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Can we make money using the game theory?

Yueyi Sun

Supervised by Professor Ben Goldys

The University of Sydney

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1 Introduction

Game theory can be defined as "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers" [4]. The theory of games allows us to model payoffs from different investments and business decisions in the presence of competition, in particular in the analysis of the oligopoly and duopoly markets. Therefore, game theory is particularly useful in economics, where we aim to analyse the behaviors of agents in real world. There are many aims of game theory, such as determining the optimal strategies for economic agents, finding an equilibrium of the market, understanding agents' choice of strategies.
In the first part of the report, I will introduce Nash equilibrium, and show the usefulness and limitations of Nash equilibrium using examples. In the second part of the report, several extensions and modifications of Nash equilibrium will be demonstrated to make up its deficiencies. In the last part of the report, I will focus on a very recent development in the game theory that uses the mean field game approach to model the order book dynamics.

2 Nash Equilibrium

2.1 Definition

Before I introduce the definition of Nash equilibrium, I should firstly introduce some concepts needed to define the Nash equilibrium.
We will consider a market that consists of n economic agents (players) $A_1, \ldots, A_n$. Each player A_i uses a strategy s_i that is expected to maximise his/her gain (or minimize loss) u_i but this gain (loss) depends on strategies $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$ of other agents as well, so that the gain of each player is finally a function of strategies chosen by all the players participating in the game:

$$u_i = u_i \left(s_1, \ldots, s_n \right) .$$

We will denote by S_i the set of all strategies available to player i. Now we are in position to define the basic object of the game theory.

Definition of Strategic Games

A strategic game consists of n players labeled $1, 2, \dots, n$ such that each player i has :
(1) *A strategy set S_i ,the elements of which are called the pure strategies of player i*
(2)*A payoff function $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$*
The players, the strategy sets and the payoff functions of a strategic game constitute a strategic game. In other words, an *n-player strategic game* is a $2n$-tuple

$$G = (S_1, \dots, S_n, u_1, \dots, u_n)$$

where the S_k and u_k are the strategy set and payoff function for player k. A sequence of strategies $(s_1, \dots, s_n)$ will be called a strategy profile.

Definition of Nash Equilibrium

A Nash Equilibrium of a strategic game $G = (S_1, \dots, S_n, u_1, \dots, u_n)$ is a strategy profile $(s_1^*, s_2^*, \dots, s_n^*)$ such that for each player i we have

$$u_i(s_1^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, \dots, s_n^*) \geq u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)$$

for all strategies $s_i \in S_i$ of player i, where $u_i(\cdot, \cdot)$ denote the payoff function of the player i

The definition provided above explains why the Nash Equilibrium provides solutions from which no player will have an incentive to deviate. Nash equilibrium provides a strategy, such that all the agents achieve the optimal gain if all the other players use the Nash strategy.

2.2 Assumptions

The appropriate use of game theory requires a clear understanding about when its assumptions make sense and when they do not. There are two assumptions of the Nash equilibrium, and only when all the players in the game follow those assumptions, the Nash equilibrium can provide a reasonable strategy. Players need to be rational and successfully predict the other players' strategies, so that the overall pattern of individual optimising behaviors can form a Nash equilibrium [1].

2.2.1 Maximisation

The aim of every player is to maximise the individual expected payoff value. Darwin's natural selection can be used to justify that market forces will drive out any player not maximising their individual profits [3].

2.2.2 Consistency

Consistency means that the players' expectations of other players' behaviors are consistent with their real behaviors. There are several ways to predict other players' behaviors, such as learning and communication.

2.3 The Usefulness of Nash Equilibrium

In this section, I will use one example to show people do play Nash equilibrium after repeating the game for a number of times. There are finitely many workers in the "stag-hunt" game. In

		minimum of other workers' efforts	
		High	Low
worker's effort	High	5,5	0,3
	Low	3,0	3,3

Table1. A "stag-hunt" played by workers in a team.

this game the workers work in teams of two. The payoff of each team is shown in Table1 [3]. I assume first that the workers need to work every day (so that this game will repeat every day), and assume that the players choose their strategies based on the payoff history. We can easily calculate the payoff value for both workers choosing high effort and low effort:

$$u(High) = 5P_{High} + 0P_{Low}$$

$$u(Low) = 3P_{High} + 3P_{Low}$$

where u denote the payoff value, P_{High} is the proportion of workers choosing high effort and P_{Low} is the proportion of workers choosing low effort.

Therefore, if more than 3/5 workers have chosen high effort at the beginning of this game, then the fraction of workers choosing high effort would increase in the next repeat of the game and finally will converge to 1 after a long enough period of time. Similarly, if less than 3/5 workers have chosen high effort at the beginning of the game, then the fraction of choosing low effort would increase next time period and finally converge to 1.
It is not hard to see that there are two Nash equilibriums in this game, either all players choose to put in high effort or all players choose to put in low effort. It remains to study the case when at the initial time the players do not start from a Nash equilibrium, and then the fraction of workers choosing high or low effort will converge to 1. In other words, the strategy profile of all the workers will converge to one of the Nash equilibriums eventually.

2.4 The Limitations of Nash Equilibrium

Nash equilibrium is useful to detect which strategy profile will be chosen by all the rational players in a game. However, there are several limitations of the concept of Nash equilibrium that restrict its applicability.

2.4.1 Non-existence

Nash equilibrium need not exist in a game with pure strategy. An example provided in Table2 below has no Nash equilibrium. Clearly, Nash equilibrium is not appropriate in the analysis of this game.

Player 2

Player 1		L	R
	T	1,0	0,1
	B	0,1	1,0

Table2. A Game with no Nash Equilibrium

2.4.2 Non-uniqueness and Instability

The "stag-hunt" game (Table1) show that a game can have more than one Nash equilibrium. An obvious question arises which of them to choose. The definition of the Nash equilibrium does not provide any answer and we need additional criteria to make a choice. We are able to analyse games with multiple Nash equilibriums if their number is relatively small. However, in some games, the number of Nash equilibriums can be very large and then it may be very difficult or impossible to decide which Nash equilibrium will be (should be) played by the players. Suppose there are finitely many players in a game and each two of them will play according to the payoff table (Table3) once. Any state in which all agents in population 2 play strategy L is a Nash equilibrium, together with the equilibrium dynamics shown by Figure1, so that it is impossible to analyse which Nash equilibrium will be played.
We can deduce from the arrows in Figure1 that all the Nash equilibriums are not stable in the game. After a small perturbation in the state space, the new game solution can never go back to the old Nash equilibrium.

Player 2

Player 1	L	R
T	1,1	1,0
B	1,1	0,0

Table3. Any state in which all agents in population 2 play *L* is a Nash equilibrium

2.4.3 Riskiness

In real life, people will make their optimal business decisions considering the combination of profit and risk. However, Nash equilibrium only guarantees the profit maximisation and neglects the risk effect. The Nash equilibrium in Table4 is (B,R), but it is obvious that player 1 would not want to play strategy B because of the enormous risk. The equilibrium of the game is more likely to be strategy profile (T,R) instead of the Nash equilibrium (B,R).

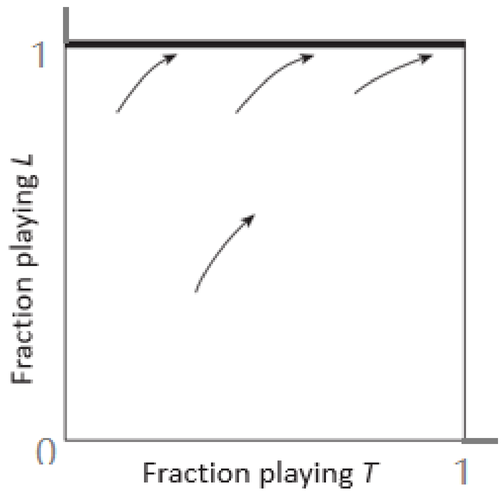

Figure1. The Nash equilibrium dynamics

Player 2

Player 1		L	R
	T	1,1	0,2
	B	-1000,0	2,2

Table4. The rational Nash equilibrium is (B,R)

2.4.4 Non-credible Threat

Nash equilibrium can depend on some non-credible threats to deter possible deviations by players in repeated game, and therefore Nash equilibrium may be implausible in some cases [1].

3 The Extensions & Modifications of Nash Equilibrium

As I have demonstrated in the last section, the Nash equilibrium is not a panacea. The limitations of Nash equilibrium restrict the usage of this concept. The question arises, what we should do in a game in which the Nash equilibrium is not useful. How should we choose an optimal strategy profile from which no player wants to deviate. In this section I will introduce several extensions and modifications of the Nash equilibrium that allow us to overcome various deficiencies of the Nash equilibrium.

3.1 Mixed Strategy

Definition of Mixed Strategy

Assume that the set S of available pure strategies is finite: $S = \{s_1, \ldots, s_m\}$. A mixed strategy for a player is a vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$, such that $p_k \geq 0$ for any k and $\sum_{k=1}^{m} p_k = 1$. In other words, a mixed strategy is a probability distribution on the set S of pure strategies.

The pure strategies are the special cases of the mixed strategies. When $p_k = 1$ and $p_l = 0$ for all $l \neq k$, then the mixed strategy $\mathbf{p} = (0, \ldots, 0, p_k = 1, 0, \ldots, 0)$ is the same as the pure strategy s_k.
We say that we have a finite game if it has a finite number of players, and a finite number of pure strategies is associated to each player.

Theorem

(Nash [5]) In every finite game there exists at least one Nash equilibrium in the set of mixed strategies.
Therefore, if we use a mixed strategy Nash equilibrium (an extension of the pure strategy Nash equilibrium), then we do not need to worry about the non-existence of the Nash equilibrium.

3.2 Maximin Strategy

Maximin payoff for a player is defined as the highest value that the player can be sure to get regardless of the other players' strategies.

Definition of Maximin Strategy

Maximin strategy is a decision rule that guarantees the maximin payoff for each player.

In Table4, the maximin payoff for player 1 and player 2 are 0 and 2, respectively. Therefore, the Nash equilibrium in Table4 is (B,R) but the equilibrium using the maximin strategy in Table4 is (T,R).
As maximin strategy can guarantee the maximin payoff for each player, it can take into ac-

count the potential risk of choosing a strategy. Therefore, maximin strategy is one of the modifications of Nash equilibrium and in the Table4 example, the solution of maximin strategy is different from the Nash equilibrium in the game. We are interested in the question when the two solutions are identical and when they are not. The equilibrium using maximin strategy coincides with the Nash equilibrium in zero-sum games and their modifications (e.g. constant sum games) [6].

3.3 Subgame Perfect Equilibrium

Definition of Subgame Perfect Equilibrium

A strategy profile is said to be a subgame perfect equilibrium if it is an equilibrium in every subgame of the repeated game.

The definition of subgame perfect equilibrium can avoid the non-credible threat in the Nash equilibrium. It is possible that there are too many Nash equilibriums in the repeated game and it would be meaningful to screen out the equilibriums which are credible in the game. The set of subgame perfect equilibriums is always a subset of the set of Nash equilibriums for that game.

It is important to note that every finite extensive game has a subgame perfect equilibrium [1].

4 Order Book Model

The last part of the report is focused on an important application of the mean field game (MFG). This is a new development in the game theory and an area of active research. I will briefly introduce MFG and then move to model the order book dynamics using the MFG approach. The idea to use the mean field game approach to model the order book dynamics is taken from the paper by Lachapelle, Lasry, Lehalle and Lions [2]. It is a very interesting paper that leads to many deep questions in economics and mathematics of game theory but contains a large number of mathematical inaccuracies. The aim at this section is to describe the order book model from [2] in a clear and consistent way.

An *order book* is an electronic list of buying and selling orders. I want to find the Nash equilibrium in the MFG in order to analyse how the market equilibrium will change.

4.1 Introduction to Mean Field Game

The reason for using the MFG approach is that when the number of players is very large then it is very difficult to analyse individually optimal strategies taking into account every other players' strategy. Therefore, we use an average strategy of all the players (including player i) to decide the i-th player's optimal strategy. The MFG is a dynamic model of self-interested players, which means that the aim of each player is to maximise the expected value of their payoffs. There are two assumptions for the players in MFG, that they are atomized and "exchangeable", and I will explain later how these two assumptions work for the order book dynamics.
The equilibrium in the MFG is achieved when the individual optimal strategy is global, which means that every player in the game chooses the same Nash strategy and is a sort of average strategy of all the players.
It is always hard to find the set of Nash equilibriums in a game (except some very simple game), so a standard method used by researchers is to analyse interesting strategy profiles and try to prove that they meet the definition of Nash equilibrium. In the order book model I will describe later, I am interested in the equilibrium of MFG as this strategy profile provides optimal strategy for every player. I will not prove this equilibrium is a Nash equilibrium in this report but there are many previous papers that have successfully proved the existence and convergence of the MFG equilibrium. Under some assumptions, the Nash equilibrium in the game played by N agents converges, as $N \to \infty$, to the strategy determined by MFG. I will assume a Nash equilibrium exist in the order book model and will try to find it.

4.2 A Simple One-sided Order Book Model

I will use the MFG approach to model the order book dynamics and start with a simple one-sided order book model. The one-sided model is only for better understanding some important concepts.

In the one-sided order book model, the *players* are the sellers in the market. When new sellers arrive, their decision to join or not to join the selling order queue is based on the size of the queue only and therefore it is the same decision for all the players (mean field). Let $u(x)$ denote the expected value of the seller's gain if he/she joins the queue of length x. We assume that the sellers are risk neutral, which means the sellers decide to join the queue or not by comparing $u(x)$ to 0. The reason we called it a one-sided order book model is because we only have the ask queue (selling order queue) in the model.

The players in the model need to satisfy the assumptions of MFG so that MFG approach is reasonable to use. Agents are atomized, so that all the agents are price takers instead of price makers. Every individual will have a nil influence on the global state (the market equilibrium). Agents are "exchangeable", that is the game is invariant for any permutation of the players. In order to satisfy these two assumptions I assume first that all the players are identical in the game, and so clearly they are "exchangeable". Secondly, I assume that the players arrive according to a Poisson process with rate λ. The arrival process of players is stochastic and will satisfy the assumption of players being atomized [2]. In addition, I assume that all the buyers are impatient in the model and they will arrive following another Poisson process with rate $\mu(x)$, $\mu(x)$ is a increasing function of queue length x. For example, $\mu(x) = 1 + logx$

Notations

- N^{λ} is the Poisson process for patient sellers
- $N^{\mu(x_t)}$ is the Poisson process for impatient buyers at time t, when the length of the queue is x_t
- the index of an anonymous player i is $i := N^{\lambda}$, which means the sellers arrive in the game following Poisson process with intensity λ
- $u_i(x)$ is the expected payoff for player i ($u_i(x)$ is the same for any player i as the sellers are identical by assumption. I will simply use $u(x)$ to denote the value function for every seller)
- x is the size of the queue
- q is the common size of each order
- c is the waiting cost per unit of time and is proportional to unit order size q, hence during a small time interval dt, the expected payoff u will be decreased by $cqdt$
- P is the price for per unit of order (q shares) when the queue size equals x. $P(x)$ is a nonnega-

tive decreasing function of the queue size. For example, $P(x) = p$, p is a constant and $P(x) = \frac{1}{x}$.

The control term R^i represents the decision process for the player i. If the seller joins the queue, then R equals 1, otherwise, R equals 0.

$$R^i(x+q) = \mathbb{1}_{\{u(x+q)>0\}} \tag{1}$$

We know that players are identical, so the control term can be anonymized. $R(x+q) := R^i(x+q) = \mathbb{1}_{\{u(x+q)>0\}}$
Note: $(x+q)$ is the size of the queue if player i chooses to join the queue, so $(x+q)$ is the mean field (average) in our model.

The size of the queue We use dx_t to denote the change of queue size at time t

$$dx_t = (dN_t^\lambda R_t - dN_t^{\mu(x)})q \tag{2}$$

The notations $dN_t^\lambda R_t^i$ represents the change of the queue size according to the i-th new seller's decision at time t. $dN_t^{\mu(x_t)}$ represents the change of queue size according to the arrival of a new buyer at time t. Note that dN_t^λ and $dN_t^{\mu(x_t)}$ can either be 0 or 1 because the probability of more than one jump in Poisson process at time point t is 0.

$$dN_t^\lambda R_t^i = \begin{cases} 1, & \text{if new seller join the queue} \\ 0, & \text{otherwise} \end{cases}$$

$$dN_t^{\mu(x)} = \begin{cases} 1, & \text{if new buyer consume the queue} \\ 0, & \text{otherwise} \end{cases}$$

After substitute the control term R into the indicator function. We have

$$dx_t = (dN_t^\lambda \mathbb{1}_{\{u(x+q)>0\}} - dN_t^{\mu(x)})q$$

I will use this indicator function to replace the control term in the rest of this section.

The matching process describes what happens if an order matches the order book. I use a pro-rata rule for the matching process. If a new buyer arrives in the game, then he will consume q shares in the ask (selling) queue. For each player waiting in the queue, $\frac{q^2}{x}$ shares will be sold. (As the total length of the queue is x shares, so we have $\frac{x}{q}$ players in the queue). The trade receive is $\frac{q^2}{x} \cdot \frac{P(x)}{q}$. The remaining value function is $\frac{u(x-q)}{q} \cdot (q - \frac{q^2}{x})$. Therefore, the new expected value for a player waiting in the queue is now $\frac{q}{x} \cdot P(x) + (1 - \frac{q}{x}) \cdot u(x-q)$

Definition of the payoff function The payoff function $J(x_t)$ is what the seller will get if he/she chooses to join the queue when the mean field (queue size) equals x_t. Then $dJ(x_t)$ is the infinitesimal change of the payoff for each agent (seller) at time t.

$$dJ(x_t) = [\frac{q}{x_t}P(x_t) + (1 - \frac{q}{x_t})J(x_t - q) - J(x_t)]dN^{\mu(x_t)} - cqdt \tag{3}$$

Note that this value for the payoff function is random as $dN^{\mu(x_t)}$ is random. So we are interested in the value function for each player - the expected value of the payoff function.

Definition of the value function

$$u(X) := \mathbb{E}J(x_T) \tag{4}$$

given $x_0 = X$, with T is "large enough". We can see from the definition of the value function that the value function is dependent on time. We assume that when T is large enough, the expected payoff has a stationary solution, which means the value function is independent of time.

Stationary solution as a fixed point of the value function After I assumed that the value function has a stationary solution, I want to find this stationary value of u. I try to find the fixed point of u by solving the equilibrium equation. The right hand side of the equation below is the change of expected value function in a small time interval dt.

$$\begin{aligned} u(x) = &(1 - \lambda \mathbb{1}_{\{u(x+q)>0\}}dt - \mu(x)dt) \cdot u(x) \Leftarrow \textit{nothing happens} \\ &+ \lambda \mathbb{1}_{\{u(x+q)>0\}}dt \cdot u(x+q) \Leftarrow \textit{seller join the queue} \\ &+ \mu(x)dt \cdot (\frac{q}{x}P(x) + (1 - \frac{q}{x})u(x-q)) \Leftarrow \textit{buyer consume the queue} \\ &- cqdt \Leftarrow \textit{waiting cost} \end{aligned} \tag{5}$$

There are three possible events in the game, and each event corresponds to a different change of the value function. Let us recall that the probability of one jump within a very small time interval dt in a Poisson process with intensity λ equals λdt. When a buyer arrives in the game, then according to the matching process I introduced earlier, there are two parts for the new value function, the trade receive and the expected value of the queue with size $(x - q)$.

Definition of the optimal expected payoff function. The optimal expected payoff function is the expected payoff if the player follows the optimal strategy. When the player follows the decision process described above, he will play the optimal trading strategy and the optimal expected payoff is

$$U(x) := max_{\delta \in \{0,1\}} \delta u(x) \tag{6}$$

4.3 Two-sided Order Book Model

In modern markets we have both selling and buying order book, so one-sided order book model is not appropriate to analyse the financial market. A healthy financial market aims to form prices based on the balance between demand and supply. The electronic order books are able to adjust the price formation process. I am interested in studying the two-sided order book dynamics in the order book model to find the equilibrium (balance) between demand and supply. I will try to find the value function dynamics.

The ask queue and the bid queue. The ask queue is the order book side for selling orders. The bid queue is the order book side for buying orders

The process. The players's actions are specified in Figure2 which show what happens when the players arrive in the game according to a Poisson process with a given intensity. Any player in the game is either a seller or a buyer, and he/she is characterized as being patient or impatient. The patient seller (buyer) will have an arrival rate λ_{sell} (λ_{buy}), and he/she will compare the different gains (costs) between joining the ask (bid) queue and consume the bid (ask) queue. The impatient players (both seller and buyer) arrive with the rate λ^{-}, and all of them will choose to consume the bid or ask queue. In other words, impatient players have

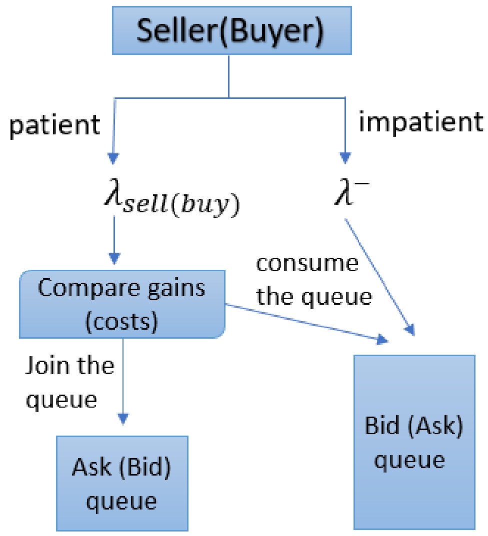

Figure2. The player actions in the order book model

infinite cost value c (the waiting cost is similar to one-sided order book model, and the cost of each order during a small time interval dt is $cqdt$).

I would like to find the Nash equilibrium (the optimal strategy) in the game, so I need to analyse how the patient players compare their gains or costs to make the optimal decision. In order to compare the gains or costs, I need to compare the transaction price with the value function.

Notations

- Q_t^a is the size of the ask queue at time t
- Q_t^b is the size of the bid queue at time t
- q is the common size of each order
- $u(Q_t^a, Q_t^b)$ is the expected value to receive if the seller chooses to join the ask queue
- $v(Q_t^a, Q_t^b)$ is the expected value to pay if the buyer chooses to join the bid queue
- c_a is the cost value in the ask queue, the waiting cost in the ask queue is $c_a qdt$ for each order during a small time interval dt
- c_b is the cost value in the bid queue, the waiting cost in the bid queue is $c_b qdt$ for each order during a small time interval dt

Definition of transaction price The balance between demand and supply will force the transaction price centered on a constant P with a *market depth* δ which means that all transactions will occur between price $P-\delta$ and $P+\delta$.

$$p_q^{buy}(Q_t^a) := P + \frac{\delta q}{Q_t^a - q},\ p_q^{sell}(Q_t^b) := P - \frac{\delta q}{Q_t^b - q} \tag{7}$$

where $p_q^{buy}(Q_t^a)$ is the price to buy an order (q shares) from the ask queue when a buying order hits the ask queue. And $p_q^{sell}(Q_t^b)$ is the price to sell an order (q shares) to the bid queue when a selling order hits the bid queue.

When the size of the ask queue increases, the price to buy will decrease. The intuitive idea is that if there are many sellers waiting to sell their shares (supply increasing), then the fair price will decrease. Also, when the size of the bid queue increases, the price to sell will increase. The intuitive idea is that if there are many buyers waiting to buy the shares (demand increasing), then the fair price will increase. Note that the boundary conditions $Q_t^a \geq 2q$ and $Q_t^b \geq 2q$ are necessary to make sure that there is no definition problem of the transaction prices, which also means the two-sided order book will never disappear.

The control. When the seller's (buyer's) expected received price (pay) of joining the ask (bid) queue is more (less) than the transaction price (the value to receive (pay) hitting the bid (ask) queue), the seller (buyer) would like to route the order to the ask (bid) queue.

I use the control term R to present the decision process.

$$\begin{aligned}
R_{+ask}(u, Q_t^a + q, Q_t^b) &:= \mathbb{1}_{u(Q_t^a+q, Q_t^b) > p^{sell}(Q_t^b)} \\
R_{+bid}(v, Q_t^a, Q_t^b + q) &:= \mathbb{1}_{v(Q_t^a, Q_t^b+q) < p^{buy}(Q_t^a)} \\
R_{-bid}(u, Q_t^a + q, Q_t^b) &:= 1 - R_{+ask}(Q_t^a + q, Q_t^b) \\
R_{-ask}(v, Q_t^a, Q_t^b + q) &:= 1 - R_{+bid}(v, Q_t^a, Q_t^b + q)
\end{aligned} \tag{8}$$

These four control terms represent four different cases. If $R_{+ask} = 1$, then the new seller chooses to join the ask queue. If $R_{+bid} = 1$, then the new buyer chooses to join the bid queue. If $R_{-bid} = 1$,then the new seller chooses to consume the bid queue. If $R_{-ask} = 1$, then the new buyer chooses to consume the ask queue. I will use these four control terms later to describe how the value function dynamics will change.

The size of the queue Similar to equation(2), we use dQ_t^a and dQ_t^b to denote the change of

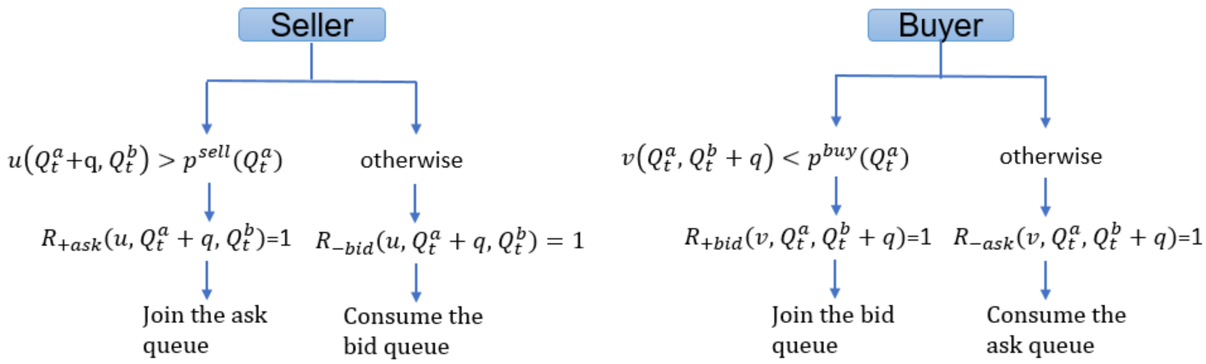

Figure3. The decision process for seller and buyer

the ask and bid queue size at time t, respectively.

$$
\begin{aligned}
dQ_t^a &= \Big(dN^{\lambda_{sell}} R_{+ask} - (dN^{\lambda_{buy}} R_{-ask} + dN^{\lambda^-})\Big) q \\
dQ_t^b &= \Big(dN^{\lambda_{buy}} R_{+bid} - (dN^{\lambda_{sell}} R_{-bid} + dN^{\lambda^-})\Big) q
\end{aligned}
\tag{9}
$$

The source of increasing the ask queue is the patient seller who is willing to join the queue, and the source of decreasing the ask queue includes patient buyer who chooses to consume the ask queue as well as the impatient buyer. Similarly, the source of increasing the bid queue is the patient buyer who is willing to join the queue, and the source of decreasing the bid queue includes patient seller who chooses to consume the bid queue as well as the impatient seller. Using the same argument as in one-sided model, the dN^{λ} can either be 0 or 1, and it represents the number of jumps in a Poisson process at time point t.

The matching process For the sake of simplicity, I use the pro-rate rule in the two-sided order book model.

Definition of the payoff function

The payoff function is what the seller will receive or what the buyer will pay if the player chooses to join the queue. Similar to the equation(3), $dJ_{ask(bid)}(Q_t^a, Q_t^b)$ is the infinitesimal change of the payoff (the payoff from joining the ask (bid) queue) for each seller (buyer) at time t.

$$\begin{aligned} dJ_{ask}(Q_t^a, Q_t^b) &= \Big[\frac{q}{Q_t^a} p^{buy}(Q_t^a) + (1-\frac{q}{Q_t^a}) J_{ask}(Q_t^a - q, Q_t^b) - J_{ask}(Q_t^a, Q_t^b)\Big] \cdot \\ &\quad (dN^{\lambda_{buy}} R_{-ask} + dN^{\lambda^-}) - c_a q dt \\ dJ_{bid}(Q_t^a, Q_t^b) &= \Big[\frac{q}{Q_t^b} p^{sell}(Q_t^b) + (1-\frac{q}{Q_t^b}) J_{bid}(Q_t^a, Q_t^b - q) - J_{bid}(Q_t^a, Q_t^b)\Big] \cdot \\ &\quad (dN^{\lambda_{sell}} R_{-bid} + dN^{\lambda^-}) - c_b q dt \end{aligned} \tag{10}$$

The significant difference between equation(9) and equation(3) is the source of consuming the queue. In the two-sided order book, not only the impatient player can consume the queue, but the patient player can choose to consume the queue as well.

Definition of the value function The value function is defined as the expected payoff function. Therefore, the value function for sellers and buyer are:

$$\begin{aligned} u(\mathcal{Q}^a, \mathcal{Q}^b) &:= \mathbb{E}\, J_{ask}(Q_T^a, Q_T^b) \\ v(\mathcal{Q}^a, \mathcal{Q}^b) &:= \mathbb{E}\, J_{bid}(Q_T^a, Q_T^b) \end{aligned} \tag{11}$$

given $(Q_0^a, Q_0^b) = (\mathcal{Q}^a, \mathcal{Q}^b)$, with T is "large enough". We assume that there exists a stationary solution for the value function, and therefore the value function is independent of time t.

Definition of the optimal expected payoff function Similarly to equation(6), the optimal expected payoff function is the expected payoff if the player follows the optimal strategy:

$$\begin{aligned} U(x) &:= \max \delta u(Q^a + q, Q^b) + (1-\delta) \cdot p^{sell}(Q^b) \\ V(x) &:= \min \delta v(Q^a, Q^b + q) + (1-\delta) \cdot p^{buy}(Q^a) \end{aligned} \tag{12}$$

The next step is to find this stationary equilibrium to prove that my assumptions are correct and with this stationary solution to find the optimal strategy for every player in the game.

Stationary solution as a fixed point of the value function In order to find the equilibrium solution, I will use the recursive equations of the value function. The idea is similar to equation(5). I will use some notations to represent the five possible events.

1. No seller or buyer arrives in the game during small time interval $dt \rightarrow$ NH
2. New seller consumes the bid queue $\rightarrow$ SCB
3. New seller joins the ask queue $\rightarrow$ SJA
4. New buyer consumes the ask queue $\rightarrow$ BCA
5. New buyer joins the bid queue $\rightarrow$ BJB

$$\begin{aligned} u(Q^a, Q^b) = &(1 - \lambda_{buy} dt - \lambda_{sell} dt - 2\lambda^- dt) \cdot u(Q^a, Q^b) \Leftarrow NH \\ &+ (\lambda_{sell} R_{-bid}(u, Q^a + q, Q^b) + \lambda^-) dt \cdot u(Q^a, Q^b - q) \Leftarrow SCB \\ &+ (\lambda_{sell} R_{+ask}(u, Q^a + q, Q^b) dt \cdot u(Q^a + q, Q^b) \Leftarrow SJA \\ &+ (\lambda_{buy} R_{-ask}(v, Q^a, Q^b + q) + \lambda^-) dt \cdot \\ &[\frac{q}{Q^a} p^{buy}(Q^a) + (1 - \frac{q}{Q^a}) u(Q^a - q, Q^b)] \Leftarrow BCA \\ &+ (\lambda_{buy} R_{+bid}(v, Q^a, Q^b + q) dt \cdot u(Q^a, Q^b + q) \Leftarrow BJB \\ &- c_a q dt \Leftarrow waiting\ cost\ in\ the\ ask\ queue \end{aligned} \tag{13}$$

$$\begin{aligned} v(Q^a, Q^b) = &(1 - \lambda_{buy} dt - \lambda_{sell} dt - 2\lambda^- dt) \cdot v(Q^a, Q^b) \Leftarrow NH \\ &+ (\lambda_{buy} R_{-ask}(v, Q^a, Q^b + q) + \lambda^-) dt \cdot v(Q^a - q, Q^b) \Leftarrow BCA \\ &+ (\lambda_{buy} R_{+bid}(v, Q^a, Q^b + q) dt \cdot v(Q^a, Q^b + q) \Leftarrow BJB \\ &+ (\lambda_{sell} R_{-bid}(u, Q^a + q, Q^b) + \lambda^-) dt \cdot \\ &[\frac{q}{Q^b} p^{sell}(Q^b) + (1 - \frac{q}{Q^b}) v(Q^a, Q^b - q)] \Leftarrow SCB \\ &+ (\lambda_{sell} R_{+ask}(u, Q^a + q, Q^b) dt \cdot v(Q^a + q, Q^b) \Leftarrow SJA \\ &- c_b q dt \Leftarrow waiting\ cost\ in\ the\ bid\ queue \end{aligned} \tag{14}$$

The solutions of these two equations provide the stationary solutions for the value functions. The value functions are independent of time t, so we can follow the process in Figure3 to compare the value functions with the transaction price to make the optimal trading strategy.

5 Future Study

I found out one important mathematical deficiency in paper [2], from which the MFG model of the order book is taken. In the one-sided order book model, they used Taylor expansion for small q to find an analytic formula for the stationary value function. While using the Taylor expansion up to order 2, they assumed that the value function u is independent of order size q. Using this assumption they derived an ordinary differential equation for u. I find this argument false, since we can easily see that u being a solution to equation(5), is dependent on the order size q, hence for each q we obtain a different solution $u(x, q)$ and then the Taylor expansion should be applied to the function $q \to u(x+q, q)$. The outcome will be very different from what is proposed in the paper. In fact we will end up with an equation containing partial derivatives of u. Another mistake in their analysis of equation(5) is that q appears as an argument of the indicator function and non-differentiability of this function is properly addressed. In addition, they did not mention anything about the value function for first two initial conditions (the value of the value function when queue size x equals 0 and q). In fact, I think it would be very difficult to choose the initial conditions, and my idea now is to collect some real data. In the two-sided order book model, there exist the similar questions. Therefore, some questions about the validity of this paper remain unresolved. Unfortunately, the summer research period was not long enough for me to solve all the difficulties, if I have chance, I will study further and try to find useful methods to answer these questions.

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