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**Nonlinear unit hydrograph models of
rainfall-streamflow events for water
quality analysis**

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Abstract

The nonlinear unit hydrograph is investigated as a possible improvement to the standard linear unit hydrograph for water quality analysis. A literature review is conducted, presenting evidence in the literature of: nonlinearity in catchment response in various conditions, evaluation of the nonlinearity of catchment response, advantages and disadvantages of nonlinear models, and water quality analysis models. The continuous- and discrete-time models of the unit hydrograph are presented and algebraic constraints on the models are given. Results from previous work solving the linear cascade model by numerical solution methods are provided and explained. Numerical methods are used to solve the single-store nonlinear unit hydrograph and the results are compared to the analytic solution. Final conclusions about the efficacy of the nonlinear model for water quality analysis will be dependent on results from the nonlinear cascade model.

Literature Review

Studies of the nonlinearity of streamflow response to rainfall have existed since the 1950s (Ding 2011). Researchers have recognised nonlinear responses as improvements to the linear hydrograph model under certain conditions.

Saghafian (2006) argues that theoretically “only peak discharge of an equilibrium hydrograph follows linear proportionality to excess rainfall intensity”, and that the available data suggests a general nonlinear response to excess rainfall. Some watersheds exhibit multiple linear unit hydrographs (UHs), dependent on effective rainfall (Minshall 1960). Amorocho (1967) proposed a quasi-nonlinear UH to remedy this issue. For storage-discharge relationships that differ with flow, used for example in groundwater modelling (the relationship is linear for confined aquifers and nonlinear for unconfined aquifers), Chapman (1999) argues that nonlinear models provide better representations, however in general linear models are effective for fitting the observed flow in most cases. Linear models can require multiple components, each with several parameters. Mathias, McIntyre, and Oughton (2016) suggest nonlinear models are an effective means of simplification, as they require less parameters. Reviews of programming methods undertaken by Mays and Taur (1982) suggest nonlinear programming (NLP) is less restrictive than linear programming (LP), which requires effective rainfall to be defined in advance.

Various methods are proposed in the literature for evaluating nonlinearity in catchment response. Ding (2011) presents a quasi-nonlinear approach to hydrograph models in the form of an input-dependent (or variable) unit hydrograph. This combines nonlinear overland flow, channel routing, and catchment runoff processes, and is characterised by a nonlinear storage-discharge equation. Saghafian (2006) proposes a method to convert UHs into general UHs via a nonlinear transformation (a rainfall ratio raised to an index less than unity). The NLP method suggested by Mays and Taur (1982) determines the rainfall losses for each event and the UH that minimises deviations between observed and derived runoff hydrographs. The model is solved using two NLP techniques,



LSGRG (large-scale generalised reduced gradient) and GRG2 (generalised reduced gradient, second version). GRG2 is appropriate for solving small to medium, dense, highly nonlinear problems, whereas LSGRG is optimal for solving large, sparse, nonlinearly constrained problems. Chapman (1999) outlines three main approaches to storm event modelling. The model used by engineering hydrologists considers the difference between base-flow and total streamflow to be relative to the causative rainfall. In systems analysis, a nonlinear loss function is applied to the rainfall data and catchment response is usually represented by a two-component linear model separating quick-flow and slow-flow. Scientific hydrology uses tracers to separate new and old flow. New flow is identified with the incoming rainfall; old flow is identified as the result of subsurface pathways.

Many advantages of nonlinear modelling are recorded in the literature. An advantage of the nonlinear instantaneous UH studied by Ding (2011) is its flexibility to fit data, even when limited to a single parameter. In addition, reversing the convolution integral used to solve the instantaneous UH in this paper allows simultaneous calibration and optimisation of parameters in more complicated models. Using the direct Bakhmeteff method to derive the instantaneous UH reproduces the ordinate and timing of its peak exactly. The transformation presented by Saghafian (2006) removes both the linearity and time-invariance assumptions of linear UHs. The resulting UH is therefore more relevant to physical data than traditional linear hydrographs. Mays and Taur (1982) observe the shortcomings of linear modelling: the initial linearity assumption results in error that neither LPs nor NLPs can rectify fully. Computational solutions are expensive and time-consuming; a nonlinear model reduces error in the preliminary stages of the modelling process.

However, nonlinear models do have shortcomings. The variable instantaneous UH model proposed by Ding (2011) is “very sensitive to change in the storm duration or computational timestep”. It is suggested that a single timestep of the full storm duration is used to best approximate the height of peak. Reducing the timestep size by more than a factor of 2 does not improve the approximation. Another limitation of the nonlinear model is its stability: changes in the inflow rate impact the model, and this impact is amplified by greater nonlinearity. Studies by Singh and Scarlatos (1978) of four catchments showed the nonlinear model to be less accurate than the linear in comparison to empirical data. This inaccuracy was caused by numerical issues. Specifically, it reveals the stability problem associated with nonlinear models: “the impact of the inflow rate is amplified by the degree of system nonlinearity” (Ding 2011, p. 407). Models of smaller catchments exhibit a time-lag between short, intense storms and the resulting hydrograph peak, a feature difficult to capture with a single-store nonlinear model. Whereas linear models preserve the mass-balance equations for the entire hydrograph, the transformations proposed by Saghafian (2006) do not preserve the mass-balance equations in the rising limb. It is suggested that this error can be resolved by proportional compensation in the remaining hydrograph. The method does preserve peak discharge, rate of rise, and time to peak (these are characteristics of interest to hydrograph modelling). Although the NLPs presented by Mays and Taur (1982) were significant improvements upon the LPs, they are disadvantaged when considering multiple



rainfall events simultaneously. Excess computation time is required, due to their complexity. A disadvantage to the nonlinear model studied by Mathias, McIntyre, and Oughton (2016) was its inability to match peak flows after parameter calibration by either method of sensitivity analysis.

The literature also presents models that overcome the shortcomings of linear UH without using nonlinearity techniques. These models are largely computational, one must be cautious of the computational cost incurred. An alternative method for improving the accuracy of linear cascade models is a generalisation of the discrete linear cascade model (DLCM) which is applied to a non-integer number of storages (Szilagyi 2006). To analyse flow peaks, Gottschalk and Weingartner (1998) suggest a probability distribution function which is derived by a combination of techniques: frequency analysis of rainfall volumes, runoff coefficients, and UHs. These alternative models are appropriate to the context of the situations to which they are applied. In these contexts, the models improve the accuracy of simulations and estimations.

There are advantages to both alternative approaches to streamflow modelling. Flow-routing techniques are still commonly used to approximate models of streamflow (Szilagyi 2006). These techniques are useful because they operate without sufficient and accurate data concerning catchment characteristics such as geometry. The DLCM is an effective flow-routing technique as it is a discrete form of the kinematic wave equation. The results from multiple simulations showed that allowing the number of stores to be a non-integer value improved the accuracy of the simulations. Furthermore, when sudden changes in catchment characteristics (such as slope or roughness) occur, allowing the final storage to have an independently determined coefficient allows the final store to respond at a different rate than the other stores of the cascade. In some cases (such as discharge flood hydrographs), this was found to significantly improve simulation accuracy. Gottschalk and Weingartner (1998) argue that a method using probability distribution functions is an advantageous approach to streamflow frequency estimation. This method allows processes generating streamflow to be considered. Streamflow variables can be related to precipitation data, antecedent moisture conditions, and basin response.

A major disadvantage of the UH for water quality analysis is inherent in its function: it is primarily concerned with characteristics of streamflow response which are not relevant to water quality analysis. Instead, the literature offers alternatives to the UH for analysing water quality. According to Barnes and Bonell (1996), rainfall-runoff response models are predominantly concerned with determining the magnitude and timing of water flux. The resulting model contrasts with the residence time distribution describing catchment response to solutes and tracers (this is a nonlinear problem). Determining the behaviour of old and new water in rainfall events is important for modelling water quality (when considering solutes), as these events disrupt the original balance of solutes and tracers. Brown et al. (1999) argues that flow-paths which dominate storm water events determine the resulting surface water chemistry during and after the event. Studies cited in this paper (Hooper & Shoemaker 1986; Wels et al. 1990; Waddington et al.1993; Pearce et al. 1986; McDonnell 1990) show that pre-event water comprises



70%-96% of the hydrograph peak in storm events. Even in dry periods, mobilised soil water and event water both contribute to streamflow. To minimise ambiguity in the model, it is necessary to use a combination of hydrometric data (defines the timing of the hydrological response) and isotopic and hydro-chemical data (determines the relative contributions of pre-event water from different locations to the resulting flow).

These models offer promising alternatives to the UH for analysing water quality. Results from the study by Barnes and Bonell (1996) suggest that their modification of the instantaneous unit hydrograph, which allows for approximate modelling of tracer data, gives a realistic simulation of concentration variations of deuterium (a solute). In a study of in-stream water quality models, Cox (2003) presents the case that computational water quality models using a lumped-parameter system of ODEs worked most effectively for modelling water quality. These models had the advantage of simplified computation, an availability of algorithms for model identification and parameter estimation, and an availability of control system methods, beneficial for management purposes.



Water Quality Analysis

Representations of flow peaks, especially the highest peaks, are important for understanding temporal variations in water quality. The unit hydrograph (UH) is a representation of catchment response to an impulse of effective rainfall. The linear unit hydrograph is typically an arrangement of linear stores. A linear store is represented by a first-order linear ordinary differential equation (ODE), and n such stores are represented by an n th-order linear ODE (see Figure 1).

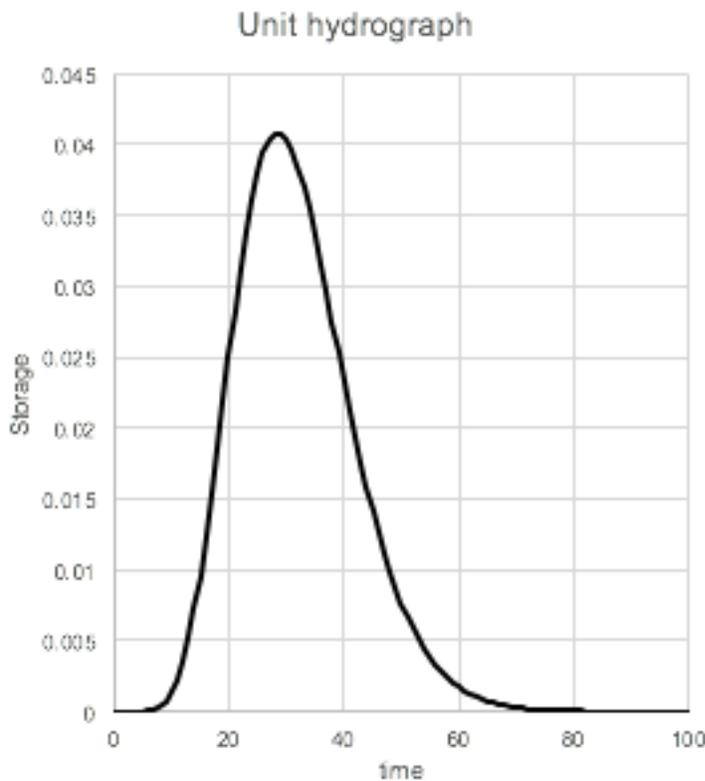


Figure 1: *The continuous-time solution to a cascade of 10 linear stores, time-constant 3.2 (solved using the method proposed by Szollosi-Nagy (1982)).*

The linear UH has difficulty in reproducing flow peaks for large events. Such events are typically sharper and exhibit a faster decline from the peak height. However, the shape of the linear form of the unit hydrograph is independent of event size. Large events dominate the transport of loads and constituents (excepting salinity). Inaccuracies in depicting large events cause prediction errors when using UHs for water quality analysis. This concern is particularly relevant under changing conditions, such as future climate scenarios. Relationships between



flow rates and the concentration of loads and constituents have been derived from empirical data; this is how the UH is used for water quality analysis and why it is important to reproduce flow rates accurately.

Streamflow models comprise a mass balance equation which estimates the effective rainfall (for example, the IHACRES CMD module) and a unit hydrograph which converts effective rainfall into streamflow. In the adopted model, the streamflow is separated into two categories, quick-flow and slow-flow. A multicomponent model is used to estimate both the quick- and slow-flow components. The quick-flow can be modelled by a linear or nonlinear equation. The discrete-time linear form is:

$$q_k = aS_k \quad (1)$$

Here q represents the mean flow over the timestep t_k to t_{k+1} , a is a constant of proportionality, and S is storage at time t_k .

The nonlinear form adopted here is:

$$q_k = \frac{S_k^b}{c + S_k^b} S_k \quad (2)$$

The nonlinear conceptual form (2) is more complex than the linear form (1) as the coefficient a from the linear form is now dependent on S . b and c are parameters to be determined from data.

The slow-flow is modelled by another nonlinear storage-discharge relationship:

$$q_k = ((1 - \alpha) \left(\frac{S_k}{\beta + S_k} \right)^2 + a\alpha) S_k \quad (3)$$

Where α , β , and a are parameters to be determined from data.

Continuous-time Formulation

The linear form of the UH in continuous-time is modelled by the ODE:

$$\frac{dS}{dt} = i - q \quad (4)$$

where $q(t) = aS(t)$ and i represents input, the initial condition.

The continuous-time homogeneous solution is:

$$S(t) = S_0 e^{-at} \quad (5)$$

This is found using the separation of variables method.

Discrete-time conversion is achieved by setting $S_k = S(t_k)$ and integrating $q(t)$ over the timestep (from t_k to t_{k+1}).

Hence the discrete-time flow for the linear UH is given by:

$$q_k = S_0 e^{-at_k} (1 - e^{-a}) \quad (6)$$



This takes the form of an exponentially decaying store, however it doesn't capture the variation in event peaks.

The nonlinear form of the UH in continuous-time is:

$$\frac{dS}{dt} = i - q \quad (7)$$

where $q = aS(t)^b$. This has the continuous-time homogeneous solution:

$$S(t) = S_0(\alpha t + 1)^\beta \quad (8)$$

Discrete-time conversion yields:

$$q_k = \frac{\alpha}{\beta} S_0^b ((\alpha t_{k+1} + 1)^\beta - (\alpha t_k + 1)^\beta) \quad (9)$$

where $\alpha = a(b - 1)S_0^{b-1}$ and $\beta = (1 - b)^{-1}$.

Algebraic Constraints

The algebraic constraints on the model are derived from considering the physical situation of the water catchment. These constraints provide guidelines for choosing and verifying a model function.

Firstly, consider the continuous-time case. The linear store model has the form:

$$q(t) = aS(t)$$

for $a > 0$. If $a = 0$, the flow is zero for any value of storage. Physically, this is equivalent to a dam. If $a < 0$, this represents the non-physical case when water in the storage flows inward and $S(t)$ tends to infinity.

Converting this to discrete-time,

$$q_k = a'S_k$$

for $0 < a' < 1$. $a' > 0$ holds for similar reasoning to continuous-time. $a' < 1$ is due to the conversion to discrete-time. If $a' = 1$, this is equivalent to infinite a in continuous-time (analogous to the catchment emptying instantaneously).

Secondly, flow increases as the volume in the store increases. Hence the rate of change of flow with respect to the store must be positive. This gives the second algebraic constraint:

$$\frac{dq_k}{dS_k} > 0$$

or, a cannot be negative.

The final constraint concerns order preservation. This is derived from considering two stores with an identical storage-flow relationship. If one store begins with a higher storage volume, it is sensible to assume that this store will remain higher than the second store as they decrease over time. It is not sensible for the function to drop off



so rapidly that the order of the two stores changes at some later time if the first store is given a greater initial impulse.

Considering this mathematically, suppose we have two stores with identical flow rates but beginning with different storage volumes. Without loss of generality, let 1S_k and 2S_k be identical stores such that

$${}^1S_k > {}^2S_k$$

This should be true for all k , hence

$${}^1S_{k+\epsilon} > {}^2S_{k+\epsilon}$$

This is true for all ϵ . It suffices to test the case when $\epsilon = 1$.

$${}^1S_{k+1} > {}^2S_{k+1}$$

$${}^1S_k - {}^1q_k > {}^2S_k - {}^2q_k$$

$${}^1S_k - {}^2S_k > {}^1q_k - {}^2q_k$$

$$\Delta S > \Delta q$$

$$\frac{\Delta S}{\Delta q} > 1$$

Taking the limit as Δ approaches 0,

$$\frac{dS_k}{dq_k} > 1 \Rightarrow \frac{dq_k}{dS_k} < 1$$

The quick- and slow-flow models aforementioned are consistent with the constraints provided $b < 1$.

Numerical Solutions

Euler's method was used to solve the unit hydrograph modelled by a continuous-time nonlinear ODE (7). The forward Euler formula is:

$$y_{k+1} = y_k + \Delta t f(t_k, y_k) \tag{10}$$

The backward Euler formula is:

$$y_{k+1} = y_k + \Delta t f(t_{k+1}, y_{k+1}) \tag{11}$$

The trapezoidal Euler formula is:

$$y_{k+1} = y_k + \frac{\Delta t}{2} (f(t_k, y_k) + f(t_{k+1}, y_{k+1})) \tag{12}$$

The forward Euler method is an explicit first-order method, the backward Euler is an implicit first-order method, and the trapezoidal Euler is an implicit second-order method.

Applying these formulae to (7) gives:

$$S_{k+1} = S_k - a\Delta t S_k^b \tag{13}$$



$$S_{k+1} = S_k - a\Delta t S_{k+1}^b \quad (14)$$

$$S_{k+1} = S_k - \frac{a\Delta t}{2}(S_k^b + S_{k+1}^b) \quad (15)$$

Solving the nonlinear ODE using the separation of variables method gives the analytic solution. This is converted to discrete-time by setting $S_k = S(t_k)$. The discrete formula is:

$$S_k = [S_0^{1-b} + a(b-1)t_k]^{\frac{1}{1-b}} \quad (16)$$

Although the analytic solution exists, numerical solutions are necessary because it is not possible to solve a system of nonlinear ODEs analytically. A system of nonlinear ODEs will be necessary to model a cascade of nonlinear stores.

Linear Cascade

Here is some previous research analysing numerical solutions of the linear ODE cascade model. The image shown below (Figure 2) is a cascade of 10 linear stores solved analytically in continuous-time (Szollosy-Nagi 1982); numerically using the forward, backward, and trapezoidal Euler methods; and using the auto-regression (using the analytical solution numerically) method. The time constant is 3.2.



Numerical and Analytic Solutions to Linear Cascade UH

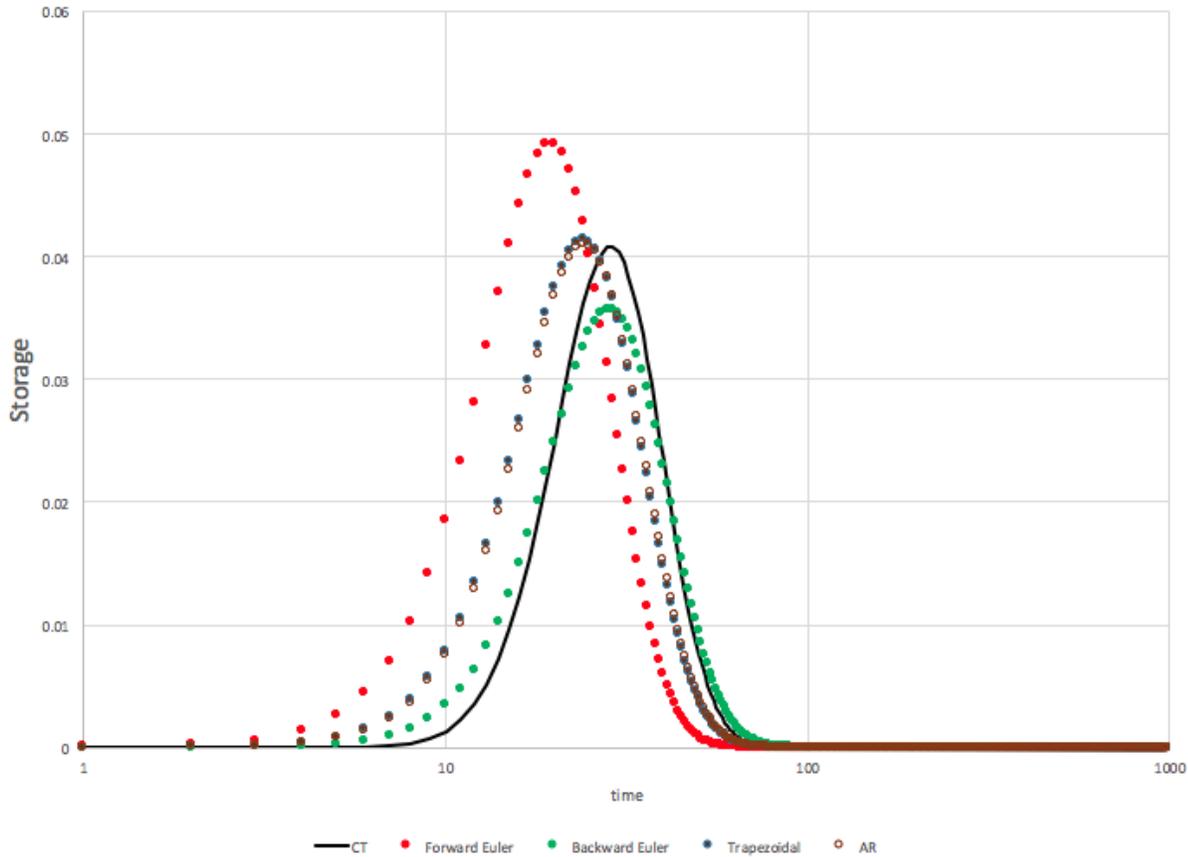


Figure 2: Numerical and analytic solutions to a linear cascade unit hydrograph of 10 stores. *N.B.:* CT is continuous-time and AR is the auto-regressive formulation of the analytic solution for a single store.

A cascade is an arrangement of stores in series. An initial impulse is given to the first store in the first timestep and the output of each store serves as an impulse to the next. At large time, underestimation by the forward solution and overestimation by the backward solution is a consistent result for a cascade of linear stores. Of the numerical solutions, the backward method predicts time of peak most accurately (with incorrect height), and the trapezoidal method predicts the shape most accurately (the timing is early). The trapezoidal method is the most accurate numerical method. The AR cascade is the best approach.

Results

Figure 3 shows results from solving a nonlinear ODE using the numerical solution methods (this is a single store).

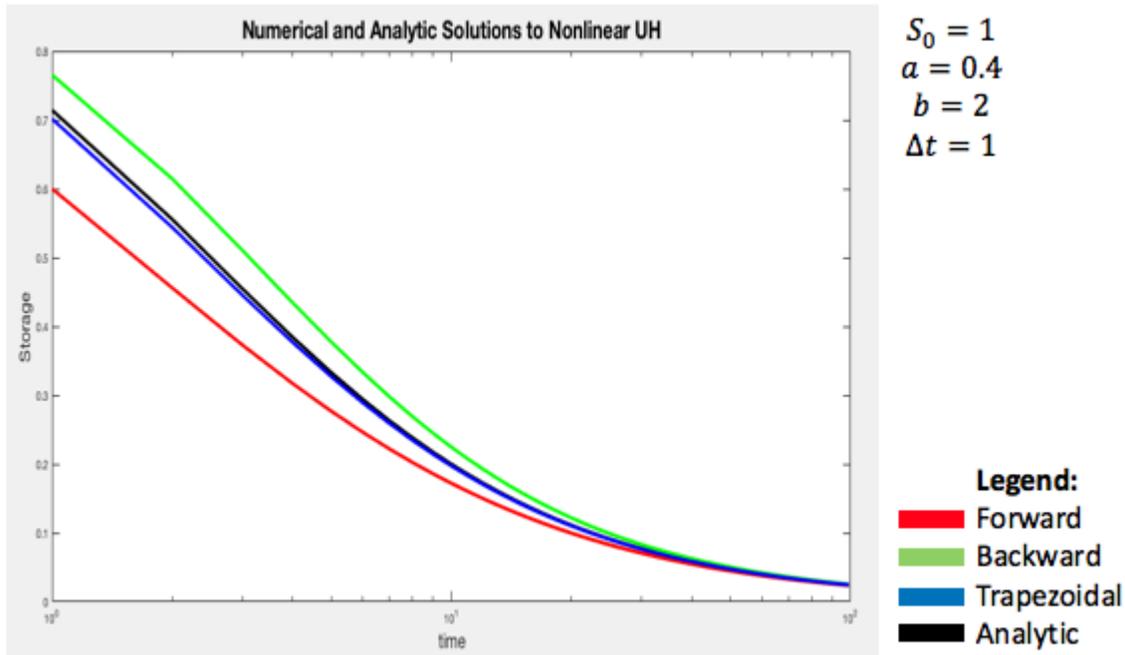


Figure 3: *Numerical and analytic solutions to a single store nonlinear unit hydrograph, x-axis in log-scale.*

As expected from analysis of numerical solutions to the linear ODE, the trapezoidal method is most consistent with the analytic solution; the forward method underestimates and the backward method overestimates. Viewing both axes in log-scale (Figure 4), the solutions converge to a power law relationship, a behaviour that is consistent with empirical analysis of catchment data.

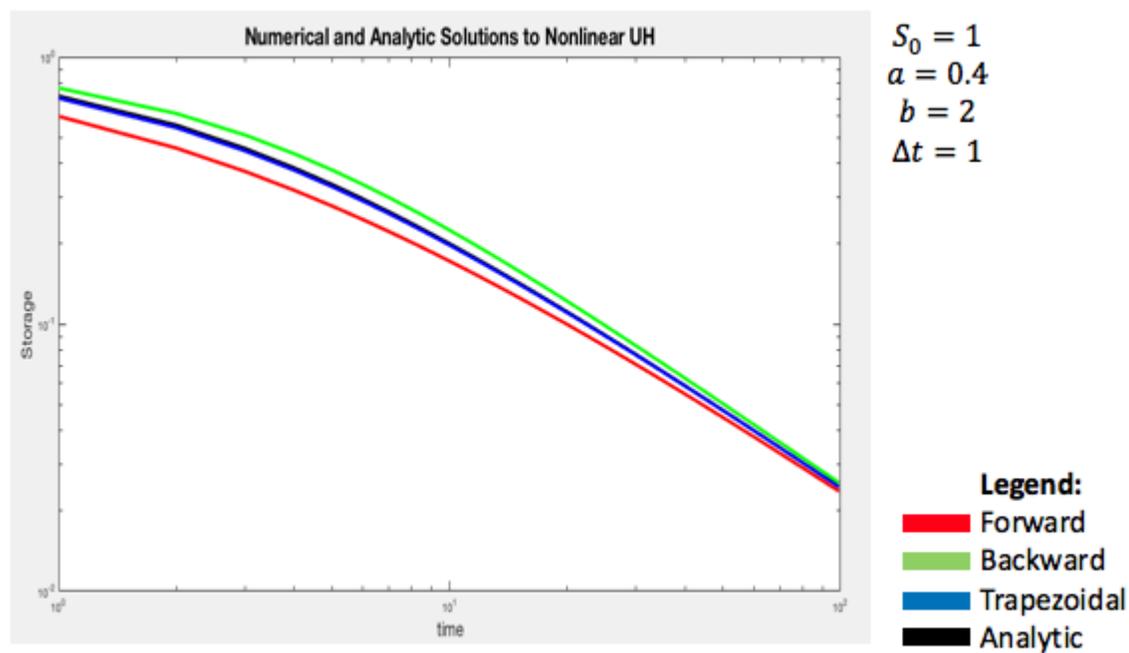


Figure 4: Numerical and analytic solutions to a single store nonlinear unit hydrograph, both axes in log-scale.

The results from the residual plot (Figure 5) are also as expected: the forward Euler has the most error (explicit, first-order method) and the trapezoidal method has the least (implicit and second-order method).

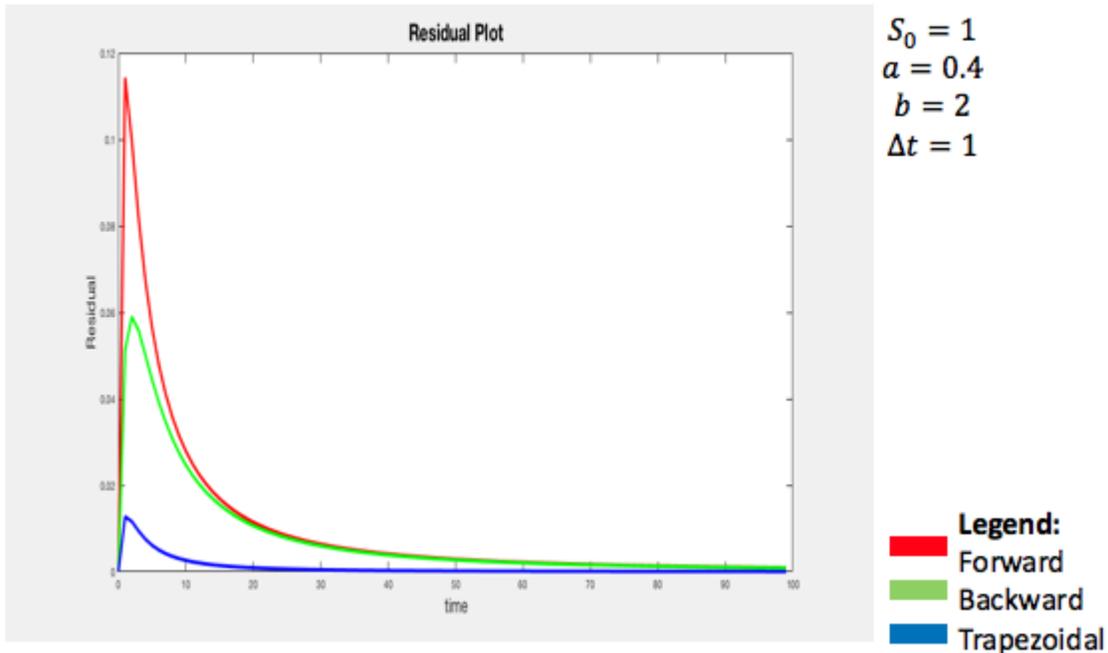


Figure 5: *Residuals of numerical solutions compared to the analytic solution.*

Future Work

Continuing this work, the numerical and analytic solutions will be applied to a cascade of nonlinear stores. A cascade is an arrangement of stores in series. An initial impulse at the beginning of the timestep initiates the first store, the output from this store will be the initial impulse for the next store. Analysis of numerical solutions to the linear UH show evidence that placing the input for each time step as an impulse at the start of the time step causes error. This is done for all stores, hence for each store the input is earlier than it should be, causing the peak to be early. In such cases, the backward solution offers the most accurate prediction of the time of the peak height and the trapezoidal solution has the best estimate of the height of peak.

A means by which this error can be minimised would be to add a delay to the cascade. There are two approaches to this. Firstly, a small delay can be added to each of the stores in the series. The input is split between t_k and t_{k+1} for each store. Having a shift on each store could potentially improve the fit for all of the stores in the cascade, however the peak will be broadened by an amount proportional to the time constant. Alternately, a large delay can be added to the first store. This can be viewed as a combined solution approach. Applied to the trapezoidal solution (which has the most accurate shape), it could shift the peak to align with the timing of peak given by the backward solution. This will not broaden the peak significantly, however the delay will only be correct for the last store in the cascade.



Conclusions

The trapezoidal solution is the preferred method due to its numerical accuracy. This is expected, as the trapezoidal method is a second-order implicit method. The numerical accuracy, however, is still dependent on the time constant, which cannot be too short.

If an analytic solution is available, this is the best approach to ensure numerical accuracy, even when cascading stores. However, it is not possible to solve a system of nonlinear ODEs analytically (such a system would be necessary for a cascade of nonlinear stores). In such cases the available evidence suggests that the trapezoidal method is best at reproducing peak height. Final conclusions about the effectiveness of the nonlinear model for water quality analysis will be dependent on the results from cascaded models.

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