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Planar Graphic Sequences

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1 Introduction

Within this report, we only consider finite, undirected, simple graphs. A graph is said to be simple if it has no loops and no multiples of the same edge. We begin with a few definitions;

Definition 1. The *degree* of a vertex is the number of edges connected to it.

Definition 2. A sequence $D = \{d_1, d_2, \ldots, d_n\}$ of non-negative integers is called a *graphic degree sequence* (respectively *planar graphic degree sequence*) if there exists a simple graph (respectively a planar simple graph) with n vertices whose degrees are $d_1, d_2, \ldots, d_n$.

The order of elements in a sequence $d = \{d_1, d_2, \ldots, d_n\}$ is unimportant; the sequences in this report will usually be presented in the non-increasing order (as it is customary in the study of degree sequences).

Example 1. Let D be the degree sequence of the graph shown in Figure 1. Then $D = \{6, 5, 4, 4, 3, 3, 2, 1\}$.

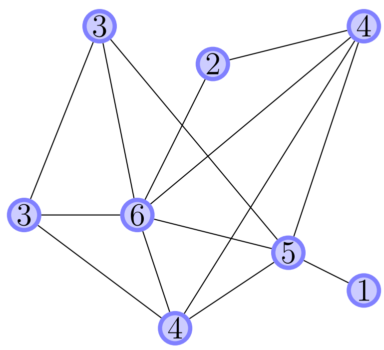

Figure 1: A graph with degree sequence D. Each vertex is labelled with its degree, an element in the degree sequence.

Throughout this report, multiples of an element in a degree sequence will be written as a power. That is, the degree sequence D in Figure 1 can also be written as $D = \{6, 5, 4^2, 3^2, 2, 1\}$.

Definition 3. A graph is said to be *regular* if every vertex has the same degree.

Example 2. Complete graphs are obviously regular, since every vertex is connected to every other vertex, and so, the degrees of each vertex are the same.

Definition 4. A graph is called *bi-regular* if the set of its degrees has two elements.

Example 3. Let A and B be the graphs shown in Figure 2, with the following respective degree sequences:

$$A : \{3^6\}, B : \{5^2, 3^4\}.$$

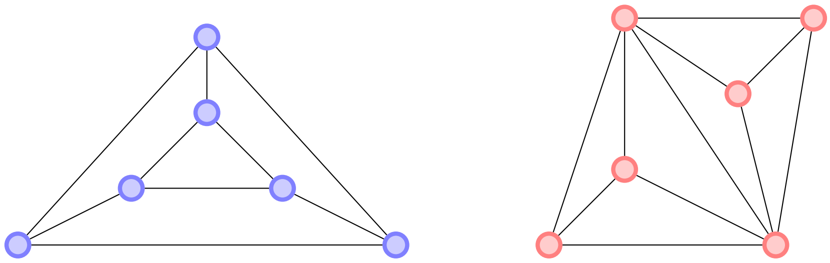

Figure 2: Diagram representations of Graphs A (left) and B (right). Graph A is regular and graph B is bi-regular

Graph A has a constant degree sequence, so it is regular. Graph B has multiples of two elements, 5 and 3. The graph B is therefore bi-regular.

Definition 5. Two vertices are *adjacent* if there is an edge connecting them.

Definition 6. A graph is called *bipartite* if its vertex set is the disjoint union of two subsets, or *parts*, such that no two vertices in the same subset are adjacent.

Definition 7. A *complete bipartite* graph is a bipartite graph where every vertex in one part of the graph is connected to every vertex in the other part of the graph. It is generally titled as $K_{a,b}$ where a and b are the number of vertices in each part.

Example 4. An example of bipartite graph (left) and complete bipartite graph (right) is shown in Figure 3. The bipartite nature of the graphs is shown in its colouring; there are no vertices of the same colour adjacent to each other. The degree sequence of the left graph is $\{6, 2^6, 1^2|5, 4^2, 3, 2^2\}$, where the $|$ symbol separates the parts of the graph. The degree sequence of the right graph is $D = \{4^5|5^4\}$ and is the $K_{4,5}$ graph.

In this report, we focus on the study of planar graphic degree sequences. Necessary and sufficient conditions for a sequence to be graphic are given by the following classical theorem (Erdos & Gallai 1960).

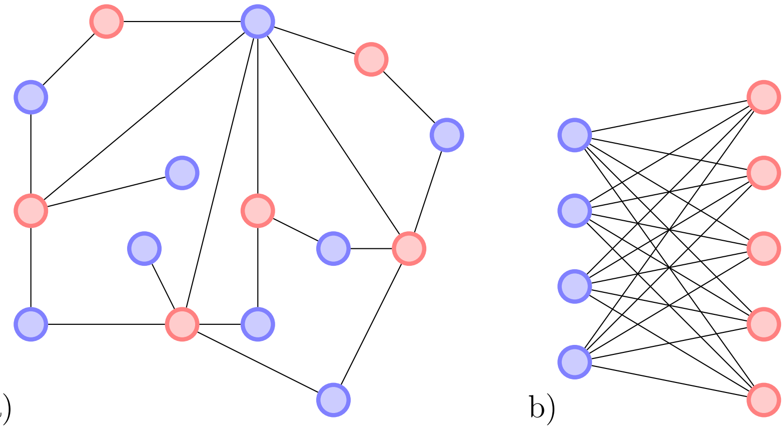

Figure 3: Examples of bipartite graphs. The graph b) is complete bipartite, the graph a) is just bipartite.

Theorem 1 (Erdös-Gallai Theorem). *A non-increasing, non-negative sequence $D = \{d_1, ..., d_n\}$ is graphic if and only if its sum is even and for every $k \in \{1, ..., n\}$,*

$$\sum_{i=1}^{k} u_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, u_i)$$

Conversely, the necessary and sufficient conditions for a sequence to be planar graphic are not known. It is in the opinion of experts in the field that the answer could be out of reach. Some partially necessary or sufficient conditions have been obtained by different authors, such as the following theorem (Schmeichel & Hakimi 1977).

Theorem 2 (Hakimi-Schmeichel Theorem). *Let $D = \{d_1, ..., d_n\}$ be a non-increasing graphic sequence. Then:*

- *If $d_3 \leq 3$, the sequence is planar graphical.*
- *If $d_3 > 3$, a necessary condition for the sequence to be planar graphical is that:*

$$\sum_{i=1}^{n} d_i \leq 6(n-2) - 2\omega(2) - 4\omega(1)$$

Investigations as to the planarity of constant degree sequences has also been of interest (Owens 1971);

Theorem 3. *Every regular, graphical sequence is planar except for $D = \{4^7\}$ and $D = \{5^{14}\}$.*

From this last theorem, we are motivated to find regular planar graphs that are also bipartite. In doing so, we have developed a method of producing bi-regular, bipartite planar graphs, constructed from a degree sequence form. In finding the bi-regular graphs, the regular graphs were also found.

2 BB Planar Graphs

Definition 8. A *BB* graph is a bi-regular, bipartite graph, with every vertex within a part of the graph having the same degree.

Example 5. The general form of a BB graph is shown in Figure 4. BB graphs have a specific degree sequence of the form $D = \{a^p | b^q\}$, where $b \geq a \geq 1$ and $p, q \geq 1$. Again, we see that the graph is bipartite in it's colouring. It is also bi-regular, there are p blue vertices each with degree a, and q red vertices each with degree b.

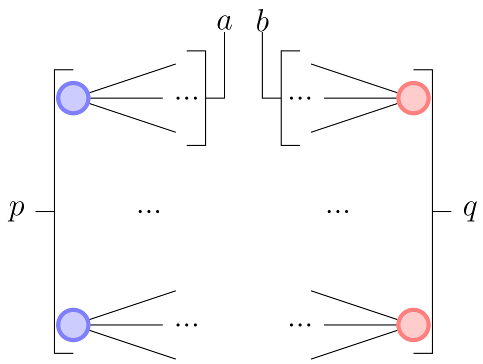

Figure 4: An example of a BB graph.

Note that a BB graph is not necessarily a bipartite complete graph, but all bipartite complete graphs are BB graphs. Since every edge in one part of the graph must extend to the other part of the graph, the total number of edges, in each part must be equal. The number of edges E and the number of vertices V in the degree seqeuence D is given by

$$V = p + q, \qquad E = ap = bq.$$

The main result of this report is which BB graphs are planar, which is described with the following theorem.

Theorem 4. *Let $1 \leq a \leq b$, $1 \leq p, q$. A degree sequence $D = \{a^p | b^q\}$ is a planar BB sequence if and only if:*

- *either $a = 1$,*
- *or $a = 2$ and,*
  - *either q is even,*
  - *or q is odd and $b = 2r$ and $p = rq$, for some $r \geq 1, q > 1$,*
- *or $a = 3$ and,*
  - *either $b = 3$ and $q = 4, 6, 7, 8, \ldots$*
  - *or $b = 4$ and $q = 3r$ for $r \geq 2$*
  - *or $b = 5$ and $q = 3r$ for $r \geq 4$.*

Note that the case for regular sequences occurs for $a = b$.

3 Proof of Theorem 4

In order to prove Theorem 4, we will first remove cases of 2 or less vertices. This case will be shown to produce a non simple graph. The case for $a > 4$ will also be shown to produce non-planar graphs. Then, we will provide the condition of planarity for bipartite graphs, which extends to BB graphs, though the use of Euler's formula plane graphs. This formula will be used extensively throughout the proof. Next, we will provide notes on unconnected graphs, and processes by which they can be connected without changing the degree sequence. Finally, by incrementing the value for a, a series of lemmas will be produced that combine to prove Theorem 4. First the case for $a = 1$ and $a = 2$ will be shown, then for $a = 3$, we will first provide the allowed values for b, and finally give constructions of the graphs in both of these cases.

From our expression for E and V, we deduce that any graph with 2 or less vertices is to be regarded trivial. Assume there exists a BB graph with 2 vertices. Then:

$$p + q = 2 \Longrightarrow p = q = 1$$

The number of edges in each part must equal, so:

$$ap = bq \implies a = b$$

Then the degree sequence of our graph is:

$$D = \{a^1|a^1\}$$

For $a = 1$ we have two single vertices connected by a single edge. With increasing a, we maintain two single vertices connected by a edges. The graph is no longer simple. This is illustrated in Figure 5. The graph of a single vertex is not bipartite or bi-regular, as there is no two mutually exclusive subsets of vertices from all the vertices of the graph. Hence, all graphs of interest will have more than 2 vertices.

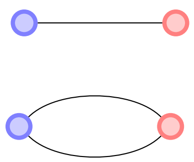

Figure 5: Graphs of two vertices with increasing degree, a. Further increase will result in more edges and the graph will still be non-simple.

Euler's formula for plane graphs states that any connected plane drawing of a graph must satisfy the following equality

$$V - E + F = 2,$$

where F is the number of faces. Furthermore, for any planar graph with no cycles of length 3 the following inequality must be true

$$E \leq 2V - 4.$$

It is characteristic for all cycles in a bipartite graph to have even length, since the pattern of vertices in the cycle must come in pairs of opposite colours. This means that the minimum length of a cycle for a bipartite graph is 4, and planar bipartite graphs must satisfy the above inequality.

Lemma 1. *Let $D = \{a^p|b^q\}$ be the degree sequence of a planar BB graph. Then $a < 4$.*

Proof. Suppose $a \geq 4$. Then by Euler's formula for bipartite planar graphs, $E \leq 2V - 4$, we have

$$4p \leq 2(p+q) - 4, \text{ and } 4q \leq 2(p+q) - 4,$$

as $E = ap = bq$, $V = p + q$ and $b \geq a$. Adding these two inequalities we get

$$\begin{aligned} 4p + 4q &\leq 2(p+q) - 4 + 2(p+q) - 4 \\ \implies 4(p+q) &\leq 4(p+q) - 8 \\ \implies p + q &\leq p + q - 2 \end{aligned}$$

which is a contradiction. $\square$

Remark 1. Now a note on connectedness. If we have a graph with two disconnected sub-graphs, in some circumstances, the edges can be rearranged to connect the sub-graphs such that the degree sequence is maintained. Beginning with the disconnected graph, select a single exterior edge on each of the sub-graph. Each of these selected edges will have a blue and red vertex, since we are considering bipartite graphs. We can replace the selected edges with an edge that extends from the blue vertex of one sub-graph to the red vertex on another sub-graph, and vice-versa.

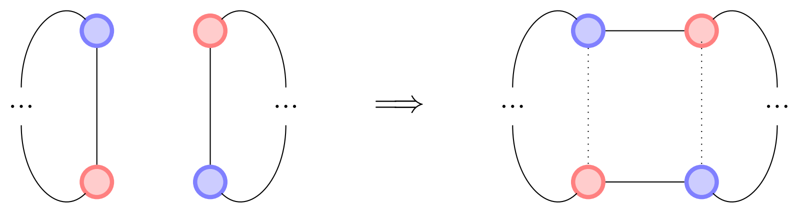

Figure 6: A diagram describing the reconnecting process of a disconnected bipartite graph. The ... represents the rest of the subgraph, and could include more edges.

Every BB graph with a degree sequence $D = \{a^p|b^q\}$ must have more than two vertices and $a < 4$. In the following lemmas we separately consider the cases $a = 1, 2, 3$.

Lemma 2. *Let D be a degree sequence of the form $D = \{a^p|b^q\}$. If $a = 1$, then D is always BB planar graphic.*

Proof. Since $a = 1$, the form of D can be rewritten as $D = \{1^{bq}|b^q\}$. For $q = 1$, the sequence can be drawn as the $K_{1,b}$ graph. Increasing q will result in q copies of the $K_{1,b}$ map. Each copy of the $K_{1,b}$ graph will be connected, but the graph as a whole will not be. Furthermore, the copies of the graph cannot be connected as per remark 1. This is illustrated in Figure 7.

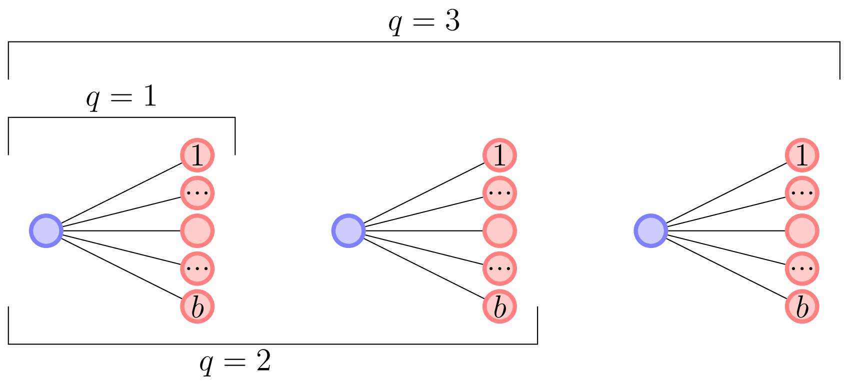

Figure 7: A diagram depicting the case for $a = 1$ in the BB degree sequence.

□

Lemma 3. *Let D be a BB degree sequence of the form $D = \{a^p|b^q\}$ with $ap = bq$. If $a = 2$ then D is planar graphic if and only if:*

- *either q is even.*
- *or q is odd, $b = 2r$ and $p = rq$ for some $r \geq 1$ and $q > 1$.*

Proof. First, consider the case that q is even. Let $q = 2s$, for $s \in \mathbb{N}$. From our expression for the number of edges, we have $ap = bq$, hence $p = bs$. Our degree sequence becomes $D = \{2^{bs}|b^{2s}\}$. For $s = 1$, the sequence can be drawn as the $K_{2,b}$ graph. Increasing s will result in s copies of the $K_{2,b}$ graph, as shown in Figure 8. As per remark 1, a graph of this degree form can be reconnected, shown in Figure 9.

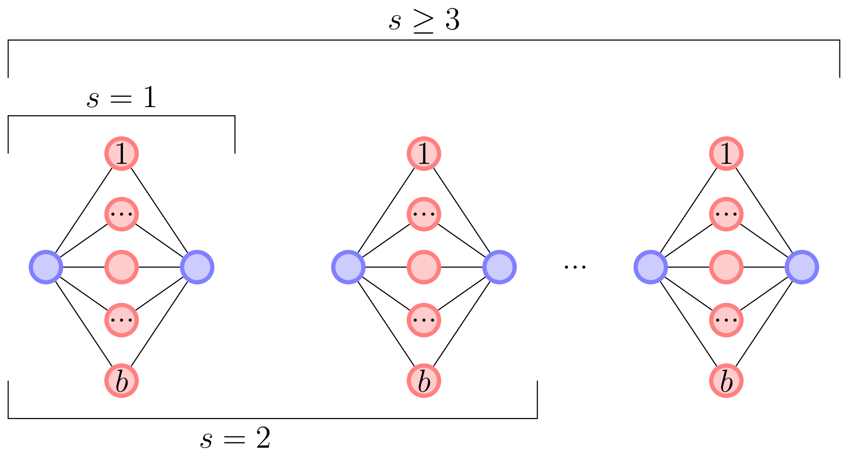

Figure 8: For $a = 2$, and $q = 2s$, the planar graph representation will be the $K_{2,b}$ graph. Increasing s will make s copies of the $K_{2,b}$ graph that are disconnected.

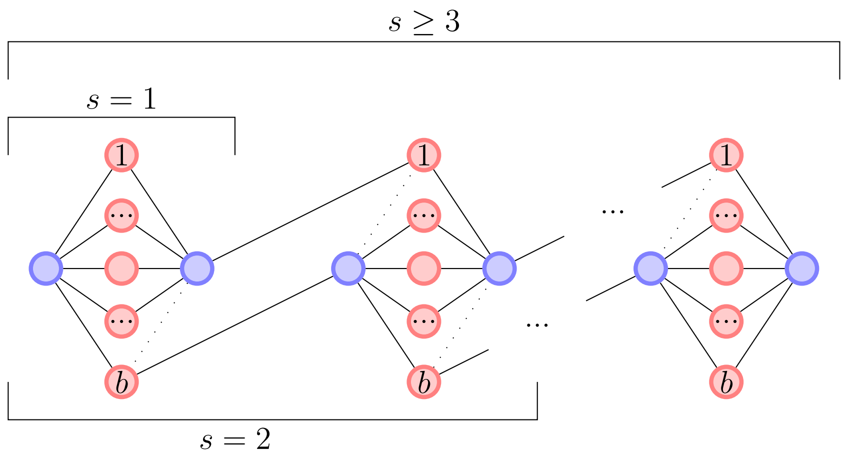

Figure 9: Replace edges from s isolated $K_{2,b}$ graphs, to produce a single connected graph, while maintaining the degree sequence.

Now consider the case for $a = 2$ and an odd q. From our expression for the number of edges, we have

$$E = ap = bq$$

$$2p = bq.$$

The LHS of this equation has a factor of 2, therefore the RHS must also have a factor of 2. Since q is odd, the b must be an even number. Let $b = 2r$, for $r \geq 1$. Continuing the expression for the number of edges:

$$2p = 2rq$$
$$p = rq$$

With new expressions for b and p in terms of r, we can rewrite the form of our degree sequence as $D = \{2^{rq}|(2r)^q\}$. Let D_r be the degree sequence of this form for a given r. The sequence D_1 can be drawn as a $2q$ cycle with alternating vertex type, as shown in Figure 10, a). Suppose there exists a planar graph D_k, for $k > 1$. By Euler's formula for planar graphs the following expressions are true.

$$\begin{aligned} E &\leq 2V - 4 \\ 2kq &\leq 2(kq + q) - 4 \\ 2kq &\leq 2kq + 2q - 4 \\ 0 &\leq 2q - 4, \qquad (*) \end{aligned}$$

From $(*)$, we also see that for this case we require $q > 1$. The degree sequence for D_{k+1} is given by:

$$\begin{aligned} D_{k+1} &= \{2^{(k+1)q}|2(k+1)^q\} \\ &= \{2^{kq+q}|(2k+2)^q\} \end{aligned}$$

If D_{k+1} is planar, then the following expression will be true by Euler's formula for planar graphs

$$
\begin{aligned}
E \leq& 2V - 4\\
2kq + 2q \leq& 2(kq + 2q) - 4\\
2kq + 2q \leq& 2kq + 4q - 4\\
0 \leq& 2q - 4
\end{aligned}
$$

which is true by $(*)$. Furthermore, increase r in the sequence $D = \{2^{rq}|2r^{q}\}$ can be thought of adding more 2 degree vertices to each group in Figure 10 ,b). Hence, a graphic sequence D is planar if:

- q is even.
- q is odd, $b = 2r$ and $p = rq$ for some $r \geq 1$ and $q > 1$.

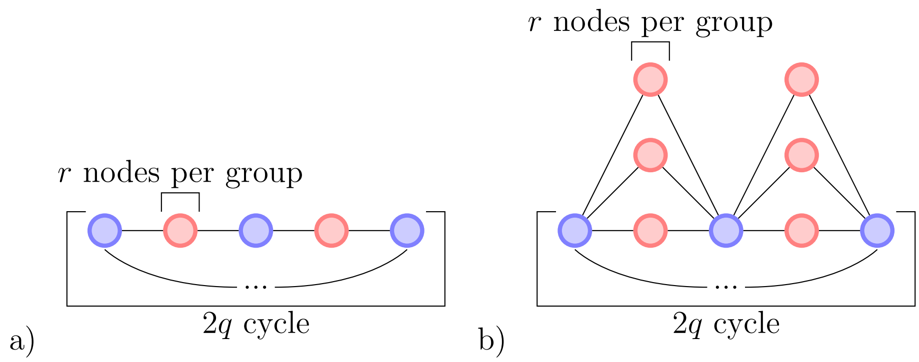

Figure 10: a) This graph has the degree sequence of D_1, a $2q$ cycle of alternating vertex type. b) This graph has the degree sequence of D_r. If each vertex of one type in D_1 is considered part of a group, then increasing r will add more vertices of that type to each group. Each vertex of a particular type in a group is connected to the same two vertices of the other type.

□

Lemma 4. *Let D be the degree sequence of the form $D = \{a^p|b^q\}$. If $a = 3$, and b is divisible by 3, then $b = 3$.*

Proof. Let $a = 3$ and $b = 3r$, for $r \geq 1$, so that b is divisible by 3. From our expression for the number of edges, we have;

$$3p = 3rq$$
$$p = rq$$

Substituting our expression for p and b in terms of r in our initial degree sequence, we get;

$$D = \{3^{rq}|(3r)^q\}$$

Next we apply Euler's condition of planarity for bipartite graphs, using our expression for V and E from our new degree sequence.

$$E \leq 2V - 4$$
$$3rq \leq 2(rq + q) - 4$$
$$rq + 4 \leq 2q$$

This inequality is only true for $r = 1$, and from our expression for b, we see that the only allowed value is $b = 3$.

□

Lemma 5. *Let D be the degree sequence of the form $D = \{a^p|b^q\}$. If $a = 3$, and b is not divisible by 3, then $b = 4$ or $b = 5$.*

Proof. Let $a = 3$. From our expression for the number of edges E, we have;

$$3p = bq$$

The LHS of this equation has a factor of 3, therefore the RHS must also have a factor of 3. Since b is not divisible by 3, the q must be a multiple of 3. Let $q = 3r$, for $r \geq 1$. Continuing the expression for the number of edges:

$$\begin{aligned} 3p &= b3r \\ p &= br \end{aligned}$$

We can now substitute our expressions for p and b in terms of r into our degree sequence;

$$D = \{3^{br}|(b)^{3r}\}$$

Again, we apply Euler's formula for planar bipartite graphs to produce the following inequality;

$$\begin{aligned} 3br &\leq 2(br + 3r) - 4 \\ 3br &\leq 2br + 6r - 4 \\ br + 4 &\leq 6r \end{aligned}$$

So $b < 6$. We also require $b \geq a$, from our initial description of BB graph degree sequence form, so $b \geq 3$. As b is not divisible by 3, we left with the two values, $b = 4$ or $b = 5$. □

Lemma 6. *Let D be the degree sequence of the form $D = \{a^p|b^q\}$. If $a = b = 3$, then D is planar graphic if and only if $q = 4, 6, 7, 8, ...$*

Proof. With $a = b = 3$, our degree sequence becomes;

$$D = \{3^p|3^q\} \implies D = \{3^q|3^q\},$$

as from our expression for the number of edges, we require $p = q$. For an even q, the planar realization of the degree sequence contains two cycles of length $\frac{q}{2}$, of alternating vertex type. By containing one cycle within another, and colouring the vertices of one cycle to be opposite of the other cycle, the associated vertices within the cycles can be connected piecewise, as shown in Figure 11.

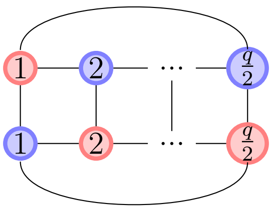

Figure 11: Two cycles of length $\frac{q}{2}$, with alternating vertex type, connected piecewise to satisfy the degree sequence.

For an odd q, we require an expansion of one vertex into many, such that the expanded formation contains only vertices of degree 3. This process of expansion is shown in figure Figure 12.

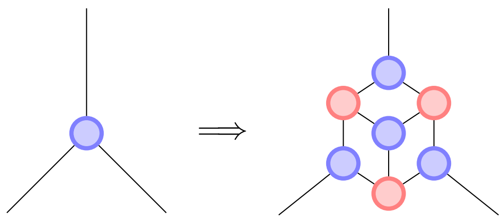

Figure 12: The process of expanding one vertex into three of each type, with the original vertex in the centre.

By expanding a single vertex into 7 vertices with degree 3, while maintaining the connections of outgoing edges, the possibility of forming planar graphs of odd q becomes possible. Notice that through the process of expansion, we only add 3 vertices of each type.

Hence, applying this transformation to a vertex on a degree sequence of the form $D = \{3^q|3^q\}$, where q is even, we will have a degree sequence of the form $D = \{3^{q+3}|3^{q+3}\}$, where $q+3$ is odd. The lowest even q possible is $q = 4$, so the lowest odd q possible is $q = 7$. For a degree sequence of the form $D = \{3^q|3^q\}$, for D to be a planar bipartite graphic sequence, then $q = 4, 6, 7, 8, ...$ □

Lemma 7. *Let D be the degree sequence of the form $D = \{a^p|b^q\}$. If $a = 3$, and $b = 4$ then D is planar graphic if and only if $q = 3r$ for $r \geq 2$.*

Proof. Recall our inequality regarding the maximum value for b when $a = 3$, shown below. Substituting $b = 4$ yields the possible values for r ;

$$br + 4 \leq 6r$$

$$4r + 4 \leq 6r$$

$$4 \leq 2r$$

So we have $r \geq 2$. Also recall the degree sequence form for when b is not divisible by 3, again, quoted below. Substituting the values for a, b we arrive at an equivalent degree sequence form;

$$D = \{3^{br}|b^{3r}\}$$

$$D = \{3^{4r}|4^{3r}\}$$

The case for $r = 2$ is shown below in Figure 13. Here we see two identical $2r$ cycles of alternating vertex type, between which are alternating groups of two vertices of one type and one vertex of the other type. There are two identical structures within the graph, shown in the alphabetical labelling of the vertices. The first structure is labelled $a - g$, while the second structure is named $a' - g'$.

By increasing r, we simply add more or the same structure to the graph, as shown in Figure 14.

□

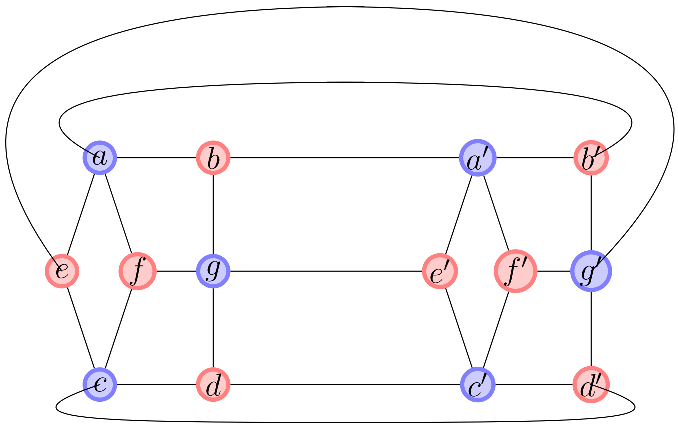

Figure 13: A graph depicting the $r = 2$ case.

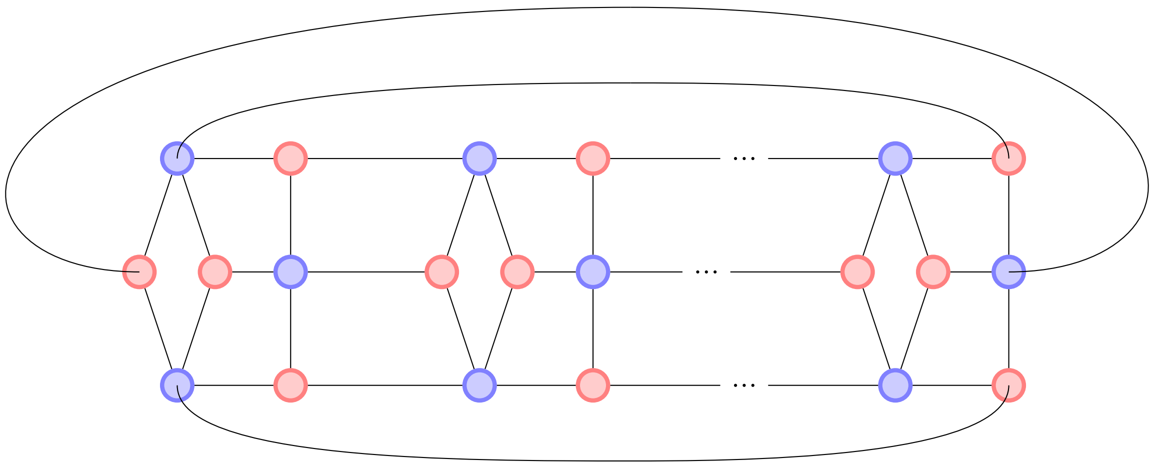

Figure 14: Appending more units to the $r = 2$ case will increase r by 1.

Lemma 8. *Let D be the degree sequence of the form $D = \{a^p|b^q\}$. If $a = 3$, and $b = 5$ then D is planar graphic if and only if $q = 3r$ for $r \geq 4$.*

Proof. We begin by again recalling the inequality from when $a = 3$ and the maximum value for b. Substituting $b = 5$, we produce different results for the values for r.

$$br + 4 \leq 6r$$

$$5r + 4 \leq 6r$$

$$4 \leq r$$

So we have $r \geq 4$.

Continuing as before, we can substitute our values for a and b into our degree sequence.

$$D = \{3^{br}|b^{3r}\}$$

$$D = \{3^{5r}|5^{3r}\}$$

The case for $r = 4$ is shown in Figure 15. Similar to the $b = 4$ case, we see two identical r cycles of alternating vertex type, between which are more complex structures of 5 and 3 degree vertices. Edges are to be extended from left to right pairwise, greek symbols have been used to indicate pairs. In the $r = 4$ case, we see two identical structures labelled $a - p$ and $a' - p'$. Within these structures, we have substructures shown in vertices $a - f$ and $g - p$, and likewise for the other main structure. By appending more main structure units to the graph, connecting pairwise to greek symbols, this increases the value of r in the degree sequence by 2.

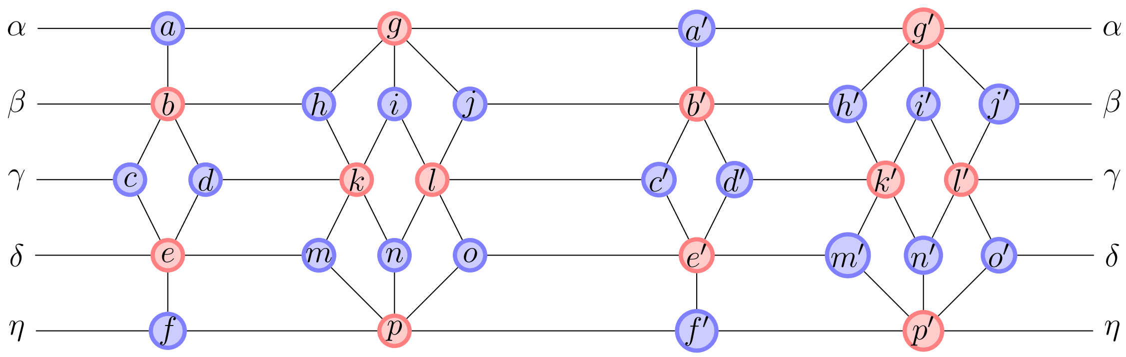

Figure 15: A graph depicting the $r = 4$ case. Appending more main structure units to this graph will increase r by 2.

By appending the main structure units in this fashion, all the even values for r greater than and equal to 4 can be drawn. To find the odd values of r, we begin by showing that the $r = 7$ case does exist, as shown in Figure 16.

An important note within this graph is that it can be contained within a 4 cycle, shown in the square nature of the drawing. Now we simply add more of the main structure units from our $r = 4$ case to achieve more odd values for r greater then 7. This is illustrated in Figure 17. If we include s main structure units, and one $r = 7$ unit, the resulting degree sequence will be the $r = 2s + 7$ graph.

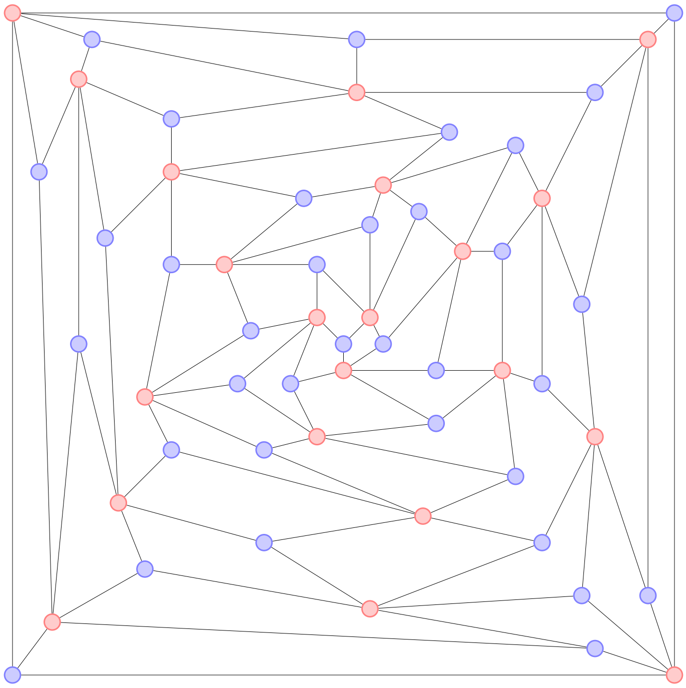

Figure 16: A diagram depicting an $r = 7$ unit.

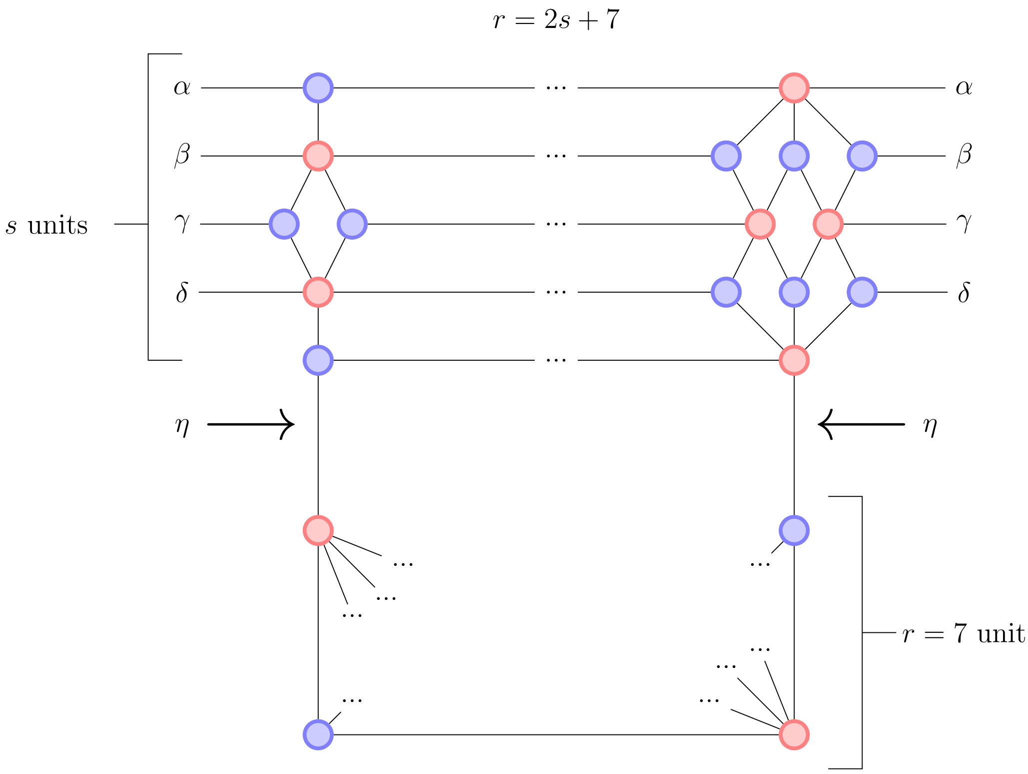

Figure 17: Appending an $r = 7$ unit to s main structure units will yield the $r = 2s + 7$ graph.

$\square$

4 Conclusion

We have introduced the concept of BB graphs, that are bi-regular and bipartite, such that every vertex in a part of the graph has the same degree. We have also found all planar BB graphs, by first finding a general degree sequence form and finding drawings of the graphs that have such a form. Possible grounds for further research could involve investigations of tripartite, or triregular graphs, and so on.

References

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