$$

Definition 3.8 (Inverse Fourier transform of functions in $S(\mathbb{R})$) : Suppose $f \in S(\mathbb{R})$ then $\hat{f}(x) \in S(\mathbb{R})$ and

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(x)e^{2\pi itx}dx$$

for all $t \in \mathbb{R}$.

Definition 3.9 (Fourier transform in $L^2(\mathbb{R})$) : The Fourier transform $\mathcal{F}$ is defined on $L^2(\mathbb{R})$ by,

$$\mathcal{F}f = \hat{f} = \lim_{k\to\infty} \hat{f}_k$$

where $\{f_k\}_{k=1}^{\infty} \subset S(\mathbb{R})$ and $||f - f_k||_2 \to 0$ as $k \to \infty$. With this definition, $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$.

The Fourier series and transform are fundamental ideas of Fourier analysis and aid in the study of functions. One of the many results that arose from Fourier analysis is the Shannon sampling theorem, which in general says that if we know the value of a band limited function at a uniform set of points on the real line, we can then determine the function over all of the real line. We begin by defining band limited functions and then after the Shannon sampling theorem.

Definition 3.10 (Band limited functions) : Given $\Omega > 0$, let

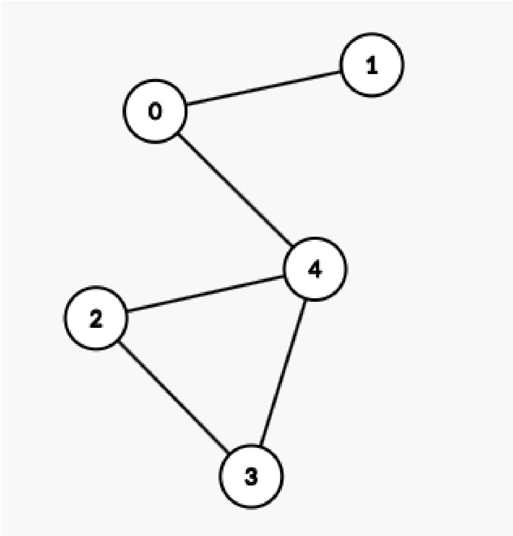
$$B_{\Omega} = \left\{ f \in L^2(\mathbb{R}); \hat{f}(x) = 0 \ \text{ for } \ |x| > \tfrac{\Omega}{2} \right\}.$$

If $f \in B_{\Omega}$ then f is called a band limited function with bandwidth Ω.

Definition 3.11 (Shanon Sampling Theorem) : if $f \in B_{\Omega}$

$$f(x) = \sum_{k=-\infty}^{\infty} f(\tfrac{k}{\Omega})\text{sinc}(\Omega x - k)$$

where $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 1: Example graph and it's Adjacency matrix

4 Graphs

A graph, $G(V, E)$, is a set of vertices $v \in V$, joined together by edges $e \in E$. The degree of a vertex is the number of incident edges to that vertex.

One of the ways we can represent a graph is by using an adjacency matrix, where the rows denote each vertex and the columns as well with the same order. If two vertices are adjacent the matrix entry will have 1 otherwise it will have 0. As an example we have the graph above in figure 1, with the vertices of this graph labeled from 0 to 4. Let's look at the row for one of the vertices, let's say the vertex labelled 0, which corresponds to the the first row of the matrix. Since this vertex is not adjacent to itself the value of the first entry of the row is 0 and since our vertex is adjacent to the vertices labelled 1 and 4 in the graph, we see that the value of the second and fifth entry is 1. Since our vertex is not adjacent to the others, we see that the remaining entries of the row have value 0. We continue in this manner down the rows of the matrix for each vertex of the graph.

Another way to represent a graph as a matrix is by a Laplacian matrix, which is denoted by L and is equal to $D - A$ where D is a diagonal matrix with the entries of the diagonal being the degree of the corresponding vertex, and A being the adjacency matrix.

5 Fourier Analysis on Graphs

We start by defining a function on a graph, $G(V, E)$, as a mapping from the vertex set V to the real numbers ($f : V \to \mathbb{R}$). Suppose the number of vertices of the graph is N and each vertex is labeled uniquely from 0 to N-1, then we can represent the function f on the graph as the following vector

$$f = [f(0), f(1), ..., f(N-1)]^T \in \mathbb{R}^N,$$

where $f(i)$, for $0 \leq i \leq N-1$, is the value of the function on the vertex labeled i. The adjacency matrix is one way of representing a graph as a matrix, and since for undirected graphs this matrix is real and symmetric, we can see using the real spectral theorem that the eigenvectors of the adjacency matrix forms an orthonormal basis. So now we can represent functions on undirected graphs as a linear combination of these eigenvectors, which we can look at as a Fourier series of a function on a graph.

Similar to Moura's definition [1] we define the graph Fourier transform of $f \in \mathbb{R}$ by,

$$\hat{f} = P^{-1} f$$

where P is a matrix whose columns are formed by the eigenvectors of the adjacency matrix. Furthermore, the eigenvectors in P are ordered in increasing order of corresponding eigenvalues. Knowing this definition we see that the inverse graph Fourier transform is,

$$f = P\hat{f}$$

6 Sampling on Graphs

In sampling we wish to determine at least an approximation of the original function given only a sample of values of the function. This was the general idea with the Shannon sampling theorem, which was the sampling of band limited functions on the real line, with the sample values being the value of the band limited function at certain uniform points on the real line, and the recovery of the original function being exact. We now define a sampling theorem on graphs. We follow Chen [2] by defining two operators, the sampling operator and the interpolation operator.

The sampling operator Ψ is a linear mapping, $\mathbb{R}^N \to \mathbb{R}^M$, where $M < N$. The sampling operator is a M×N matrix where M is the number of entries needed to be sampled. Each row of the sampling operator matrix has a 1 in the column if that entry is needed to be sampled and 0 for the rest. For example, suppose we wanted to take the first and third entry of a column vector f, then Ψ would be a 2×N matrix, where in the first row it would have a 1 in the first column and 0 in the rest and in the second row it would have a 1 in the third column and 0 in the rest. Multiplying the sampling operator, Ψ with f gives us a sample vector Ψf.

The interpolation operator Φ is a linear mapping, $\mathbb{R}^M \to \mathbb{R}^N$. This operator will interpolate the sample values of the functions that we have and recover at least an approximation of the original function over all vertices of the graph. The operator is a N×M matrix and is multiplied to the sample vector, Ψf.

On the real line, the Shannon sampling theorem applies to functions that are band limited. If a function on the graph is band limited we can use the sampling theorem for graphs to give us a perfect recovery. We again follow Chen [2] by first defining band limited functions on a graph and then stating the graph sampling theorem.

Definition 6.1 (Band limited functions on graphs) : A function $f : V \to \mathbb{R}$ is said to be $K-$*bandlimited* if for some $K \in \{0, 1, ..., N-1\}$,

$$\hat{f}(k) = 0 \text{ for all } k \geq K.$$

If f is a $K-$*bandlimited* function, then we say that K is the bandwidth of f. Furthermore, the set of all functions in $\mathbb{R}^N$ with bandwidth at most K is denoted by BL_K

Now that we have established band limited functions on graphs, we can proceed to the graph sampling theorem. Let $P_{(K)}$ denote the matrix formed by the first K columns of P.

Theorem 6.1 (Graph Sampling Theorem) : For all $f \in BL_K$, we can perfectly recover $f = \Phi\Psi f$ by choosing

$$\Phi = P_{(K)}U,$$

where U is such that $U\Psi P_{(K)}$ is a $N \times N$ identity matrix.

We note that the sample size needs to be at least the bandwidth of the function on the graph.

Example : Consider the labeled undirected graph with its respective adjacency matrix in figure 1 . Note that the values displayed for the entries in the matrices are an approximation. We begin first by finding the eigenvectors of the adjacency matrix to help construct matrix P, which for this graph is

$$P = \begin{pmatrix} -0.5930 & 0 & -0.3620 & -0.6325 & -0.3425 \\ 0.3540 & 0 & 0.6714 & -0.6325 & -0.1547 \\ -0.2390 & 0.7071 & 0.3094 & 0.3162 & -0.4972 \\ -0.2390 & -0.7071 & 0.3094 & 0.3162 & -0.4972 \\ 0.6394 & 0 & -0.4762 & 0 & -0.6037 \end{pmatrix}.$$

Now let's suppose we have the following function on the graph,

$$f = [-0.3327, 0.24414, 0.05286, -0.22998, 0.27208]^T.$$

We first would like to know whether or not this function is band limited and by finding the graph Fourier transform we see that,

$$P^{-1}f = \hat{f} = [0.5, 0.2, 0.1, 0, 0]^T$$

and so since $\hat{f}(3)$ and $\hat{f}(4)$ equal 0, we now know that our function f is band limited with bandwidth 3. Now suppose our sample was of the first, second and fourth entry of f, then our sampling operator Ψ would be a $3{\times}5$ matrix, where the first row would have a 1 in the first column and 0 the rest, the second row would have a 1 in the second column and 0 the rest, and finally the third row will have a 1 in the fourth column and 0 in the rest. As we can see, after multiplying the sampling operator with f we get our sample vector.

$$\Psi f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -0.3327 \\ 0.24414 \\ 0.05286 \\ -0.22998 \\ 0.27208 \end{pmatrix} = \begin{pmatrix} -0.3327 \\ 0.24414 \\ -0.22998 \end{pmatrix}.$$

By the graph sampling theorem, we can get a perfect recovery of f from our sample if we choose the interpolation operator to be $\Phi = P_{(3)}U$, and since $U = (\Psi P_{(3)})^{-1}$, we can express the interpolation operator as follows,

$$\Phi = P_{(3)}(\Psi P_{(3)})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \\ -2.21 & -1.9 & 0 \end{pmatrix}.$$

So we should see now that if we use the interpolation operator upon the sample, we should get a perfect recovery.

$$\Phi\Psi f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \\ -2.21 & -1.9 & 0 \end{pmatrix} \begin{pmatrix} -0.3327 \\ 0.24414 \\ -0.22998 \end{pmatrix} = \begin{pmatrix} -0.3327 \\ 0.24414 \\ 0.05286 \\ -0.22998 \\ 0.27208 \end{pmatrix}.$$

As we can see, we have achieved a perfect recovery from our sample.

7 Conclusion

In this project we successfully explored the concepts of Fourier analysis on discrete graphs. After revising the ideas of Fourier series and Fourier transformations as well as sampling of functions on the real line, and further also revising elementary concepts of graph theory, we studied an analogous way to describe Fourier series on graphs, which was by representing functions on graphs as a linear combination of the adjacency matrix. Furthermore by defining a matrix with its columns being the eigenvectors of the adjacency matrix, we could define a way to represent the Fourier and inverse Fourier transform on a graph. Finally, by using these ideas we could then proceed to understand a way of expressing sampling with functions on graphs.

References

[1] Aliaksei Sandryhaila and José M.F Moura (2013). Discrete Signal Processing on Graphs, *IEEE Transactions on Signal Processing*, 61:1644-1656.

[2] Siheng Chen, Rohan Varma, Aliaksei Sandryhaila and Jelena Kovačević (2015). Discrete Signal Processing on Graphs: Sampling Theory, *IEEE Transactions on Signal Processing*, 63:6510-6523.