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Random Walks on Derived Graphs

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1 Introduction

A random walk is a Markov process; that is one whose future behaviour does not depend on its past only its present position, which charts successive steps on some mathematical space chosen according to a probability distribution. A random walk is called *simple* if the size of each step is fixed, and *symmetric* if the probabilities of each possible step at a given position are equal. For example, we may study a simple symmetric random walk on the d-dimensional integer lattice. The path that the random walk takes is formed by starting at 0 in $\mathbb{Z}^d$ and randomly adding or subtracting an element of the standard basis of $\mathbb{Z}^d$ at each step. The usefulness of random walks extends to many different disciplines in the modelling of stochastic processes, or processes that may not be random in nature but are best analysed from a probabilistic perspective. For instance, the recurrence property of a simple random walk on the integers can be used to model the financial concept of Gambler's Ruin, and show that a gambler who plays a fair game (one with equal probability of winning or losing each round) with a finite amount of wealth will almost surely be bankrupted if they play forever. Random walks also find application in many physical sciences. The theory of Diffusion-Limited Aggregation uses random walks to describe the aggregation of particles in any system where natural diffusion is the primary force of movement such as electrodeposition, Hele-Shaw flow, mineral deposits, and dielectric breakdown. Brownian motion, or the random movements of particles suspended in liquid caused by transfer of kinetic energy from and to the molecules of that liquid, is also modelled by a random walk.

Therefore, it is of interest to define random walks on as general a space as possible. *Groups* are one such space on which random walks have been well studied. We will review some of the existing theory of random walks on groups in this report and then generalise it further.

2 Preliminaries

A group is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G,\ (g_1, g_2) \mapsto g_1 g_2$ such that the following properties hold:

1. For every $g_1, g_2, g_3 \in G$, $(g_1 g_2) g_3 = g_1(g_2 g_3)$.
2. There exists an element $e \in G$ such that $eg = ge = g$ for every $g \in G$.
3. For every $g \in G$ there exists an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

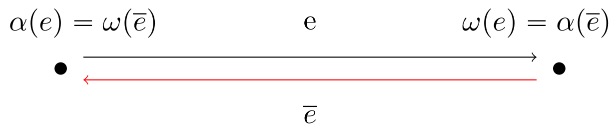
Groups form an important part of modern mathematics. Examples include the integers, real and complex numbers, and the symmetries of other objects given by reversible operations on each group. Each group is able to be represented by a particular type of *graph* called a *Cayley graph*.

Firstly, we say what precisely is a graph: A graph Γ is defined to be a 5-tuple $\Gamma = (V(\Gamma), E(\Gamma), \alpha, \omega, \overline{\cdot})$ where:

- $V(\Gamma)$ is the set of nodes or *vertices* of the graph,
- $E(\Gamma)$ is the set of *directed edges* travelling from one vertex to another,
- $\alpha(e) : E(\Gamma) \to V(\Gamma)$ gives the starting point of the edge e,
- $\omega(e) : E(\Gamma) \to V(\Gamma)$ gives the end point of the edge e,

The map $\overline{\cdot} : E(\Gamma) \to E(\Gamma),\ e \mapsto \overline{e}$ satisfies the following:

1. $\overline{e} \neq e \ \forall \ e \in E(\Gamma)$
2. $\overline{\overline{e}} = e$
3. $\alpha(\overline{e}) = \omega(e)$



In black: the edge e with starting point $\alpha(e)$ and ending point $\omega(e)$, in red: the corresponding reverse edge $\overline{e}$

Secondly, for a group G, a set of *generators* of G is a set S such that S is symmetric, that is $S = S^{-1}$, for the identity e $e \notin S$ and for each $g \in G$ there are $(n_1, n_2, ...n_k) \in \mathbb{N}$ and $(s_1, s_2, ..., s_k) \in S$ such that $s_1^{n_1} s_2^{n_2} ... s_k^{n_k} = g$. That is, every element in the group is equal to a finite product of elements in the generating set.

The Cayley graph is given by $\Gamma(G, S)$ with $V(\Gamma(G, S)) = G$, $E(\Gamma(G, S)) = \{(g, gs) : g \in G, s \in S\}$, $\alpha(g, gs) = g$, $\omega(g, gs) = gs$, and $(\overline{g, gs}) = (gs, g)$. Intuitively, it is the graph with a vertex for each element in the group and an edge from one vertex to another with corresponding group elements a, b respectively if there is a generator s such that $b = as$. Cayley graphs may be finite or infinite.

Example 1. *The d-dimensional integer lattice is the Cayley graph of the group* $\mathbb{Z}^d$.

Note that the random walk on a group coincides with the random walk on its Cayley graph.

We next look at a construction of a graph called the *derived graph* which generalises the Cayley graph in that each Cayley graph may be expressed as a derived graph.

Suppose we have a graph Γ and for some group G a function $c : E(\Gamma) \to G$ such that $c(\overline{e}) = c(e)^{-1}$ and call it the *voltage assignment*. For each edge in Γ we assign a *voltage* from the group G. The pair (Γ, c) is called a *voltage graph*. The *derived graph* is written $\Gamma \times_c G$. Its vertices are given by

$V(\Gamma \times_c G) = V(\Gamma) \times G$. Its edges are given by $E(\Gamma \times_c G) = E(\Gamma) \times G$. For each $e \in E(\Gamma)$ and $g \in G$, $\alpha((e,g)) := (\alpha(e), g)$ and $\omega((e,g)) := (\omega(e), gc(e))$, and $\overline{(e,g)} = (\overline{e}, gc(e))$. We now show that the definition of $\overline{(e,g)}$ satisfies the definition of $\overline{\cdot}$ given previously:

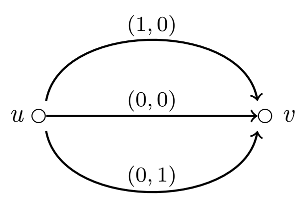
Proposition 1. *$\Gamma \times_c G$ is a graph.*

Proof. 1) $\overline{(e,g)} = (\overline{e}, gc(e)) \neq (e,g)$ since $e \neq \overline{e}$ by definition.
2) $\overline{\overline{(e,g)}} = \overline{(\overline{e}, gc(e))} = (\overline{\overline{e}}, gc(e)c(\overline{e}) = (e, gc(e)c(e)^{-1}) = (e,g)$.
3) $\alpha(\overline{(e,g)}) = (\alpha(\overline{e}), gc(e)) = (\omega(e), gc(e)) = \omega((e,g))$. □

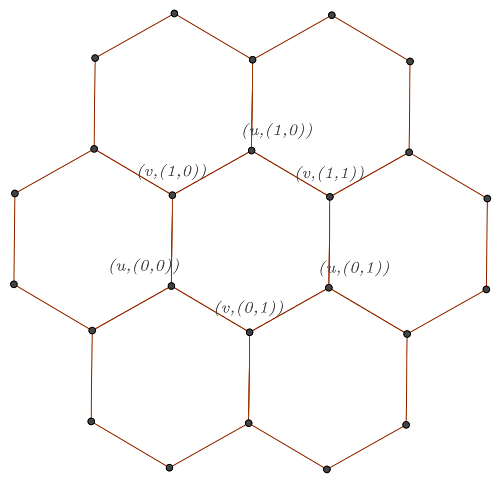
Example 2. *By taking the voltage graph Γ to be a single vertex with a loop for each element and its inverse in the generating set S of a given group G, the derived graph of the voltage graph $\Gamma \times_c G$ coincides with the Cayley graph of the group G. For instance, given a voltage graph consisting of a single vertex and two loops, assigned $(1,0),(0,1) \in \mathbb{Z}^2$ respectively, the derived graph is the 2-dimensional integer lattice.*

Example 3. *The next most simple, and more interesting, example is given pictorially by the voltage graph below labelled with $(1,0),(0,0),(0,1) \in \mathbb{Z}^2$.*

The voltage graph Γ



The derived graph $\Gamma \times_c \mathbb{Z}^2$



Therefore, this report seeks to generalise in the same way random walks on groups to random walks on derived graphs, to study the properties of both.

3 Random Walks on Groups

Let $\{X_i\}_{i=1}^{\infty}$ be a set of topological spaces and $\prod_{i=1}^{\infty} X_i := \{(x_1, x_2, ...)|x_i \in X_i\}$. Then the *product topology* has the base of open sets $\{U = \prod_{i=1}^{\infty} U_i \mid each\ U_i \subseteq X_i\ is\ open,\ U_i = X_i\ for\ all\ but\ finitely\ many\ i\}$ For example, let each $X_i = \mathbb{Z}_2 = \{0, 1\}$ equipped with the discrete topology $\mathcal{T} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Then $X^{\mathbb{N}} = \prod_{i=1}^{\infty}\{0, 1\} = \{x_1, x_2, ...|x_i \in \{0, 1\}\}$. Let $\lambda = \{\lambda_i\}_{i=1}^{n}$ be a finite sequence then $\mathcal{T}$ has a base given by sets $Z(\lambda) = \{\{x_i\}_{i=1}^{\infty}|x_i = \lambda_i\ for\ 1 \leqslant i \leqslant n\}$.

Let G be a countable group. Give G the Borel sigma algebra generated by the discrete topology, and let μ be a probability measure on the Borel sets. Then (G, μ) is a measure space such that the support of μ generates G, with the support defined as follows:

$$supp(\mu) = \{g \in G : \mu(g) \neq 0\}$$

Then let $G^{\mathbb{N}} = \prod_{i\in\mathbb{N}} G$ be equipped with the product topology and let Σ be the Borel sigma-algebra on $G^{\mathbb{N}}$. We write $\mu^{\mathbb{N}}$ to mean the product measure on the Borel sigma algebra on the product topology on $G^{\mathbb{N}}$ where for U_i open in G, $\mu^{\mathbb{N}}(\prod_{i=1}^{\infty} U_i) = \prod_{i=1}^{\infty} \mu(U_i) \in [0, 1]$. Let (X, Σ_X, μ) be a measure space, $f : X \to Y$. For a function we write

$$f_*(\mu^{\mathbb{N}})(g) = \mu^{\mathbb{N}}(f^{-1}(g))$$

to mean the pushforward measure. The pushforward sigma algebra is $\Sigma_Y = \{A \subseteq Y|f^{-1}(A) \in \Sigma_X\}$ Define the map

$$\begin{array}{llll} \phi : & G^{\mathbb{N}} & \longrightarrow & G^{\mathbb{N}} \\ & (h_1, h_2, h_3, ...) & \longmapsto & (h_1, h_1h_2, h_1h_2h_3, ...) \end{array}$$

Remark 1. *In fact, ϕ is a homeomorphism.*

Then we may define a probability measure on $G^{\mathbb{N}}$, $\mathbb{P} = \phi_*(\mu^{\mathbb{N}})$.

Example 4. *Let $G = \mathbb{Z}$, $\mu(\{1\}) = \frac{1}{2}$, $\mu(\{-1\}) = \frac{1}{2}$. We may use the probability measure we have defined to find the probability of the simple random walk on $\mathbb{Z}$ being at a fixed point at a fixed time. Let a random walk $U = \mathbb{Z} \times \{2\} \times \mathbb{Z} \times \mathbb{Z} \times ... \in \Sigma$, then*

$$\begin{aligned} \mathbb{P}(U) &= \mu^{\mathbb{N}}(\phi^{-1}(U) \\ &= \mu^{\mathbb{N}}(\{(n, n_2, ...)|n_2 = 2 - n\}) \\ &= \mu(\{1\}) \cdot \mu(\{1\}) + \mu(\{3\})\mu(\{-1\}) \\ &= \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{4} \end{aligned}$$

Which is clearly the probability of the random walk being at 2 in its second step.

For each $g \in G$, $\{g_n\}_{n=1}^{\infty} \in G^{\mathbb{N}}$ we define the action of G on $G^{\mathbb{N}}$ to be

$$g(g_1, g_2, ...) = (gg_1, gg_2, ...)$$

If $A \subseteq \Sigma$ we have the following function

$$(g_*\mathbb{P})[A] = \mathbb{P}[g^{-1}A]$$

which is the walk starting at g instead of the identity of $G^{\mathbb{N}}$, since for each $h_i \in G$ and sequence starting from g, $A = (gh_1, gh_1h_2, gh_1h_2h_3, ...)$, $g_*\mathbb{P}[A] = \mathbb{P}[g^{-1}A] = \mathbb{P}[(h_1, h_1h_2, h_1h_2h_3, ...)]$ which reduces to our previous case. Let $\mathcal{T}_n \subset \Sigma$ be a sigma-algebra where $\mathcal{T}_n = \{(g_{n+1}, g_{n+2}, ...)|\{g_n\}_{n=1}^{\infty} \in G^{\mathbb{N}}\}$, then we define $\mathcal{T} = \bigcap_n \mathcal{T}_n$ to be the *tail sigma-algebra*. Elements of the tail sigma-algebra are events which are independent of any finite prefix of the sequence.

These are the class of random walks which are of interest to us, as we want to study the *eventual* behaviour of random walks.

First, we define a μ harmonic function.
A function is μ harmonic if for all $g \in G$

$$f(g) = \sum_{h \in G} f(gh)\mu(h)$$

Which, since we require G to be the group generated by $supp(\mu)$ intuitively means that $f(g)$ takes the μ average of the value of the function around g.

In fact, every bounded tail measurable random variable is harmonic with respect to our probability measure μ. For T a bounded tail measurable random variable, if we define

$$f(g) = g_*\mathbb{E}[T] = \int_{G^{\mathbb{N}}} T(gh_1, gh_2, ...)d\mathbb{P}(h_1, h_2, ...)$$

$f(g)$ is μ harmonic in the sense defined above.

Let $H^{\infty}(G, \mu)$ denote the bounded μ harmonic functions on the group G and $L^{\infty}(G^{\mathbb{N}}, \mathcal{T}, \mathbb{P})$ the bounded tail measurable random walks.

The inverse mapping is the *Furstenberg transform* given by

$$\begin{aligned} \Phi : H^{\infty}(G, \mu) &\longrightarrow L^{\infty}(G^{\mathbb{N}}, \mathcal{T}, \mathbb{P}) \\ f &\longmapsto \lim_n f(Z_n) \end{aligned}$$

The Furstenberg transform turns out to be bijective, allowing us to identify the bounded tail random variables with the μ harmonic functions.

We have the following result about μ harmonic functions on abelian groups:

Theorem 1. *Let G be an abelian group with a probability measure μ. If f is a bounded μ-harmonic function then f is trivial.*

This means that given a tail event T and a μ harmonic function $f = \Phi^{-1}(\mathbb{1}_T)$, since f is constant, so too is $\mathbb{1}_T$. But then this means that T is the whole event space and has probability one, or has probability 0.

4 Random Walks on Derived Graphs

We will follow much the same steps as in the case of random walks on groups. Let $\Gamma \times_c G$ be a derived graph with voltage graph (Γ, c) labelled by a group G. Let

$$V^\infty = \{(v_i, g_i)_{i=0}^\infty \in V(\Gamma \times_c G)^{\mathbb{N}} \mid \forall i \geqslant 1 \ \exists (e_i, g_i) \ \alpha(e_i, g_i) = (v_{i-1}, g_{i-1}), \ \omega(e_i, g_i) = (v_i, g_i)\}$$

be the set of paths of vertices through the derived graph. Let

$$E^\infty(\Gamma \times_c G) = \{(e_i, g_i)_{i=1}^\infty \in E(\Gamma \times_c G)^{\mathbb{N})} \mid \omega(e_i) = \alpha(e_{i+1}, \ g_{i+1} = g_i c(e_i)\}$$

be the set of paths of edges through the derived graph. We may then define the following function which maps edges in the derived graph to sequences of vertices:

$$\begin{aligned} \phi \ : \ E^\infty(\Gamma \times_c G) &\longrightarrow V^\infty \\ (e_i, g_i)_{i=1}^\infty &\longmapsto (\alpha(e_i), g_i)_{i=1}^\infty \end{aligned}$$

And we define the function which maps edges in the voltage graph to edges in the derived graph:

$$\begin{aligned} \pi \ : \ E^\infty(\Gamma) &\longrightarrow E^\infty(\Gamma \times_c G) \\ (e_i)_{i=1}^\infty &\longmapsto (e_i, g_i)_{i=1}^\infty = ((e_1, 0), (e_2, c(e_1)), (e_3, c(e_2)c(e_1))...) \end{aligned}$$

Suppose μ is a probability measure on the group G generated by S with $supp(\mu) = S$ and $range(c) = supp(\mu)$.

On the voltage graph, we give each edge exiting a given vertex a normalised probability ν: $\nu(e) = \frac{\mu(c(e))}{w_{\alpha(e)}}$ Where $w_v = \sum_{\alpha(e)=v} \mu(c(e))$

Then we equip the paths of vertices in the derived graph $V^\infty(\Gamma \times_c G)$ with the probability measure $\mathbb{P} = \phi_* \pi_* \nu^{\mathbb{N}}$. We will write $\mathbb{P}_{(v,g)}$ to mean the measure of walks starting at (v, g).

A tail random variable is defined in the same way as in the case of random walks on groups, considering $\mathcal{T}_n = \{(v_{n+1}, g_{n+1}, v_{n+2}, g_{n+2}, ...)|(v_n, g_n)_{n=1}^{\infty} \in V(\Gamma \times_c G)\}$ and $\mathcal{T} = \bigcap_n \mathcal{T}_n$ the tail sigma-algebra.

We define the μ harmonic function f on a derived graph:

$$f(v, g) = \sum_{\alpha(e)=v} f(\omega(e), gc(e))\nu(e)$$