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Quaternions and Octonions

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Contents

Abstract

The quaternions and octonions are the two largest of the four normed division algebras. Despite their quirks of the quaternions being noncommutative and octonions even nonassociative, they continue to find uses in fields such as algebra, geometry, topology and number theory. We investigate the natural questions of why there exists only four normed division algebras, and why properties such as commutativity or associativity are lost in higher-dimensions by classifying these algebras in a classical theorem by Hurwitz. We also provide examples of quaternions and octonions uses by investigating classical results such as Lagrange's four square theorem and the Hopf fibration, among others.

1 Introduction

Quaternions and octonions are two of the four normed division algebras that extend the familiar concepts of real and complex numbers. The quaternions were first discovered on the 16th of October, 1843 by William Rowan Hamilton during his search for a three-dimensional number system analogous to the complex numbers. With years of work and no success, during a walk with his wife along the Royal Canal on his way to a meeting with the Royal Irish Academy in Dublin, Ireland, Hamilton had his now famous idea for quaternions. So pleased with his discovery, he carved the fundamental formula for quaternionic algebra:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

into the stone of the Brougham bridge, which today bears a plaque of the historic moment and the above equation. John Graves only a couple of months later wrote to Hamilton of his discovery of an eight-dimensional normed division algebra which he called the "octaves". However in March of 1845, Arthur Cayley published a paper in which he briefly included a description of the octonions. The octonions subsequently became known also as the "Cayley Numbers", despite much protestation by Graves. The excitement for quaternions among mathematicians, physicists and engineers since their discovery has not ceased, and they continue to find many interesting algebraic, geometric and physical phenomena. While the octonions have somewhat languised in obscurity in comparison, they too find many mathematical applications, and have experienced something of a rebirth in recent times. We shall consider just a few of these uses for quaternions and octonions, primarily focusing on geometric applications. For our purposes, we may initially define quaternions as the set:

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\} \tag{2}$$

where i, j, k satisfy equation 1, and the octonions defined analogously (in eight-dimensions). With addition and quaternionic multiplication (as we do with the complex numbers), these form a vector space and an algebra. Together with a map known as *conjugation* $^* : \mathbb{H} \to \mathbb{H}$ such that $a + bi + cj + dk = q \mapsto q^* = a - bi - cj - dk$, we can then define a *norm* on $\mathbb{H}$, such that for any $q \in \mathbb{H}$, $\|q\| = qq^* = q^*q$, and likewise for octonions. Finally, the norm on these algebras has the property that $\|xy\| = \|x\| \|y\|$, which is referred to as the normed property. This is a property that is explored further in section 3.

These should be familiar to the reader, who we assume has a working understanding of real and complex numbers. We shall try to provide sufficient information to follow, but naturally some understanding of algebra, geometry, and topology that would be typical of an undergraduate student in their penultimate or final year of study may be required.

2 Number Theoretic Applications

Having now defined the quaternions and octonions analogously to the complex numbers, we turn our attention to a few immediate consequences and applications of these algebras. Our first point of contact will be in number theory, where we aim to illustrate that the quaternions, without imposing any additional structure, can already aid in calculations that are otherwise more complicated. The section will conclude in a proof of Lagrange's four square theorem, which states that any natural number can be expressed as the sum of four square integers. We first provide an example of number theoretic results accessible via quaternions through a proof of Euler's four square theorem, just using the definitions given in the introduction.

Proposition 2.1. *If two numbers can be written as a sum of four squares, then so too can their product.*

Proof. Suppose that two natural numbers, p and q say, can be written as the sum of four square integers. That is $p = p_1^2+p_2^2+p_3^2+p_4^2$ and $q = q_1^2+q_2^2+q_3^2+q_4^2$, where $p, q \in \mathbb{N}$ and $p_i, q_i \in \mathbb{Z}$. Note then that we can write these as the norms of two quaternions $p = \|p_1 + p_2 i + p_3 j + p_4 k\|^2 = \|\alpha\|^2$ and $q = \|q_1 + q_2 i + q_3 j + q_4 k\|^2 = \|\beta\|^2$, where $\alpha, \beta \in \mathbb{H}$. Now if we consider their product $pq = \|\alpha\|^2 \|\beta\|^2 = \|\alpha\beta\|^2$ by the normed property of the quaternions. But note now that as $\alpha, \beta \in \mathbb{H}$, $\alpha\beta \in \mathbb{H}$ also, say $\alpha\beta = x_1 + x_2 i + x_3 j + x_4 k$. Then we have $pq = \|x_1 + x_2 i + x_3 j + x_4 k\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Note that each of these $x_i \in \mathbb{Z}$, as the product of two quaternions whose coefficients are integral will itself have integer coefficients, as they are simply sums and products of the coefficients, and the integers are closed under addition and multiplication. □

2.1 Quaternionic Integers and Primes

We continue with some definitions for the quaternions that give us some further structure for us to explore, namely defining quaternionic analogues for how the set of integers sit within the real numbers, as well as quaternionic primes. These definitions are common throughout the literature, but here we use those given by Conway [2] (chapter 8 of Stillwell [9] provides these also).

Definition 2.2. The integer analogue for the quaternions are known as the **Hurwitz integers**, and are given as

$$H = \left\{ a + bi + cj + dk \in \mathbb{H} : a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}$$

While it may seem as though the quaternionic analogue for $\mathbb{Z}$ should be those with only integer coefficients (these are the Lipschitz integers), it turns out that having those with only half-integer coefficients provides richer properties. These Hurwitz integers also have an analogue to the classic prime numbers (which henceforth will be referred to as *rational primes* to avoid confusion), which we define below:

Definition 2.3. We say a Hurwitz integer is a **Hurwitz prime** if it is one whose norm is a rational prime. That is, $q \in H$ is Hurwitz prime if $\|q\|$ is prime.

Proposition 2.4. *For any Hurwitz prime q, the only factorisations into two Hurwitz integers must have the form $q = u \times v$, where either $\|u\| = p$ and $\|v\| = 1$, or vice versa.*

Proof. Suppose $q \in H$ is a Hurwitz prime, so $\|q\|$ is a rational prime, say p. Now suppose that $q = u \times v$ is a factorisation of q into two Hurwitz integers. Then $\|q\| = \|u \times v\| = \|u\| \|v\|$ by the normed property of $\mathbb{H}$, and hence $\|u\| \|v\| = p$. But then as p is prime, one of $\|u\|$ or $\|v\|$ must be p, and the other must be 1. □

This shows that we must also consider the Hurwitz integers of norm 1, which are the *Hurwitz units*. Classifying these is thankfully quite simple, which we demonstrate with the following lemma, the proof of which is again thanks to Conway [2].

Lemma 2.5. *There are precisely 24 Hurwitz units, namely the 8 Lipschitz units $\pm 1, \pm i, \pm j, \pm k$, and the 16 others $\pm\frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k$.*

Proof. Suppose $q = a + bi + cj + dk \in H$ is a Hurwitz unit, so $\|q\| = 1$. Then either $a, b, c, d \in \mathbb{Z}$ or $a, b, c, d \in \mathbb{Z} + 1/2$. In the first case $|a|, |b|, |c|, |d| \geq 1$ (as q has integer coefficients), but of course the equation $a^2 + b^2 + c^2 + d^2 = 1$ implies that exactly one of a, b, c, d is ± 1, and the other three are 0, so we obtain the eight Lipschitz units $\pm 1, \pm i, \pm j, \pm k$. In the second case where $a, b, c, d \in \mathbb{Z} + 1/2$, we must have $|a|, |b|, |c|, |d| \geq 1/2$, and since $a^2 + b^2 + c^2 + d^2 = 1$, all of a, b, c, d must be $\pm 1/2$, giving us the other 16 possible Hurwitz units $\pm\frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k$. □

2.2 Rational Primes vs. Hurwitz Primes

We now have ample structure defined to be able to further study the similarities and differences between rational primes and Hurwitz primes, which we begin by stating and proving a classical result of Lagrange on rational primes, the proof of which is of course well-known and can be found in, for example, [9].

Lemma 2.6. *If p is an odd rational prime then there are integers l and m such that p divides $1 + l^2 + m^2$.*

Proof. Let p be any odd rational prime, that is $p = 2n + 1$, where $n \in \mathbb{N}$. First note the squares l^2, m^2 of any two $l, m \in \{0, 1, \ldots n\}$, with l, m distinct, are incongruent modulo p, as if $l^2 \equiv m^2 \bmod p$ then we would have $(l - m)(l + m) \equiv 0 \bmod p$. But since $0 < l + m < p$, we conclude that $l \equiv m \bmod p$. Thus the $n + 1$ possible values of l, m give $n + 1$ incongruent values of l^2 and m^2. Likewise we then have $n + 1$ incongruent values for $-1 - m^2$. But as $p = 2n + 1$, we only have a possible $2n + 1$ incongruent values, and hence for some $l, m \in \{0, 1, \ldots, n\}$ we must have $l^2 \equiv -1 - m^2 \bmod p$ and thus $1 + l^2 + m^2 \equiv 0 \bmod p$. □

While on the surface this result of Lagrange's appears to only concern rational primes, it will be crucial in our eventual proof of Lagrange's four-square theorem. We continue by stating and proving an analogue of the prime divisor property for Hurwitz integers. However, since multiplication in $\mathbb{H}$ is somewhat more subtle, mostly due to the noncommutativity, we must first establish some background, which is largely an abridged version of that given by Stillwell [9].

Call α a *right divisor* of some Hurwitz integer q if $\alpha = \gamma\delta$ for some $\gamma \in H$. If we now consider the case where $\alpha = \gamma\delta$ and $\beta = \varepsilon\delta$ have a common right divisor δ (some $\gamma, \varepsilon \in H$), then let $\mu \in H$ and consider $\rho = \alpha - \mu\beta = (\gamma - \mu\varepsilon)\delta$. This shows that just as in the traditional Euclidean algorithm, the remainder term we get when right dividing α by β also has δ as a right divisor. Thus if we continue to divide on the right as in the Euclidean algorithm, we will eventually obtain the *greatest common right divisor* of α and β, which we shall denote by right gcd(α, β). Just as in the traditional Euclidean algorithm, it then follows that

$$\text{right gcd}(\alpha, \beta) = \mu\alpha + \nu\beta \tag{3}$$

for some $\mu, \nu \in H$. We now have sufficient tools to state the proof for the aforementioned prime divisor property analogue, as is given in [9].

Lemma 2.7. *If p is a rational prime, and p divides a product $\alpha\beta$ of Hurwitz integers $\alpha, \beta \in H$, then p divides α or p divides β.*

Proof. Suppose that p is a rational prime that divides a product $\alpha\beta$ of Hurwitz integers $\alpha, \beta \in H$, but does not divide α. Then we must have that $1 = \text{right gcd}(p, \alpha) = \mu p + \nu\alpha$. Right multiplying by β then gives that $\beta = \mu p\beta + \nu\alpha\beta$. But since p clearly divides $\beta\mu p$, and also divides $\alpha\beta$ by hypothesis, we conclude that p must divide β. □

This result is certainly of great value to us, as it shows that the Hurwitz integers really are like an analogue of the traditional integers, being that they even have a prime divisor property and Euclidean algorithm! Together with Lemma 2.6, this result will also be essential in being able to put all of our lemmas and theorems together to prove Lagrange's four-square theorem.

2.3 Sums of Four Squares

We are now ready to switch gears a bit and begin a proper attack on the desired theorem.

Lemma 2.8. *If p is an odd rational prime but not a Hurwitz prime, then $p = a^2 + b^2 + c^2 + d^2$ where $2a, 2b, 2c, 2d \in \mathbb{Z}$.*

Proof. Suppose p is an odd real prime, but not a Hurwitz prime. Then as p is not a Hurwitz prime (but is a Hurwitz integer), it must have a non-trivial Hurwitz prime factorisation. Hence we can write $p = (a + bi + cj + dk)\gamma$ where $2a, 2b, 2c, 2d \in \mathbb{Z}$ and $\gamma \in H$. Then as $p = \bar{p}$ (since p is a rational prime), we have that $p^2 = p\bar{p} = (a + bi + cj + dk)\gamma\bar{\gamma}(a - bi - cj - dk) = (a^2 + b^2 + c^2 + d^2)\,\|\gamma\|^2$. But as p is prime, p^2 must have only p as its factors. Hence we conclude that $p = a^2 + b^2 + c^2 + d^2$, where $2a, 2b, 2c, 2d \in \mathbb{Z}$. □

Note that if in the above calculation we have that p was the sum of four square integers, that is $a, b, c, d \in \mathbb{Z}$, then we have proven that any prime is the sum of four square integers. If however p was the sum of four square half-integers, that is $a, b, c, d \in \mathbb{Z} + \frac{1}{2}$, then we can use the result to show that we can still write any odd rational prime p as the sum of four square integers.

Lemma 2.9. *Any odd rational prime that is the sum of four square half-integers is also the sum of four square integers.*

Proof. First we note that if $\alpha \in H$ has half-integer coefficients, i.e. $\alpha = a + bi + cj + dk$ where $a, b, c, d \in \mathbb{Z} + \frac{1}{2}$, then we may alternatively write $\alpha = \omega + a' + b'i + c'j + d'k$ where $\omega = (\pm 1 \pm i \pm j \pm k)/2$ and a', b', c', d' are even integers. Now suppose p is the sum of squares of four half-integers, so we have $p = a^2 + b^2 + c^2 + d^2 = (a + bi + cj + dk)(a - bi - cj - dk)$ with $a, b, c, d \in \mathbb{Z} + \frac{1}{2}$. Then using the above result, we may write $p = (\omega + a' + b'i + c'j + d'k)(\bar{\omega} + a' - b'i - c'j - d'k)$ where $\omega \in H$ and a', b', c', d' are even integers as above. Noting that ω has norm 1, we may then write

$$p = (\omega + a' + b'i + c'j + d'k)\bar{\omega}\omega(\bar{\omega} + a' - b'i - c'j - d'k)$$

But as $\omega\bar{\omega} = 1 = \bar{\omega}\omega$ and a', b', c', d' are all even integers, the Hurwitz integer $(\omega + a' + b'i + c'j + d'k)\bar{\omega}$ will have all integer coefficients, say A, B, C, D. Further to this, the Hurwitz integer $\omega(\bar{\omega} + a' - b'i - c'j - d'k)$ is simply its conjugate. Hence we have $p = (A + Bi + Cj + Dk)(A - Bi - Cj - Dk) = A^2 + B^2 + C^2 + D^2$. □

We now combine Lemmas 2.8 and 2.9 to give us a conclusive theorem on rational primes as sums of squares. This will naturally be the result that underpins the proof we give of Lagrange's four-square theorem.

Theorem 2.10. *Any rational prime number is the sum of four square integers.*

Proof. Suppose p is some rational prime. Then p is either 2, or odd; if $p = 2$, then $p = 0^2 + 0^2 + 1^2 + 1^2$, and hence is expressible as the sum of four square integers. Else p is odd, so by Lemma 2.6, there exists some integers l and m such that p divides $1 + l^2 + m^2$. But note that $1 + l^2 + m^2 = (1 + li + mj)(1 - li - mj)$, and so by Lemma 2.7, if p is a Hurwitz prime, p must divide one of these Hurwitz integers. But this is not possible, as neither of the quaternions $1 \pm li/p \pm mj/p$ are Hurwitz integers. Hence the rational prime p cannot be a Hurwitz prime, and so by Lemma 2.8, p must be either the sum of four square integers, or the sum of four square half-integers. If it is the former, the proof is complete. If it is the latter, then by Lemma 2.9 p must also be the sum of four square integers. □

We now conclude by providing the proof of the central theorem for this section: Lagrange's four-square theorem.

Theorem 2.11. *Any natural number can be written as the sum of four square integers.*

Proof. Let $n \in \mathbb{N}$ be some natural number. Then by the Fundamental Theorem of Arithmetic, n is a product of prime numbers. But by Theorem 2.10 any prime number can be written as the sum of four square integers, and by Lemma 2.1 if any two numbers can be written as the sum of four square integers, then so can their product. □

3 Normed Division Algebras

Having established that the quaternions find some nice, curious applications in fields such as number theory, as well as possessing certain structure that resembles that of the real and complex numbers, it is natural to ask questions about the deeper structure underlying each of these algebras. The definitions of both the complex numbers, quaternions and octonions involved constructions via imaginary units satisfying certain equations - can this type of construction using imaginary units be performed *ad infinitum*? To answer this question is one of the goals of this section, and we begin by formalising the process by which we construct new algebras from old. This formalism gives us a framework for the other goal of this section - to further investigate the abstract algebraic structure of the quaternions and octonions. The definitions we use are found commonly throughout the literature on this subject, and we use those given by Baez [1] and Harvey [3].

3.1 Formal Definitions

Definition 3.1. An **algebra** A will be a vector space over a field F, equipped with a bilinear map $\cdot : A \times A \to A$ called *multiplication* and a non-zero element $1 \in A$, called the *multiplicative unit.* We shall abbreviate multiplication via juxtaposition, writing $x \cdot y = xy$.

Note that unless otherwise indicated, we shall be considering only finite-dimensional algebras over the field of real numbers, and which are also endowed with a positive-definite quadratic form $\langle \cdot,\cdot \rangle$, which then induces a norm $\|x\| = \sqrt{\langle x,x \rangle}$. We now further restrict the types of algebras we shall primarily be concerned with via the following definition, as well as defining a number of tools that we will require in our study of these algebras.

Definition 3.2. If an algebra A satisfies the properties that $\|xy\| = \|x\| \|y\|$ for all $x, y \in A$ we say it is **normed**, and if $xy = 0 \implies x = 0$ or $y = 0$ for all $x, y \in A$, we say it has the **division property**. An algebra with both of these properties is a **normed division algebra**.

Definition 3.3. The **commutator** $[\cdot, \cdot] : A \times A \to A$ is an alternating bilinear map such that $[x, y] = xy - yx$, measuring the noncommutativity of an algebra A. Similarly the **associator** $[\cdot, \cdot, \cdot] : A \times A \times A \to A$ is a trilinear map such that $[x, y, z] = (xy)z - x(yz)$, measuring the nonassociativity of an algebra A. Say an algebra is **power-associative** if the subalgebra generated by any one element is associative, **alternative** if the subalgebra generated by any two elements is associative, and of course associative if the subalgebra generated by any three elements is associative.

Thanks to a theorem by Emil Artin, it is known that an algebra A is alternative if and only if the associator is alternating - this provides us with a useful characterisation of alternative algebras - however we omit the proof for the sake of conciseness (see [3] or [8] for more details).

3.2 The Cayley-Dickson Process

From an algebra A, we define a new algebra $A(+) = A \oplus A$ using the direct sum. Addition in this new algebra is done component-wise, and multiplication is done according to the following rule:

$$(a, b)(c, d) = (ac - bd^*, a^*d + cb) \tag{4}$$

where $(a, b)^* = (a^*, -b)$ is known as *conjugation*, and the multiplicative unit is $(1, 0)$. Further, observe that the element $(0, 1)^2 = -1$. This method for constructing new, higher-dimensional algebras from old is known as the *Cayley-Dickson Process*, and gives us an alternative framework to view the construction of the quaternions and octonions. Naturally, we have that the algebra A must be a subalgebra of $A(+)$, and so by applying the Cayley-Dickson Process iteratively starting with $\mathbb{R}$ we obtain the inclusion $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, illustrating how each of these algebras sits inside the next, like a set of Russian babushka dolls. Continuing in this way, we use our newly formalised construction of new algebras from old to begin exploration of deeper questions, such as why the quaternions are noncommutative and octonions nonassociative, and for how long we are able to continue this process of creating normed division algebras. We begin to do this by first identifying a key lemma that collects a number of formulae relating to key properties. These formulae are provided to us by Harvey [3], but follow directly from the definition of multiplication given for the Cayley-Dickson process (equation 4).

Lemma 3.4. *Suppose A is a normed division algebra, and let $A(+)$ denote the algebra obtained via the Cayley-Dickson process. Let a denote $(a, 0)$, ε denote $(0, 1)$, and $x = a + \alpha\varepsilon, y = b + \beta\varepsilon, z = c + \gamma\varepsilon$ all be elements of $A(+)$.*

$$\frac{1}{2}[x, y] = \frac{1}{2}[a, b] + \Im(\bar{\alpha}\beta) + (\beta\Im(a) - \alpha\Im(b))\varepsilon \tag{5}$$

$$[x, y, z] = [a, \bar{\gamma}\beta] + [b, \bar{\alpha}\gamma] + [c, \bar{\beta}\alpha] + \alpha[\bar{b}, \bar{c}]\varepsilon + \beta[a, \bar{c}]\varepsilon + \gamma[a, b]\varepsilon + (\alpha\bar{\beta}\gamma - \gamma\bar{\beta}\alpha)\varepsilon \tag{6}$$

$$[x, \bar{x}, y] = [a, \bar{\beta}, \alpha] + [\alpha, \bar{b}, a]\varepsilon \tag{7}$$

$$\|x\|\,\|y\| - \|xy\| = 2\langle a, [\bar{\beta}, \alpha, \bar{b}]\rangle \tag{8}$$

Noting that equation 7 given in Lemma 3.4 assumed that the algebra A was associative. These formulae then lead us directly to the following corollary, which explicitly tells us which properties are lost and at which point throughout the Cayley-Dickson process.

Corollary 3.5. *Suppose that $A(+)$ is the algebra defined by applying the Cayley-Dickson process to a normed division algebra A.*

$A(+)$ is commutative $\iff$ A is real. (9)

$A(+)$ is associative $\iff$ A is commutative and associative. (10)

$A(+)$ is alternative, $A(+)$ is normed, and A is associative are all equivalent. (11)

Proof. For statement 9, both directions follow from equation 5. For statement 10, both directions again follow directly from equation 6. Statement 11 follows from consideration of equations 7 and 8. $\square$

Applying the above corollary 3.5 to our construction of the four normed division algebras earlier, we have the immediate corollary:

Corollary 3.6.

$\mathbb{C} = \mathbb{R}(+)$ *is commutative, associative and normed.* (12)

$\mathbb{H} = \mathbb{C}(+)$ *is not commutative but is associative and normed.* (13)

$\mathbb{O} = \mathbb{H}(+)$ *is neither commutative nor associative, but is alternative and normed.* (14)

$\mathbb{O}(+)$ *is not commutative, associative, alternative or normed.* (15)

This result provides a great deal of insight into why we observed the loss of certain properties for these higher dimensional algebras, and leads us to our final consideration of this section - the classification of normed division algebras.

3.3 Hurwitz Theorem

The complete classification of normed division algebras is given by a theorem due to Hurwitz, which was published posthumously in 1923. We now state the theorem, and subsequently provide a proof, based on that given by Harvey [3].

Theorem 3.7. *The only normed division algebras are* $\mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $\mathbb{O}$.

Proof. Suppose A is a normed division algebra, and let $A_1 = \Re(A) = \mathbb{R}$. If $A_1 = A$ we are done, else $\exists \varepsilon_1 \in A_1^{\perp}$ such that $-\varepsilon_1^2 = \|\varepsilon_1\| = 1$, so define $A_2 = A_1 + A_1\varepsilon_1$. Then A_2 must be a normed subalgebra of A isomorphic to $\mathbb{C}$. If $A_2 = A$ we are done, else $\exists \varepsilon_2 \in A_2^{\perp}$ such that $-\varepsilon_2^2 = \|\varepsilon_2\| = 1$, so let $A_3 = A_2 + A_2\varepsilon_2$. Note now that $\dim(A_1) = 1$, $\dim(A_2) = 2$ and thus $\dim(A_3) = 4$, so A_3 must be a normed subalgebra of A isomorphic to $\mathbb{H}$. Now if $A_3 = A$ we are done, else $\exists \varepsilon_3 \in A_3^{\perp}$ a such that $-\varepsilon_3^2 = \|\varepsilon_3\| = 1$, so take $A_4 = A_3 + A_3\varepsilon_3$. Then A_4 will have dimension 8, and must be a normed subalgebra isomorphic to $\mathbb{O}$. Now if $A_4 = A$ we are finished, and if not then by repeating the Cayley-Dickson process we must obtain a 16-dimensional algebra with $\mathbb{O}(+)$ as a normed subalgebra. But by corollary 3.6 statement 15, this can not be a normed algebra, and hence can not be a normed division algebra. □

Hurwitz Theorem naturally has many consequences being a statement definitively classifying normed division algebras, as these are algebras which possess a large amount of important structure and thus appear in many fields of study. One such corollary is that which we give (and subsequently prove) below, in which Hurwitz Theorem is used to prove a well-known result specifying which n-spheres permits a group structure.

Corollary 3.8. *The only spheres which permit a group structure are* S^0, S^1 *and* S^3.

Proof. Suppose that $S^{n-1} \subset \mathbb{R}^n$ is some $(n-1)$-sphere which permits a group structure. Then $\mathbb{R}^n$ is a normed vector space with the operations of *addition* and *scalar multiplication* as usual, and the standard Euclidean norm. Now for multiplication define $x \cdot 0 = 0 \cdot x = 0$ for all $x \in \mathbb{R}^n$, and for any nonzero $x, y \in \mathbb{R}^n$ define multiplication by:

$$x \cdot y = \|x\| \, \|y\| \left(\frac{x}{\|x\|} \cdot \frac{y}{\|y\|} \right) \tag{16}$$

where the multiplication $\frac{x}{\|x\|} \cdot \frac{y}{\|y\|}$ is obtained by the fact that each element is in S^{n-1} and thus has a multiplication as defined by the group structure (by assumption). For arbitrary $x \in \mathbb{R}^n$ multiplicative inverses are then given by:

$$x^{-1} = \frac{1}{\|x\|} \left(\frac{x}{\|x\|} \right)^{-1} \tag{17}$$

where again we have obtained the inverse element $\left(\frac{x}{\|x\|} \right)^{-1}$ from the inverse as defined by the group structure. It can then be shown, also using the properties of $\mathbb{R}^n$ as a vector space, that this multiplication is compatible with left and right distributivity, as well as scalar multiplication. Now with this defined multiplication, we have that:

$$\|xy\| = \left\| \|x\| \, \|y\| \left(\frac{x}{\|x\|} \frac{y}{\|y\|} \right) \right\| = \|x\| \, \|y\| \left\| \frac{x}{\|x\|} \frac{y}{\|y\|} \right\| = \|x\| \, \|y\|$$

where we have used the fact that the group is closed, and so $\frac{x}{\|x\|} \frac{y}{\|y\|} \in S^{n-1}$ and thus has unit norm, as well as that norms have the property of absolute homogeneity. Hence we have obtained the norm property for our algebra. We also have that $xy = 0 \implies \|x\| = 0$ or $\|y\| = 0 \implies x = 0$ or $y = 0$ as norms must be positive definite, and so we also have the division property. Thus, we conclude that $\mathbb{R}^n$ must be a normed division algebra, and thus $n = 1, 2, 4$ or 8. But if $n = 8$, then the algebra must have been the octonions which are non-associative, and hence we could not have had a group structure. Thus, the only spheres which permit a group structure are S^0, S^1 and S^3. □

4 Quaternionic Rotations

We now continue in the same geometric fashion with which we wrapped up the last section by considering one of Hamilton's original intents for quaternions - three-dimensional rotations. However, before exploring how quaternions can perform rotations in three-dimensions, we first need to establish a couple of very useful results, for which we thank Porteous [7].

Proposition 4.1. *A quaternion is real if and only if it commutes with every quaternion. In other words, $\mathbb{R}$ is the centre of $\mathbb{H}$.*

Proof. Let $Z(\mathbb{H}) = \{q \in \mathbb{H} : \forall r \in \mathbb{H}, rq = qr\}$ denote the centre of $\mathbb{H}$. ($\Rightarrow$) Suppose $r \in \mathbb{H}$ is real. Then clearly as $rq = qr$, we have $r \in Z(\mathbb{H})$. ($\Leftarrow$) Let $q = a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ be a quaternion which commutes with every other quaternion (i.e. $q \in Z(\mathbb{H})$). Then since q commutes with i: $ai - b + ck - dj = iq = qi = ai - b - ck + dj$ and so $2(ck - dj) = 0 \implies c = d = 0$. Similarly, since q commutes with j: $aj - bk - c + di = jq = qj = aj + bk - c - di$, and since $d = 0$ from above, we have $b = 0$, and so $q = a \in \mathbb{R}$. □

Lemma 4.2. *A quaternion is pure if and only if its square is a non-positive real number.*

Proof. ($\Rightarrow$) Consider the imaginary quaternion $q = bi + cj + dk$, where $b, c, d \in \mathbb{R}$. Then $q^2 = -(b^2 + c^2 + d^2)$ which is real and non-positive. ($\Leftarrow$) Consider $q = a+bi+cj+dk$ where $a, b, c, d \in \mathbb{R}$ such that q^2 is a non-positive real number. Then note that $q^2 = a^2 - b^2 - c^2 - d^2 + 2a(bi + cj + dk)$. So for q^2 to be real, we require either $a = 0$ or $b = c = d = 0$. But if $b = c = d = 0$ then $q^2 = a^2 \geq 0$, and so q^2 is not non-positive. Hence we conclude that $a = 0$ and so $q = bi + cj + dk$ must be imaginary. □

4.1 Construction for rotations in $\mathbb{R}^3$

We are now ready to begin considering how we can perform rotations in $\mathbb{R}^3$ by utilising quaternions.

Proposition 4.3. *Let $R_q : \Im(\mathbb{H}) \to \Im(\mathbb{H})$ be such that $v \mapsto qv\bar{q}$, where $q \in S^3$ is a unit quaternion. Then the map R_q is linear and a rotation in 3-space, i.e. element of $SO(3)$.*

Proof. First let q be a unit quaternion, and note that the map R_q as above is clearly linear: for $\lambda, \mu \in \mathbb{R}$ and $a, b \in \Im(\mathbb{H})$, we have $R_q(\lambda a + \mu b) = q(\lambda a + \mu b)\bar{q} = (\lambda qa + \mu qb)\bar{q} = \lambda qa\bar{q} + \mu qb\bar{q} = \lambda R_q(a) + \mu R_q(b)$. Now write $q = q_0 + uq_1$, where $q_0 = \Re(q)$ and $q_1 = \Im(q)$ with $u \in \Im(\mathbb{H}) \cap S^3$. Given a vector $v \in \mathbb{R}^3$, associate it with an imaginary quaternion using the natural identification of $\mathbb{R}^3$ with the span of i, j, k, and decompose it into $v = \alpha + \beta$, where α is the component of v along $q - q_0$ and β is the component of v orthogonal to $q - q_0$ (the quaternion $q - q_0$ is imaginary and hence can be associated with the corresponding vector (using the aforementioned identification) in $\mathbb{R}^3$). First consider the action of R_q only on α. Since α is the component of v along $q - q_0$, we can write $\alpha = \gamma(q - q_0)$ for some $\gamma \in \mathbb{R}$. Then $R_q(\alpha) = R_q(\gamma(q - q_0)) = q\gamma(q - q_0)\bar{q} = \gamma(qq\bar{q} - q_0q\bar{q}) = \alpha$, and so we conclude that α is invariant under R_q. Now we consider the action of R_q on β: first note that $q \in S^3 \implies q_0^2 - q_1^2u^2 = 1$. But by Lemma 4.2 and since $u \in \Im(\mathbb{H}) \cap S^3$, we must have that $u^2 = -1$, and hence $q_0^2 + q_1^2 = 1$. We can then parameterise this by $q_0 = \cos(\theta), q_1 = \sin(\theta)$ for $\theta \in \mathbb{R}$. Recall $q = \cos(\theta) + u\sin(\theta)$, where u is a unit imaginary quaternion and $\theta \in \mathbb{R}$. Thus:

$$R_q(\beta) = (\cos(\theta) + u\sin(\theta))\beta(\cos(\theta) - u\sin(\theta)) = \beta\cos(2\theta) + (u \times \beta)\sin(2\theta)$$

Noting that a number of trigonometric and vector product identities are used in the above simplification. Hence we conclude that the map R_q is rotating the plane defined by β and $u \times \beta$ (that is, the plane $(q - q_0)^{\perp}$) by an angle of 2θ, where $\theta = \arccos(\Re(q))$. By linearity of the map R_q as shown earlier, we see that R_q will thus be a rotation of any given vector v about the axis given by $q - q_0$ by angle 2θ. □

That this map can perform any rotation of $\mathbb{R}^3$ is clear, and under the operation of composition it can be seen that the set of all such maps forms a group. Suppose $q, r \in \mathbb{H} \cap S^3$, then $R_q \circ R_r(v) = q(rv\bar{r})\bar{q} = qrv\overline{qr} = R_{qr}$, so the set is closed under the operation of composition, which is of course associative. The group inverse $(R_q)^{-1}$ of an element R_q is given by the rotation $R_{\bar{q}}$, and the group identity is the identity rotation map R_1. Define the map $\rho : \mathbb{H} \cap S^3 \to SO(3)$ such that $q \mapsto R_q$. Then ρ is clearly surjective, and is a group homomorphism as $\rho(ab) = R_{ab} = \rho(a) \circ \rho(b)$. Furthermore, if we consider the kernel of ρ, we see that $qv\bar{q} = v \implies qv = vq$, and hence the kernel of ρ is simply the centre of $\mathbb{H}$ intersect S^3, which by proposition 4.1 is just $\mathbb{R} \cap S^3 = \{-1, 1\}$. By the first group isomorphism theorem we then have that $(\mathbb{H} \cap S^3)/\{-1, 1\} \cong SO(3)$.

4.2 Advantages and Applications

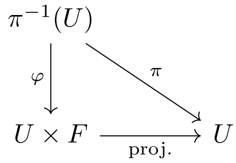
As we have seen in the construction above, quaternions clearly provide a convenient way of performing three-dimensional spatial rotations. If we consider other common methods of rotation, such as rotation matrices or Euler angles (roll, pitch and yaw), this quaternionic approach actually has a number of advantages. For example, to specify the representation of a certain rotation with quaternions we need only 4 numbers, whereas 3×3 rotation matrices require 9. While in both cases we can reduce via the governing equations the number of numbers *actually* required to only 3, the representation remains more compact. Similarly, while in all three of these rotation systems the axis and angle of rotation can be determined, it is much less cumbersome to do so with quaternions. Quaternionic rotations also have the very pragmatic advantage of being more computationally efficient. When performing multiple rotations through the use of technology, rounding error will necessarily accumulate with each successive rotation; with quaternions this is not a significant problem, as in order to maintain rotation we simply divide our defining quaternion by its norm and again have a unit quaternion to rotate with. In comparison, using rotation matrices becomes much more difficult as we require the condition of orthogonality (and with positive determinant), something that can in practice be much harder to renormalise for. Finally, the quaternion based approach has the benefit of not suffering from certain singularities in rotation. When using Euler angles for example, one can suffer from the phenomena known as "gimbal lock"; when the pitch is rotated $90°$ up or down, the yaw and roll rotations will then correspond to the same motion, and so we lose a degree of freedom. This is not seen in quaternionic rotation systems, which is crucial as such scenarios may occur in, for example, aerospace navigation systems, where a plane may find itself in this scenario during particularly steep ascent or descent.

5 The Hopf Fibration

We now move on to our final geometrically-flavoured application of the normed division algebras - the Hopf fibration. This structure is a fundamental object of study in the field of topology, as it enables us to describe higher dimensional spheres in terms of lower dimensional spheres. While we try to provide sufficient definitions and background where necessary, for those unfamiliar with topology and its concepts we recommend Lee [5] as a fantastic reference for an introduction to topology.

Definition 5.1. A **fibration** is a quadruple (T, B, F, π), where T, B, F are topological spaces and $\pi : T \to B$ is a continuous, surjective map satisfying the conditions that:

1. Any point $b \in B$ admits a neighbourhood U such that the preimage $\pi^{-1}(U)$ is homeomorphic to $U \times F$.
2. The homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ is consistent with π. That is, the following diagram must commute:



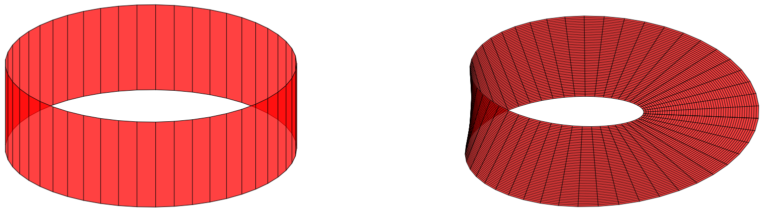
The first of these conditions will be referred to as the condition of being *locally trivial*.

The spaces T, B and F will be referred to as the *total space*, *base space*, and the *fibre* respectively, while the map π will be referred to as the *fibre map*. Fibrations as described in the above definition will be denoted by $F \to T \xrightarrow{\pi} B$.

Remark. The preimage of a point $\pi^{-1}(b)$ is homeomorphic to the fibre F, because for any $b \in B$ the restriction of the homeomorphism φ^{-1} to $\{b\} \times F$ demonstrates that $\{b\} \times F$ is homeomorphic to $\pi^{-1}(b)$, and $\{b\} \times F$ is homeomorphic to F.

We now aim to provide the reader with some intuition behind these fibrations and what kinds of spaces they can create. Let B, F be topological spaces, and take $T = B \times F$, with π the projection map. Then (T, B, F, π) clearly form a fibration, and so certainly are locally trivial. Moreover, this fibration has the total space T being homeomorphic to $B \times F$ not only locally but also globally. These are referred to as *trivial* fibrations, as they are quite simply Cartesian products. Fibrations which are not globally trivial can in some sense be thought of as "twisted" versions of these trivial fibrations, which we now illustrate with some brief examples.

Example 5.2. The Möbius strip is the total space of a fibration with base space S^1 and fibre I some open interval. Taking the corresponding trivial example of a fibration in which the total space $T = S^1 \times I$ results in a cylinder. Of course the Möbius strip can be realised as a rectangle with a pair of opposite sides of the rectangle glued together in the opposite orientation (i.e. a cylinder with a twist in it), via the simple equivalence relation $\sim$ on $[0,1] \times [0,1]$ such that the top and bottom sides are identified by $(x, 0) \sim (1 - x, 1)$ for $0 \leq x \leq 1$. This example is illustrated below in figure 1, which was plotted via MATLAB (code in Appendix).



(a) Cylinder (b) Möbius Strip

Figure 1: The Cylinder (left) and Möbius Strip (right) plotted via MATLAB (code in Appendix).

Example 5.3. The Klein bottle is the total space of a fibration with base space S^1 and fibre S^1. Taking the corresponding trivial example of a fibration in which the total space $T = S^1 \times S^1$ results in the 2-torus $\mathbb{T}^2$. Again, the Klein bottle can be thought of as a 2-torus with a "twist" in it.

5.1 Construction via Projective Geometry

We now turn our attention to one of the most fundamental early examples of a fibre bundle thanks to Heinz Hopf in 1931, the Hopf fibration. The Hopf fibration $S^1 \to S^3 \xrightarrow{\pi} S^2$ is a fibering of spheres by spheres, in the sense that for every point $p \in S^2$ the preimage $\pi^{-1}(p)$ is homeomorphic to S^1 in the form of a great circle of S^3. This fibration allows one to in some sense "see" the 3-sphere using visualisations via stereographic projection, but also finds a number of other applications in, for example, quantum mechanics and electromagnetics [10].

First observe that the 3-sphere can be realised as a subset of $\mathbb{C}^2$ as $S^3 = \{(z, v) \in \mathbb{C}^2 : z\bar{z} + v\bar{v} = 1\}$. We now define a space that will be crucial in our construction of the Hopf fibration.

Definition 5.4. Define the equivalence relation $\sim$ on $\mathbb{C}^2$ such that $x \sim y \iff x = \lambda y$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. Then the **complex projective line**, denoted by $\mathbb{CP}^1$, is the quotient space $(\mathbb{C}^2\backslash\{\mathbf{0}\})\backslash \sim$.

Under this definition it becomes clear that $\mathbb{CP}^1$ is simply the set of lines through the origin of $\mathbb{C}^2$. Note that to denote an element of $\mathbb{CP}^1$ we use *homogeneous coordinates* $[z : v]$, which are such that if the homogeneous coordinates of a point are multiplied by a non-zero scalar they represent the same point; that is $[z : v] = [\lambda z : \lambda v]$ for any $\lambda \in \mathbb{C}/\{0\}$. This is of course consistent with the fact that an element of $\mathbb{CP}^1$ is simply an equivalence class of $\sim$. The purpose of $\mathbb{CP}^1$ will hopefully now become clear through the following claim, a more detailed proof of which can be found, for example, thanks to Liu [6].

Proposition 5.5. *The complex projective line $\mathbb{CP}^1$ is homeomorphic to the 2-sphere S^2.*

Proof. Observe that $\mathbb{CP}^1 = \{[z : 1] \in \mathbb{CP}^1 : z \in \mathbb{C}\} \cup [1 : 0]$. That $\mathbb{CP}^1$ is homeomorphic to the extended complex plane (also known as the Riemann sphere) $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ should now become clear, as it is simply a copy of $\mathbb{C}$ together with an additional point (as is the case with the Riemann sphere). This is also demonstrated by the continuous maps $\phi : \mathbb{CP}^1 \to \widehat{\mathbb{C}}$ such that $[u : v] \mapsto u/v$ if $v \neq 0$ and $\{\infty\}$ if $v = 0$, and $\phi^{-1} : \widehat{\mathbb{C}} \to \mathbb{CP}^1$ such that $z \mapsto [z : 1]$ if $z \neq \{\infty\}$ and $[1 : 0]$ if $z = \{\infty\}$. But of course $\widehat{\mathbb{C}}$ is homeomorphic to S^2 through stereographic projection, and thus $\mathbb{CP}^1$ must be homeomorphic to the 2-sphere. Alternatively as an explicit approach, one can simply verify that the map $\varphi : \mathbb{CP}^1 \to S^2$ such that:

$$[z : v] \mapsto \frac{(2\Re(2z\bar{v}), 2\Im(2z\bar{v}), z\bar{z} - v\bar{v})}{z\bar{z} + v\bar{v}} \tag{18}$$

is indeed a homeomorphism between the two spaces. Note that this map is simply that due to stereographic projection, and so the two approaches are fundamentally the same. □

This homeomorphism is crucial to our verification of the Hopf fibration, as it allows us to work with the base space being $\mathbb{CP}^1$. We now define the fibre map π for our fibre bundle, which we shall refer to as the *Hopf map*. Let $\pi : S^3 \to \mathbb{CP}^1$ be the map such that $(z, v) \mapsto [z : v]$. Then consider the preimage of a point $[z : v] \in \mathbb{CP}^1$:

$$\pi^{-1}([z : v]) = \{(\hat{z}, \hat{v}) \in S^3 : \hat{z} = \lambda z, \hat{v} = \lambda v, \lambda z, v \in \mathbb{C}\}$$

But note that as $(\hat{z}, \hat{v}) \in S^3$, we must have that $\hat{z}\bar{\hat{z}} + \hat{v}\bar{\hat{v}} = 1$, and hence $\lambda\bar{\lambda}\,(z\bar{z} + v\bar{v}) = 1$. But seeing as $(z, v) \in S^3$ also, we must have $\lambda\bar{\lambda} = 1$. Of course any such $\lambda = e^{i\theta}$ for some $-\pi < \theta \leq \pi$, and so the preimage of a point $[z : v] \in \mathbb{CP}^1$ is seen to be a great circle of S^3, that is, the fibres are homeomorphic to S^1. We now verify the condition of local triviality, i.e. that for any $[z : v] \in \mathbb{CP}^1$ there is a neighbourhood U such that $\pi^{-1}(U) \cong U \times S^1$.

Let $U_1 = S^2 \backslash (0, 0, -1)$ and $U_2 = S^2 \backslash (0, 0, 1)$. These are clearly seen to be open as S^2 has the subspace topology, and $S^2 \backslash (0, 0, \pm 1) = S^2 \cap (\mathbb{R}^3 \backslash (0, 0, \pm 1))$ with $\mathbb{R}^3 \backslash (0, 0, \pm 1)$ open (if $\mathbf{x} \in \mathbb{R}^3 \backslash (0, 0, \pm 1)$, then taking $\delta = \|\mathbf{x} - (0, 0, \pm 1)\| / 2$ shows $B_\delta(\mathbf{x}) \subset \mathbb{R}^3 \backslash (0, 0, \pm 1)$). Hence U_1 and U_2 form an open covering of S^2. Note that thanks to the homeomorphism obtained in proposition 5.5, when we refer to a point in either U_1 or U_2, we shall commonly refer to it by its corresponding element in $\mathbb{CP}^1$ or $\widehat{\mathbb{C}}$.

Now define $\varphi_1 : \pi^{-1}(U_1) \to U_1 \times S^1$ such that

$$(z, v) \mapsto \left(\frac{v}{z}, \frac{z}{\sqrt{z\bar{z}}} \right) \tag{19}$$

and $\varphi_2 : \pi^{-1}(U_2) \to U_2 \times S^1$ such that

$$(z, v) \mapsto \left(\frac{z}{v}, \frac{v}{\sqrt{v\bar{v}}} \right) \tag{20}$$

Each of these maps are continuous, as they are simply compositions of continuous functions. Further, they have inverses:

$$\varphi_1^{-1}\left(\lambda, e^{i\theta}\right) = \frac{\left(e^{i\theta}, \lambda e^{i\theta}\right)}{\sqrt{1 + \lambda\bar{\lambda}}} \tag{21}$$

$$\varphi_2^{-1}\left(\lambda, e^{i\theta}\right) = \frac{\left(\lambda e^{i\theta}, e^{i\theta}\right)}{\sqrt{1 + \lambda\bar{\lambda}}} \tag{22}$$

which are also continuous themselves being compositions of continuous functions. That these are indeed inverses is a simple verification left to the reader. The existence of a continuous inverse function for both φ_1 and φ_2 implies that these are indeed homeomorphisms, and hence that $\pi^{-1}(U_1) \cong U_1 \times S^1$ and $\pi^{-1}(U_2) \cong U_2 \times S^1$. Thus we have verified the condition of local triviality for the Hopf fibration. Finally, as the homeomorphisms φ_1 and φ_2 are both consistent with the Hopf map π (i.e. the diagram given in definition 5.1 commutes), we conclude our verification that the Hopf fibration is indeed a fibering of spheres by spheres.

That this is not an example of a trivial fibration can be shown in a number of ways, however it can most simply be seen by the following proof via the fundamental group, for which we thank Hatcher [4]. We wish to show that while the 3-sphere is locally homeomorphic to the product of the 1-sphere with the 2-sphere, globally this is not true. Note that the fundamental group of the 3-sphere $\pi_1(S^3)$ is trivial as S^3 is simply connected, while $\pi_1(S^1 \times S^2) \cong \pi_1(S^1) \cong \mathbb{Z}$. Hence as they do not have the same fundamental group, the two spaces can not be homeomorphic.

5.2 Generalisations via Quaternions and Octonions

We now return to the world of quaternions and octonions to introduce the higher-dimensional analogues of the Hopf fibration. In fact we shall see that we do not really need to adjust much of our above construction at all to do this. Just as we defined S^3 as a subset of $\mathbb{C}^2$, define S^7 and S^{15} as subsets of $\mathbb{H}^2$ and $\mathbb{O}^2$ respectively:

$$S^7 = \{(p, q) \in \mathbb{H}^2 : p\bar{p} + q\bar{q} = 1\}$$
$$S^{15} = \{(o, u) \in \mathbb{O}^2 : o\bar{o} + u\bar{u} = 1\}$$

And just as we defined $\mathbb{CP}^1$ by taking the quotient of $\mathbb{C}^2\backslash\{\mathbf{0}\}$ with the equivalence relation $\sim$, define $\sim_{\mathbb{H}}$ and $\sim_{\mathbb{O}}$ such that for any $p, q \in \mathbb{H}^2$ and any $o, u \in \mathbb{O}^2, p \sim_{\mathbb{H}} q \iff p = \alpha q$ for some $\alpha \in \mathbb{H}\backslash\{0\}$ and $o \sim_{\mathbb{O}} u \iff o = \omega u$ for some $\omega \in \mathbb{O}\backslash\{0\}$. Then we can define the **quaternionic projective line** $\mathbb{HP}^1$ and the **octonionc projective line** $\mathbb{OP}^1$ as the quotient spaces $(\mathbb{H}^2/\{\mathbf{0}\})/\sim$ and $(\mathbb{O}^2/\{\mathbf{0}\})/\sim$ respectively. Again we use homogeneous coordinates to denote an element of these spaces. We now present similar homeomorphisms to those used in the original construction.

Proposition 5.6. *The quaternionic projective line $\mathbb{HP}^1$ and the octonionic projective line $\mathbb{OP}^1$ are homeomorphic to the 4-sphere S^4 and the 8-sphere S^8 respectively.*

The proof of the above proposition is identical to the proof that $\mathbb{CP}^1 \cong S^2$ - even the same map given in equation 18 can be used - and so we omit it. Further, we can get even more mileage out of the original construction in our generalisation: the fibre map π is the same! Hence define $\pi_{\mathbb{H}} : S^7 \to \mathbb{HP}^1$ and $\pi_{\mathbb{O}} : S^{15} \to \mathbb{OP}^1$ such that $(\alpha, \beta) \mapsto [\alpha : \beta]$ in both cases. Consideration of the preimages in the same fashion as earlier then shows that $\pi_{\mathbb{H}}^{-1}([p : q]) = S^3$ and $\pi_{\mathbb{O}}^{-1}([o : u]) = S^7$ - these are the fibres for our higher-dimensional Hopf fibrations.[1] Thus we have determined the fibrations $S^3 \to S^7 \xrightarrow{\pi} S^4$ and $S^7 \to S^{15} \xrightarrow{\pi} S^8$.

6 Conclusion and Further Study

Hopefully it is now clear that the quaternions and octonions truly are a fascinating object of study within mathematics that also find a number of practical applications. Their uses in fields such as number theory, algebra, geometry and topology emphasises the wide range of phenomena in which the normed division algebras can be used, and indeed understanding them can lead to fantastic insight and new perspectives. These connections are of course much deeper and richer than we are able to provide in detail, and as such this leads us to consider possible directions for further study. The most notable of course is the well-known application of octonions to Lie groups and algebras, as the smallest exceptional Lie group G_2 can be realised as the automorphism group of the octonions. The octonions also find a number of curious applications in supersymmetry and quantum mechanics which are not yet well-understood.

Acknowledgements

I would like to thank my supervisor Dr. Thomas Leistner for his support and guidance, without whom this research project would not have been possible.

[1]Verifying that $\pi_{\mathbb{H}}$ and $\pi_{\mathbb{O}}$ are both homeomorphisms (and the diagram given in definition 5.1 commutes) is once again the same as before, and so for the sake of conciseness we omit the details.

Appendices

Here we provide the MATLAB code used to produce figure 1 seen in section 5. Note that this is only one of many ways to produce such a diagram illustrating a cylinder and a Möbius strip.

```matlab
% Author: James McCusker
% Date: 21st February, 2019
% AMSI VRS 2018/19 - Quaternions and Octonions
% Supervisor: Dr. Thomas Leistner
% University of Adelaide
% Cylinder and Mobius strip surface plots
numpoints = 50; % select number of points to create mesh with

r = 1; h = 1; % radius and height of cylinder
theta = linspace(0,2*pi,numpoints); % theta from 0 to 2pi
% we use parametric equations for circular cylinder
xCyl = repmat(r*cos(theta),2,1);
yCyl = repmat(r*sin(theta),2,1);
zCyl = [zeros(1,numpoints) ; h*ones(1,numpoints)];
figure(1)
hold on
axis([-1.5 1.5 -1.5 1.5 -0.5 1.5]); % set axis for viewing
set(gca, 'Visible', 'off') % remove axis and axis ticks from plot
cylinder = surf(xCyl,yCyl,zCyl,'FaceColor','r','FaceAlpha',0.75,'LineStyle','-')
% range of parameters u, v for Mobius strip
u = linspace(0,2*pi,numpoints);
v = linspace(-1,1,numpoints);
[u,v] = meshgrid(u,v);
% use parametric equations for mobius strip
xMob = (1+(v/2).*cos(u/2)).*cos(u);
yMob = (1+(v/2).*cos(u/2)).*sin(u);
zMob = (v/2).*sin(u/2);
figure(2)
hold on
axis([-1.5 1.5 -1.5 1.5 -0.5 1.5]); % set axis for viewing
set(gca, 'Visible', 'off') % remove axis and axis ticks from plot
mobius = surf(xMob,yMob,zMob,'FaceColor','r','FaceAlpha',0.75,'LineStyle','-')
```

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