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Forecasting Realized Volatility in a Financial Market

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1 Introduction

Realized Volatility RV is an important measure of price variations in a financial market. Each price variation is measured by taking the difference of the logarithm of two instantaneous prices, sub-sampled at a certain time interval. For our study, we used prices sub-sampled at 5 minutes.[1] Each price variation measured in this way is often referred as a 5-minute intra-day return. Assume that there are M number of 5 minutes in a trading day. Index the date by t. Then the jth 5-minute intra-day return is $r_{t,j}$. *Realized Volatility* for day t is defined as[2]:

$$RV_t = \sqrt{\sum_{j=1}^{M} r^2{}_{t,j}}. \tag{1}$$

The motivations for forecasting *Realized Volatility* is two-fold. Firstly, price variations entail uncertainty and risk. Thus, forecasting *Realized Volatility* is important for risk management. In creating portfolios, realized covariance matrix for different assets is essential for assessing the risk level involved in the portfolio. *Realized Volatility* is the diagonal component in the realized covariance matrix. Moreover, models used for *Realized Volatility* can often be extended in to a multivariate model and used to forecast realized covariance matrix. Secondly, forecasting *Realized Volatility* is important in pricing, especial *option pricing*. An *option* is a financial derivative that represents a contract that offers the buyer the right to buy or sell a financial asset at an agreed-upon price during a certain period of time or on a specific date. The Black-Shores-Merton model is one of the most well-known option pricing model. It uses a partial differential equation to describe the price evolution of an European option under the Black-Scholes-Merton model. Let the spot price of the underlying asset be S, the risk-free rate be r and the volatility of the asset price be σ. The *option* price V follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V^2}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The above Black-Shores-Merton equation implies that, if one can forecast the volatility of the underlying financial asset, one can forecast the unique option price. Hence, forecasting *Realized Volatility* is of great use to both risk management and pricing.

In this report, we describe two representative models for forecasting *Realized Volatility*: the popular *Heterogeneous Autoregressive* HAR model and the more recent *Heterogeneous Autoregressive State Space* HARS model. The aim of comparing these two models is to test if adding a state variable is justifiable. To our knowledge, there is not much research on parameter estimations of the *Heterogeneous Autoregressive State Space* model. We will compare two fitting methods: the *Kalman Filter* and the more recent *Standardised Self-Perturbed Kalman Filter*. The aim of such a comparision is to test if the claimed improvement in efficiency by the *Standardised Self-Perturbed Kalman Filter* entails any accuracy loss in either estimation or forecasting.

[1] See Liu et al. (2015) for evidence that 5-minute Realized Volatility is sufficiently good.

[2] We adopt the definition from Corsi (2009) but adjust the notations slightly for simpler presentation.

2 Models

Realized Volatility RV is determined by market paricipants' trading activities. Usualy, any financial market is filled with market participants with a large spectrum of trading frequencies. For example, there are intraday speculators for instantaneous profits and there are also institutional investors for risk hedging. Thus a good model for *Realized Volatility* ought to be heterogeneous, that is, it should contain different volatility components. In addition, it is observed that volatility over long time intervals has larger influence on volatility over short time intervals than conversely. Intuitively, intraday speculators care about the long-term volatility because the long-term volatility affects the expected trend. On the other hand, the long-term traders are not much affected by the activities of the intraday speculators. Accouting for these observations about *Realized Volatility*, Corsi (2009) introduced the *Heterogeneous Autoregressive* HAR model. It is an 'almost autogressive' volatility cascade that contains three volatility components: the daily *Realized Volatility* RV_{t-1}, the weekly mean *Realized Volatility* $RV_{t-1:t-5}$ and the monthly mean *Realized Volatility* $RV_{t-1:t-20}$[3]. The HAR model has the following form:

$$RV_t = \beta_0 + \beta_1 RV_{t-1} + \beta_2 RV_{t-1:t-5} + \beta_3 RV_{t-1:t-20} + \epsilon_t,$$

where ϵ_t accounts for the measurement error and nuisance. It is assumed that $\epsilon_t \sim N(0, \sigma_\epsilon^2)$.

However, larger realized volatility implies greater price variations within any time interval. When we sample prices at 5-min intervals and use them to caculate the *Realized Volatility*, more information about the price variations may be lost. This means, larger realized volatility increases the likelihood of having larger measurement errors. Thus, measurement error should be heteroscedastic. On the other hand, the HAR model uses fixed autoregressive parameter. The underlying assumption of using fixed autoregressive parameter is that future *Realized Volatility* depends on past *Realized Volatility* to the exactly same degree across time. This is called *constant persistence*. Heteroscedastic measurement errors suggest that the assumption of constant persistence should be relaxed. In a largely volatile period, when *Realized Volatility* is measured with large uncertainty, the parameter of RV_{t-1} should decrease. To make the model more realistic, Bekierman and Manner (2017) introduced a time varying component to the autoregressive parameter:

$$RV_t = \beta_0 + (\beta_1 + \lambda_t) RV_{t-1} + \beta_2 RV_{t-1:t-5} + \beta_3 RV_{t-1:t-20} + \epsilon_t. \tag{2}$$

The model is named the HARS model where S stands for state space because the model contains a state variable λ_t. The state variable is driven by a latent gaussain process:

$$\lambda_t = \phi \lambda_{t-1} + \eta_t, \qquad \eta_t \sim N(0, \sigma_\eta^2). \tag{3}$$

The introduction of λ_t should be able to capture additional sources of variations.

[3]More precisely, $RV_{t-1:t-j} = \frac{1}{j} \sum_{i=1}^{j} RV_{t-i}$

3 Fitting Methods

Kalman Filter is a natural choice to fit a state space model such as the HARS model. We also used *Kalman Filter* to fit the HAR model. *Kalman Filter* is an algorithm that uses a series of observed *Realized Volatilities* over time to estimate the a series of the unobserved state variables and a series of the mean squared errors of the estimates. To simplify the notation, we denote the set of of unknown parameters,i.e., β_0, β_1, β_2, β_3, σ_ϵ, ϕ and σ_η, by a vector $\boldsymbol{\theta}$. Assume that a set of parameter values are given. Take time step t as an example. We observe all the *Realized Volatilities* up until time $t-1$ but we are unable to observe the state variable. The best estimate of λ_t is the expected value of λ_t given the most recent estimate, that is $\lambda_{t-1|t-1}$. Using Equation (3):

$$\lambda_{t|t-1} = \phi\lambda_{t-1|t-1}. \tag{4}$$

The variance of $\lambda_{t|t-1}$ depends on both the variance of $\lambda_{t-1|t-1}$, denoted by $R_{t-1|t-1}$, and σ_η^2 in Equation (3):

$$R_{t|t-1} = \phi^2 R_{t-1|t-1} + \sigma_\eta^2. \tag{5}$$

Having $\lambda_{t|t-1}$, we can use Equation (2) to obtain an estimate for RV_t:

$$RV_{t|t-1} = \beta_0 + (\beta_1 + \lambda_{t|t-1})RV_{t-1} + \beta_2 RV_{t-1:t-5} + \beta_3 RV_{t-1:t-20}. \tag{6}$$

The variance of $RV_{t|t-1}$ depends on the variance of $\lambda_{t|t-1}$ and the value of σ_ϵ^2:

$$Q_{t|t-1} = RV_{t-1}^2 R_{t|t-1} + \sigma_\epsilon^2. \tag{7}$$

Note that if $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ have a joint normal distribution:

$$\begin{bmatrix}\boldsymbol{z}_1\\ \boldsymbol{z}_2\end{bmatrix} \sim N\left(\begin{bmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{bmatrix}, \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{bmatrix}\right) \tag{8}$$

Then, the distribution of $\boldsymbol{z}_2$ conditional on $\boldsymbol{z}_1$ is $N(\boldsymbol{m}, \Sigma)$ where:

$$\boldsymbol{m} = \boldsymbol{\mu}_2 + \Omega_{21}\Omega_{11}^{-1}(\boldsymbol{z}_1 - \boldsymbol{\mu}_1) \tag{9}$$

$$\Sigma = \Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12} \tag{10}$$

$\boldsymbol{m}$ is the optimal forecast of $\boldsymbol{z}_2$ conditional on $\boldsymbol{z}_1$ and Σ is the mean squared error of this forecast.

Using this result, the distribution of (RV_t, λ_t) conditional on all the information up to $t-1$ is:

$$N\left(\begin{bmatrix}RV_{t|t-1}\\ \lambda_{t|t-1}\end{bmatrix}, \begin{bmatrix}Q_{t|t-1} & RV_{t-1}R_{t|t-1}\\ RV_{t-1}R_{t|t-1} & R_{t|t-1}\end{bmatrix}\right).$$

So the distribution of λ_t given the new information RV_t also follows a normal distribution with mean $\lambda_{t|t}$ and variance $R_{t|t}$:

$$\lambda_{t|t} = \lambda_{t|t-1} + \frac{R_{t|t-1}RV_{t-1}}{Q_{t|t-1}}(RV_t - RV_{t|t-1}), \tag{11}$$

$$R_{t|t} = R_{t|t-1} - \frac{(R_{t|t-1}RV_{t-1})^2}{Q_{t|t-1}}. \tag{12}$$

These two equations defines the updating stage: $\lambda_{t|t}$ is the optimal estimate of the state variable in time t and $R_{t|t}$ is the mean squared error of this estimate. Note that in the entire prediction stage, we needed two estimates from previous time step: $\lambda_{t-1|t-1}$ and $R_{t-1|t-1}$. They come from the previous updating stage. If we define the error and the Kalman gain as the following:

$$v_t = RV_t - RV_{t|t-1}, \tag{13}$$

$$K_t = \frac{R_{t|t-1}}{Q_{t|t-1}} RV_{t-1}. \tag{14}$$

We can see that, in the updating stage, we are tuning the estimates for λ_t using K_t as a weight:

$$\lambda_{t|t} = \lambda_{t|t-1} + K_t v_t. \tag{15}$$

The variance of $\lambda_{t|t}$ is also adjusting using K_t as a weight:

$$R_{t|t} = R_{t|t-1} - K_t R_{t|t-1} RV_{t-1}. \tag{16}$$

The prediction and updating stages are repeated for each time step. To start the process, we need some guesses for the initial value of state variable and its variance. We denote them by λ_0 and R_0. Denote the actual values of *Realised Volatility* data up until t by $\boldsymbol{Y_t}$. Assume that there are t time steps up to t. Then the log-likelihood of the parameter values being $\boldsymbol{\theta}$ and the initial values of the state variable and its variance being λ_0 and R_0, is:

$$l(\boldsymbol{\theta}, \lambda_0, R_0 | \boldsymbol{Y_t}) = \frac{1}{2} \sum_{i=1}^{t} -log(2\pi) - log(Q_{i|i-1}) - \frac{v_i^2}{Q_{i|i-1}}. \tag{17}$$

We estimate the parameter values including λ_0 and R_0 by maximizing Equation (17).

Note that there are 9 parameters to estimate for the HARS model. The HARS has 1 state variable λ_t. So σ_η is one-dimensional. A model for realized co-variance could have multiple state variables and thus σ_η is multi-dimensional. Imagine a model for a realized-covariance matrix that contains $n > 1$ assets. There will be $\frac{n(n+1)}{2}$ covariance components to estimate. So there are potentially $\frac{n(n+1)}{2}$ state variables. The covariance matrix for the state variables is $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$. Even if all the state variables are independent, σ_η would still contain $\frac{n(n+1)}{2}$ terms. As n increases, the number of parameters to estimate increases as n^2 and would eventually become computationally infeasible.

The *Standardised Self-Perturbed Kalman Filter* avoids this problem by changing Equation (5) to:

$$R_{t|t-1} = \phi^2 R_{t-1|t-1}, \tag{18}$$

so that the need to estimate σ_η is completely eliminated. When there are more than one state variables, the *Standardised Self-Perturbed Kalman Filter* requires to estimate fewer parameters and thus should be more efficient in terms of algorithm running time compared to the *Kalman Filter*.

Moreover, note the relation that $R_{t-1|t-1}$ affects the value of $R_{t|t-1}$, which then affects the value of K_t. K_t is used as a weight to adjust the value of $\lambda_{t|t}$. Thus, we can tune the value of $\lambda_{t|t}$ by tuning $R_{t-1|t-1}$. The *Standardised Self-Perturbed Kalman Filter* sets the updating stage for $R_{t-1|t-1}$ as:

$$R_{t|t} = R_{t|t-1} - K_t R_{t|t-1} RV_{t-1} + \zeta \max(0, FL(\frac{v_t^2}{\sigma_\epsilon^2} - 1)), \tag{19}$$

where ζ is a speed-adjustment parameter. The addtional term has two benefits. Firstly, if the squared error v_t^2 is large, $R_{t-1|t-1}$ is tuned up so that in the next step $\lambda_{t|t}$ will change to capture the additional sources of variation in the *Realised Volatility* data. That is the model's self-perturbation is activated. Secondly, if the squared error v_t^2 is largely from the variability of the *Realised Volatility* data, spurious variation of $\lambda_{t|t}$ is avoided. This can be seen from the $\max(0, FL(\frac{v_t^2}{\sigma_\epsilon^2} - 1))$ term. It ensures that the self-perturbation is activated only when squared error v_t^2 is large relative to the measurement error variance σ_ϵ^2. Moreover, ζ controls the speed of self-perturbation. If ζ is large, the change in $R_{t-1|t-1}$ is large and thus in the next time step change in $\lambda_{t|t}$ is large. Once squared error becomes small, $\max(0, FL(\frac{v_t^2}{\sigma_\epsilon^2} - 1)) = 0$ the self-perturbation is deactivated.

4 Data

To test if the introduction of the state variable is justifiable and to test if the claimed improvement in efficiency entails large accuracy loss, we use S&P 500 Index Realised Volatilty (5-min sub-sampled) from Oxford-Man Institute's 'realised library' dated from 2004-01-02 to 2017-10-20. We divide the data into three subsamples. The first one is the pre-GFC period dated from 2004-01-02 to 2007-02-28. The second one is the GFC period dated from 2007-03-31 to 2010-03-30. The third one is the post-GFC period dated from 2010-04-01 to 2017-10-20. The data we use is very representative. The pre-GFC period is characterised by very flat *Realised Volatility* data. The GFC period is characterised by very volatile *Realised Volatility* data. The post-GFC period is the intermediate between the two. Our experiments are on four sets of data: the entire period from 2004-01-02 to 2017-10-20, pre-GFC period from 2004-01-02 to 2007-02-28, GFC period from 2007-03-31 to 2010-03-30 and the post-GFC period from 2010-04-01 to 2017-10-20. For each set of data, we use the first 70% as in-sample for parameter estimations. With the estimated parameters, we first compare the estimated *Realised Volatility* with the real data. Then we make *Realised Volatility* forecasts using the estimated parameters and compare them with the last 30% of the real data. Fig. 1 shows a visualization of the data. The red lines represents the movement of the *Realised Volatilities*. The portions with the white backgrounds are the in-sample data while the portions with the yellow backgrounds are the out-of-sample data. The out-of-sample data for the entire period is shaded orange. Table 1 shows a summary of the data. The standard deviation values support our characterisation of different subsample periods: the GFC period is the most volatile while the pre-GFC period is the least volatile. Moreover, the data across all periods are right-skewed. The distribution of most of the data sets have more and also more extreme outliers than does the normal distribution except the out-of-sample in the GFC period when the *Realised Volatilities* starts to stabilise.

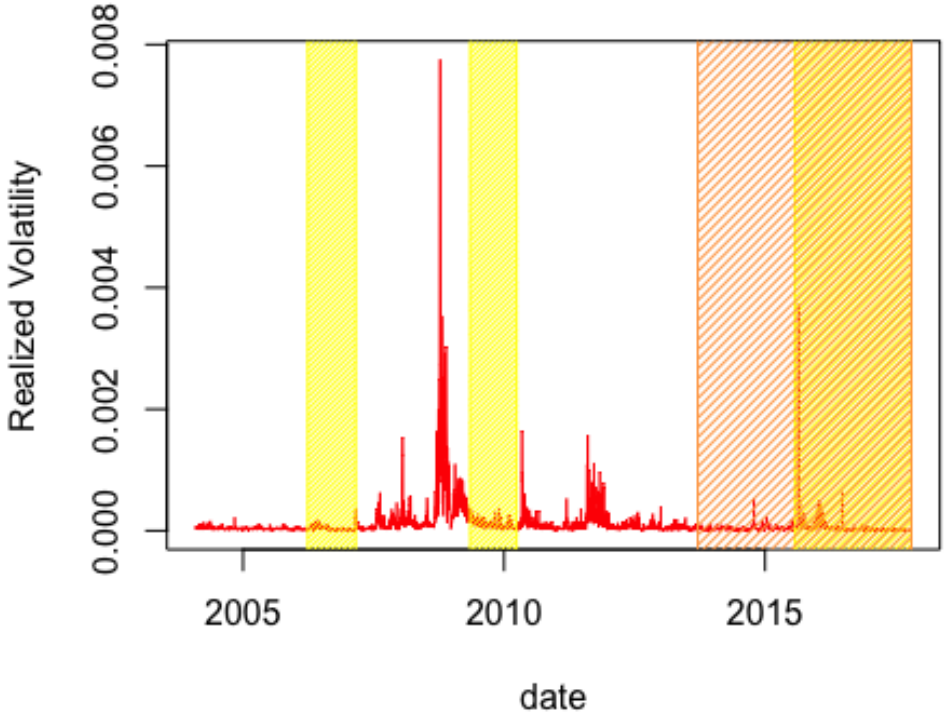

Figure 1: S&P 500 Realized Volatility in 2004-2017.

		Min.	Max.	Mean	Std.Dev.	Skewness	Kurtosis
2004-2017	Entire Sample	0.000002	0.007748	0.000105	0.000276	11.3370	215.2204
	In-Sample	0.000005	0.007748	0.000130	0.000315	10.0206	170.3786
	Out-of-Sample	0.000002	0.003732	0.000045	0.000132	21.5695	581.4445
Pre-GFC	Entire Sample	0.000006	0.000275	0.000038	0.000026	2.73891	13.3775
	In-Sample	0.000006	0.000213	0.000038	0.000024	2.2061	8.0038
	Out-of-Sample	0.000006	0.000275	0.000037	0.000031	3.2252	16.3945
GFC	Entire Sample	0.000008	0.007748	0.000263	0.000511	6.5939	69.7880
	In-Sample	0.000008	0.007748	0.000334	0.000596	5.6091	50.6476
	Out-of-Sample	0.000012	0.000370	0.000098	0.000068	1.3840	2.1699
Post-GFC	Entire Sample	0.000002	0.003732	0.000070	0.000149	11.4029	219.2453
	In-Sample	0.000002	0.001645	0.000078	0.000135	6.0469	49.7269
	Out-of-Sample	0.000002	0.003732	0.000052	0.000175	16.9209	344.4760

Table 1: Data Summary.

Parameters for the HAR model are estimated using the Ordinary Least Square method and parameters for the HARS model are estimated using the maximum likelihood method. The estimated parameters for different data sets are disclosed in the appendix.

5 Result

We use the estimated parameters to estimate the first 70% of *Realised Volatilities* in each data set and to forecast the last 30% of *Realised Volatilities* in each data set. Fig.2 is a summary of different models' fitting with the acutual *Realised Volatilities* across four data sets. The first row of figures is for the entire period; the second row is for the pre-GFC period; the third row is for the GFC period; the last row is for the post-GFC period. The first column is for the HARS model fitted using SPKF; the second column is for the HARS model fitted using KF; and the third column is for the HAR model fitted using OLS.

For each figure, the red lines represent the actual *Realised Volatilities*; the green dashed lines represent the estimated *Realised Volatilities*; and the blue dashed lines represent the forecasted *Realised Volatilities*.

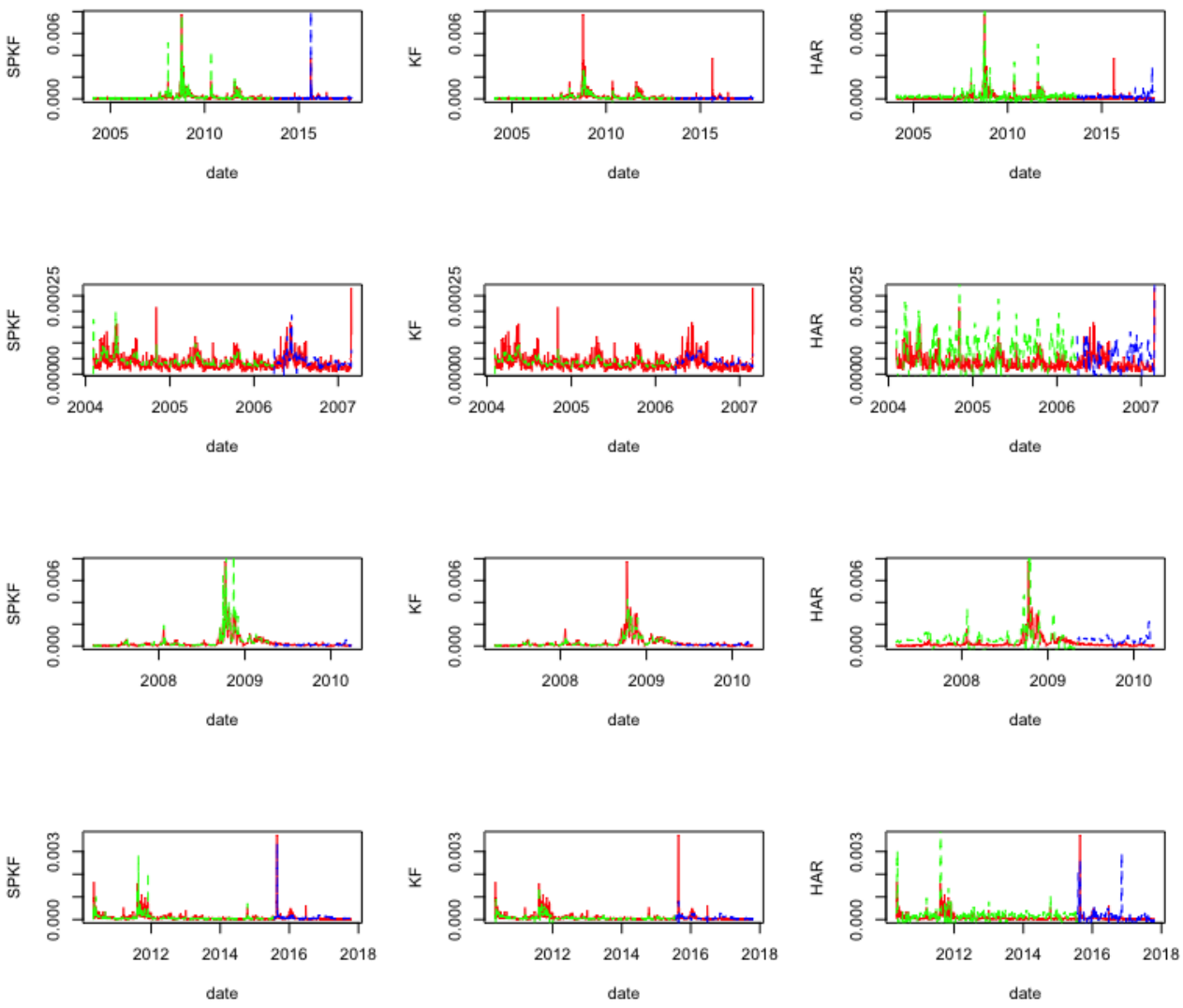

Figure 2: A Summary of the Models' Fitting with Actual data.

By the figure itself, we can make two observations. Firstly, the HARS model fitted using either method seems to perform better than the HAR model. Secondly, it is hard to tell whether the HARS model fitted using KF is better than the HARS model fitted using SPKF.

To assess the result quantitatively, we used two performance measure. *Squared errors* are commonly used. Squared error SE at time t is defined to be:

$$SE_t = (\widehat{RV_t} - RV_t)^2.$$

Since we are likely to have larger error when RV_t is largely, we also use *Qlike loss* to measure the relative 'distance' between the actual *Realised Volatility* and the estimated or forecasted *Realised Volatility*. It is defined

to be:

$$Qlike_t = \frac{RV_t}{\widehat{RV_{t|t-1}}} - log(\frac{RV_t}{\widehat{RV_{t|t-1}}}) - 1. \tag{20}$$

Patton (2011) proved that *Qlike loss* is a robust measure to compare different realized volatility models and it is less sensitive to outliers or very extreme values. As a rule of thumb, the smaller the *Qlike loss*, the better is the model for the specific data set.

We calculated the mean and median *Qlike loss* for the four data sets and for three models: the HAR model fitted using KF, the HARS model fitted using SPKF, and the HARS model fitted using KF. Table 2 gives the results for in-sample estimation and Table 3 gives the results for out-of-sample forecasting. We also include the median values as the median values are more informative in case of an extreme outlier. Note that we scale the squared errors by a factor of 10^8 for simple presentation. The minimum median and mean value of *Qlike losses* and *MSE* are coloured in red.

Time Period	Model	SE		QLIKE	
		Median	Mean	Median	Mean
2004-2017	HAR	2.95	75.10	0.7778	1.5020
	HARS(SPKF)	0.08	9.16	0.1148	0.2285
	HARS(KF)	0.05	4.09	0.0836	0.1966
Pre-GFC	HAR	0.32	0.82	0.3547	0.9557
	HARS(SPKF)	0.02	0.06	0.0658	0.1297
	HARS(KF)	0.01	0.05	0.0580	0.1229
GFC	HAR	23.16	253.23	0.9107	1.6745
	HARS(SPKF)	0.30	170.25	0.0745	0.9675
	HARS(KF)	0.30	16.88	0.0741	0.1715
Post-GFC	HAR	2.03	19.88	0.7486	2.5499
	HARS(SPKF)	0.05	1.81	0.1396	0.2792
	HARS(KF)	0.04	1.29	0.1177	0.2475

Table 2: Performances in Estimation.

Time Period	Model	SE		QLIKE	
		Median	Mean	Median	Mean
2004-2017	HAR	2.62	11.27	1.1962	1.3058
	HARS(SPKF)	0.08	229.28	0.2751	0.5428
	HARS(KF)	0.04	4.87	0.1918	0.4372
Pre-GFC	HAR	0.36	1.12	0.4237	1.0315
	HARS(SPKF)	0.02	0.10	0.1004	0.8095
	HARS(KF)	0.02	0.09	0.0941	0.2224
GFC	HAR	23.80	34.09	0.9467	1.0755
	HARS(SPKF)	0.29	1.32	0.1937	0.3931
	HARS(KF)	0.25	0.79	0.1398	0.3026
Post-GFC	HAR	2.05	39.84	1.0934	1.2302
	HARS(SPKF)	0.13	4.05	0.4605	0.6819
	HARS(KF)	0.07	13.04	0.3711	0.5441

Table 3: Performances for Forecasting.

The calculated *Qlike losses* and *MSE* support our first observation from the figures: the HARS model fitted using either method has much smaller *Qlike losses* and *MSE* than the HAR model in both estimation and forecasting. This provides empirical evidence for introducing the state variable in the model. In terms of the fitting methods, KF most of the time outperformes the SPKF. This comes at no surprise as SPKF is introduced for models with multiple state variables. Assume that we want to model the covariance matrix of n assets. Then the model requires $N := \frac{n(n+1)}{2}$ state variables. Then, Σ_η contains $K := \frac{N(N+1)}{2} = \frac{n(1+n)(2+n+n^2)}{8}$ unknown parameters. Using SPKF means that we can avoid estimating the K parameters and replace them with the speed-adjustment parameter ζ. We find that in the univariate case, SPKF is still comparable to KF in terms of accuracy. This leads us to explore more about SPKF in fitting a multivariate model.

6 Conclusion

In this report, we discussed the importance of forecasting *Realised Volatility* and *Realised Covariances*. We described two models for forecasting *Realised Volatility*: the HAR model and the HARS model with an additional state variable. We compared two algorithm to fit the HARS model: the *Kalman Filter* and the *Standardised Self-Perturbed Kalman Filter*. Using the data from S&P 500 index Realised Volatility, we find that: firstly, the introduction of the state variable is justifiable; secondly, the *Standardised Self-Perturbed Kalman Filter* has some accuracy loss but could potentially be computationally feasible for estimating models with multiple state variables. Our interest in forecasting *Realised Covariances* drives us to test the SPKF with a model that

contains multiple state variables.

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Appendix

	β_0	β_1	β_2	β_3	$log(\sigma_\epsilon)$
2004-2017	0.0002	-0.0017	6.34	-6.99	-2.03
Pre-GFC	-0.0337	1.0050	4.90	-9.27	11611.02
GFC	-5.1912	1.0001	4.68	-6.38	91.73
Post-GFC	-0.2029	1.0011	4.76	-7.66	-69.00

Table 4: Parameter Estimation for the HAR model using Kalman Filter.

	β_0	β_1	β_2	β_3	$log(\sigma_\epsilon)$	ϕ	λ_0	$log(R_0)$	$log(\zeta)$
2004-2017	1.4×10^{-5}	0.3666	0.1090	0.5047	-9.41	0.8215	1.58	2.26	3.71
Pre-GFC	6×10^{-6}	0.2199	0.4513	0.2925	-10.42	0.9155	1.35	1.05	4.28
GFC	1×10^{-6}	0.2492	0.7480	0.1364	-9.12	0.9043	-0.65	-3.18	-2.01
Post-GFC	1.7×10^{-5}	0.3065	0.0148	0.3514	-9.93	0.4559	6.24	5.84	0.21

Table 5: Parameter Estimation for the HARS model using SPKF.

	β_0	β_1	β_2	β_3	$log(\sigma_\epsilon)$	ϕ	λ_0	$log(R_0)$	$log(\sigma_\eta)$
2004-2017	5×10^{-6}	0.5981	0.2614	0.1469	-10.71	-0.1288	3.25	-3.60	-0.2133
Pre-GFC	1×10^{-5}	0.1546	0.5313	0.0535	-11.16	0.0000	18.32	16.09	-0.9949
GFC	6×10^{-6}	0.6022	0.4245	0.0151	-10.66	-0.0628	5.63	-34.56	-0.2676
Post-GFC	3×10^{-6}	0.8339	0.1165	0.1612	-11.09	-0.1828	2.43	-4.18	0.0208

Table 6: Parameter Estimation for the HARS model using KF.