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Cayley Graphs of Finite Semigroups

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This project will seek to find interesting and relevant properties of the Cayley graphs of finite semigroups.

We will investigate how the algebraic properties of semigroups relate to their graph theoretic properties. We will start with small cases, primarily a survey of semigroups of sizes 2 and 3 and then selected examples from semigroups of size 4-6. Where possible we will extend these results to arbitrarily large or even infinitely large semigroups.

1 Background

1.1 Semigroup Theory

A semigroup is a set S with an operation $*$ such that

(i) For all $a, b \in S, a * b \in S$. (Closure)

(ii) For all $a, b, c \in S, a * (b * c) = (a * b) * c$. (Associativity)

For convenience we will omit $*$ so that $a * b$ will be written ab. We will introduce some important concepts in semigroup theory here. However, some definitions and basic results will be provided when discussing the findings of the project.

If for every $a, b \in S, ab = ba$ then we say S is commutative. If there exists an element $a \in S$ such $as = a[sa = a]$ for all $s \in S$ then we call a a left [right] zero. An element that is both a left and right zero is called a zero. If every element is a left [right] zero then S is called a left [right] zero semigroup.

An element e of a semigroup S is a left [right] identity if for all $s \in S$, $es = s$ $[se = e]$ and a (two-sided) identity if $es = se = s$ for all $s \in S$. We will make use of the notation S^1 to mean a semigroup with an adjoined identity if it does not already have one.

We also commonly use the letter e to denote an idempotent; that is an element whose square is itself ($e^2 = e$). Unlike in group theory where every group has one and only one idempotent, namely the identity, a semigroup can have as many idempotents as elements (or none at all, however a finite semigroup must have at least one idempotent). A semigroup where every element is idempotent is called a band.

An element a of a semigroup S is said to be left [right] cancellable if, for any $x, y \in S$, $ax = ay$ $[xa = ya] \Rightarrow x = y$. If every element in S is left [right] cancellable, then we say S is left [right] cancellative.

If for an element a of a semigroup S there exists $x \in S$ such that $axa = a$ then we say that a is regular. If every element of S is regular we say S is a regular semigroup. Two elements $a, b \in S$ are said to be inverses of each other if $aba = a$ and $bab = b$. In fact if a is regular in S and $axa = a$ then a has at least one inverse, namely xax (this is easily verified). S is called an inverse semigroup if every element of S has a unique inverse.

If I is a non-empty subset of a semigroup S and $SI \subseteq I$ [$IS \subseteq I$], then I is called a left [right] ideal of S. An ideal of S is a non-empty subset that is both a left and right ideal of S. For an element $a \in S$, we call S^1a [aS^1] the left [right] principal ideal generated by a, and S^1aS^1 the principal two-sided ideal.

This brings us onto an important concept in semigroup theory, that of Green's relations:

- $\mathscr{L}$-relation: $a\mathscr{L}b \iff S^1a{=}S^1b$ i.e. $\exists u, v \in S^1$ such that $ua = b$ and $vb = a$.
- $\mathscr{R}$-relation: $a\mathscr{R}b \iff aS^1{=}bS^1$ i.e. $\exists u, v \in S^1$ such that $au = b$ and $bv = a$.
- $\mathscr{D}$-relation: $a\mathscr{D}b \iff \exists c \in S$ such that $a\mathscr{L}c$ and $c\mathscr{R}b$.
- $\mathscr{J}$-relation: $a\mathscr{J}b \iff S^1aS^1{=}S^1bS^1$ i.e. $\exists s, t, u, v \in S^1$ such that $sau = b$ and $tbv = a$; and finally,
- $\mathscr{H}$-relation: $a\mathscr{H}b \iff a\mathscr{L}b$ and $a\mathscr{R}b$. i.e. $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$.

For our purposes, it is important to note that:

Proposition 1. *In a finite semigroup S, $\mathscr{J} = \mathscr{D}$.*

Proof. Before we proceed we mention that for any element x in a finite semigroup S there exists $m, n \in \mathbb{N}$ $(m \neq n)$ such that $x^m = x^n$ since otherwise we would have infinite distinct powers of x. Then for some $r \in \mathbb{N}$, $x^{m+r} = x^m$ and the subset $\{x^m, x^{m+1}, \cdots, x^{m+r-1}\}$ is a cyclic subgroup of S and so S contains an idempotent.

We only need to show that $\mathscr{J} \subseteq \mathscr{D}$ since by definition $\mathscr{D} \subseteq \mathscr{J}$. Suppose $a\mathscr{J}b$ in S, then there exists $s, t, u, v \in S^1$ such that $sau = b$ and $tbv = a$. By repeated substitutions we have $a = (ts)^m a(uv)^m$ for some $m \in \mathbb{N}$ such that $(ts)^m$ and $(uv)^m$ are idempotent. Then $a = (ts)^m(ts)^m a(uv)^m = (ts)^m a$. We have $b = [s]au$ and $au = (ts)^m au = (ts)^{m-1}tsau = [(ts)^{m-1}t]b \Rightarrow b\mathscr{L}au$. Similarly, $a = a(uv)^m$ so $a = au[v(uv)^{m-1}]$ and $au = a[u] \Rightarrow au\mathscr{R}a$. Hence $a\mathscr{D}b$ and $\mathscr{D} = \mathscr{J}$. In fact this proof shows $\mathscr{D} = \mathscr{J}$ in any semigroup where every element has finite order (obvioulsy this is a necessary condition for a semigroup of finite cardinality). ☐

It is often useful to imagine the $\mathscr{D}$-class structure of a semigroup by using an egg box diagram. Imagine a grid, where each row is an $\mathscr{R}$-class, each column an $\mathscr{L}$-class and each cell a $\mathscr{H}$-class (see figure 1). We often use the notation R_a to represent the $\mathscr{R}$-class which a is in. Obviously if $b \in R_a$ then $R_a = R_b$. Similarly we use L_a, H_a, D_a, J_a for the $\mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}$-classes. We usually denote an idempotent element by an asterix.

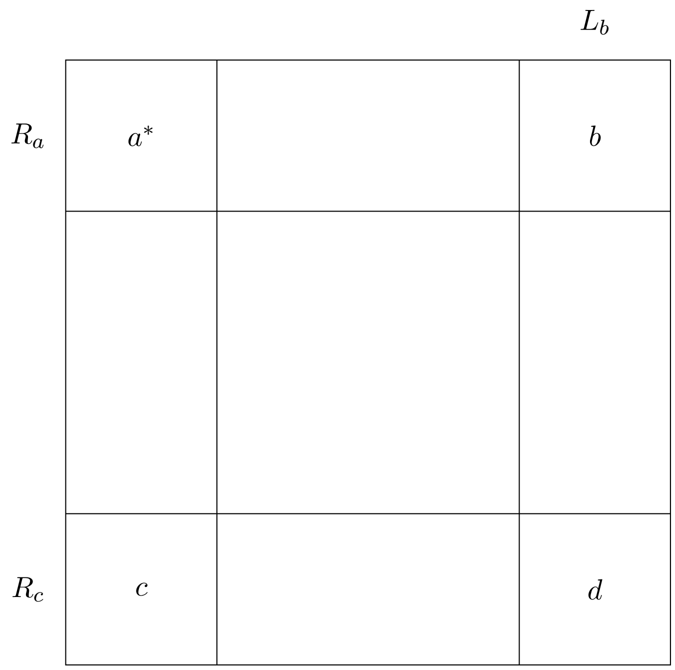

Figure 1: An example of a $\mathscr{D}$-class. Each row is an $\mathscr{R}$-class, each column an $\mathscr{L}$-class and each cell a $\mathscr{H}$-class.

1.2 Cayley Graphs

We denote a directed graph G by $G(V, E)$ where V is the set of vertices and E the set of ordered pairs such that if $a, b \in V$ and $(a, b) \in E$ then there is an edge from a to b.

A left Cayley graph of a semigroup S, $Cay(T, S)$, is a graph whose vertex set consists of the elements S and whose edge set consists of ordered pairs (a, b) if and only if $a \neq b$ and $sa = b$ for some $s \in T \subseteq S$. Note that we do not include (a, a) in the edge set. We can define a right Cayley graph analogously, however we will only be investigating the left Cayley graphs of semigroups. If not specifically mentioned, the generating set, T, will be the whole semigroup itself. For more information on Cayley graphs see Keralev [1] and Cain [2].

We will make use of some graph theoretic terms throughout this paper, however we will introduce any definitions and required background when an unfamiliar term is presented.

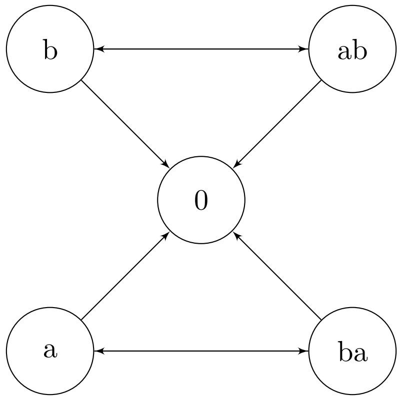

Figure 2: The Cayley graph of the 5-element Brandt Semigroup $\mathbb{B}$ with generating relation $\mathbb{B}_2 = \{a, b \mid a^2 = b^2 = 0, aba = a, bab = b\}$

For example, take the Cayley graph $Cay(\mathbb{B}_2, \mathbb{B}_2)$ of the 5-element Brandt Semigroup $\mathbb{B}_2 = \{a, b \mid a^2 = b^2 = 0, aba = a, bab = b\}$ in figure 2. We can see that $(s, 0) \in E$ for all $s \in \mathbb{B}_2$, $0s = 0$. Similarly $(a, ba), (ba, a) \in E$ since $a\mathscr{R}ba$ i.e. $b[a] = ba$ and $a = a[ba]$.

2 Results

2.1 A survey of semigroups of size 2 and 3

When counting the number of semigroups, it is customary to exclude not only isomorphic but anti-isomorphic semigroups. Semigroups S,T are anti-isomorphic if there exists a bijection ϕ from S to T such that $(ab)\phi = b\phi a\phi$ $(a, b \in S)$. There are 4,18,126,1160 semigroups of sizes 2,3,4 and 5 respectively [3]. However, the Cayley graphs of a semigroup and its anti-isomorphic partner are not necessarily graph isomorphic. Take for example, a left-zero semigroup S and its anti-isomorphism copy T, a right-zero semigroup. $Cay(S, E)$ is a complete graph (every vertex has an edge to every other vertex) while $Cay(T, E)$ is a null graph (it has no edges). We will exclude the study of self-dual semigroups, those who are isomorphic to their anti-isomorphic partners. For example, all commutative semigroups are self-dual by the homomorphism property.

Where necessary we will include the Cayley table of a semigroup. A Cayley table gives all products of a finite semigroup in a way similar to a multiplication table and is read as such (Figure 3).

	a	**b**	**ab**	**ba**	**0**
a	0	ab	0	a	0
b	ba	0	b	0	0
ab	a	0	ab	0	0
ba	0	b	0	ba	0
0	0	0	0	0	0

Figure 3: The Cayley Table for the Brandt Semigroup of Order 5

2.1.1 Semigroups of Order 2

We have 4 distinct non-[anti]isomorphic semigroups of order 2. They are: the cyclic group of order 2, a left-zero semigroup, a semilattice and the null-semigroup. However, we will also include the right-zero semigroup (which is anti-isomorphic to the left-zero semigroup). Immediately, its obvious that the Cayley graph of any null-semigroup will have edges pointed only towards the zero element. Similarly, in a left-zero semigroup each element has an edge to every other element. A right-zero semigroup will have a null graph since $sa = a$ for all $s, a \in S$.

When we talk about a semilattice, we specifically refer to a lower or meet-semilattice. A meet-semilattice is a partially ordered set which has a meet for every nonempty finite subset. It turns out that a commutative band is a lower semilattice with respect to the natural partial ordering ($a \leq b \Leftrightarrow ab = ba = a$) and conversely, a semilattice is a commutative band with respect to the meet operation [4]. If a relation ρ on a semigroup S satisfies $(a, b) \in \rho \Rightarrow (ac, bc), (ca, cb) \in \rho$ for every $c \in S$ then ρ is called a congruence on S. Semilattices play a large role in semigroup theory; in fact, there exists a finest congruence η such that the equivalence classes mod η $(S \backslash \eta)$ is a semilattice [5]. Howie and Lallament showed that when S is regular η is the congruence generated by the $\mathscr{D}$-classes [6]. How these algebraic properties relate to the graph theoretic properties of Cayley graphs of finite semilattices will be made clear in the next section.

Proposition 2. *The Cayley graph of a group is complete.*

Proof. Let G be a group. Let $a, b, c \in G$, then if $ac = bc$ then $a = b$ by the fact that $c^{-1} \in G$. In other words, each row and column of the Cayley table of a group contains each element once and only once. Therefore every element has an edge to each other element, and so Cay(G,E) is complete. □

In a more general sense we can extend this result to any $\mathscr{L}$-class in S, and since there exists an idempotent in any finite semigroup, there is at least one complete subgraph of Cay(S,S) (if we decide to include (a,a) in the edge set for any $a \in S$).

We also remark that the Cayley graphs of different semigroups need not be distinct. In other words, we cannot conclude that if two Cayley graphs are isomorphic that their generating semigroups are isomorphic. For example, a left-zero semigroup is complete, as is the Cayley graph of a group.

2.1.2 Semigroups of Order 3

There are 18 non-[anti]isomorphic semigroups of order 3 [3]. We will consider all 18 and include an additional 6 anti-isomorphic partners. Of these 24 semigroups, 1 is a group, 2 semi-lattices, 8 non-commutative bands, 12 are commutative, 9 are regular and 5 are inverse. There are also two cyclic semigroups. A cyclic semigroup is a semigroup that is generated by one element, denoted $\langle a \rangle$ = M(m,r) = $\{a, a^2, \cdots, a^m, \cdots, a^{m+r-1}\}$. We call m the index and r the period where $m, r \in \mathbb{N}$ and $a^m = a^{m+kr}$ for all $k \in \mathbb{N}$. [1] We remind the reader that $\{a^m, a^{m+1}, \cdots, a^{m+r-1}\}$ a cyclic group. Apart from the 3- element group, there are are 2 cyclic semigroups of order 3, namely M(2,2) and M(3,1). Figure 4 shows their Cayley graphs.

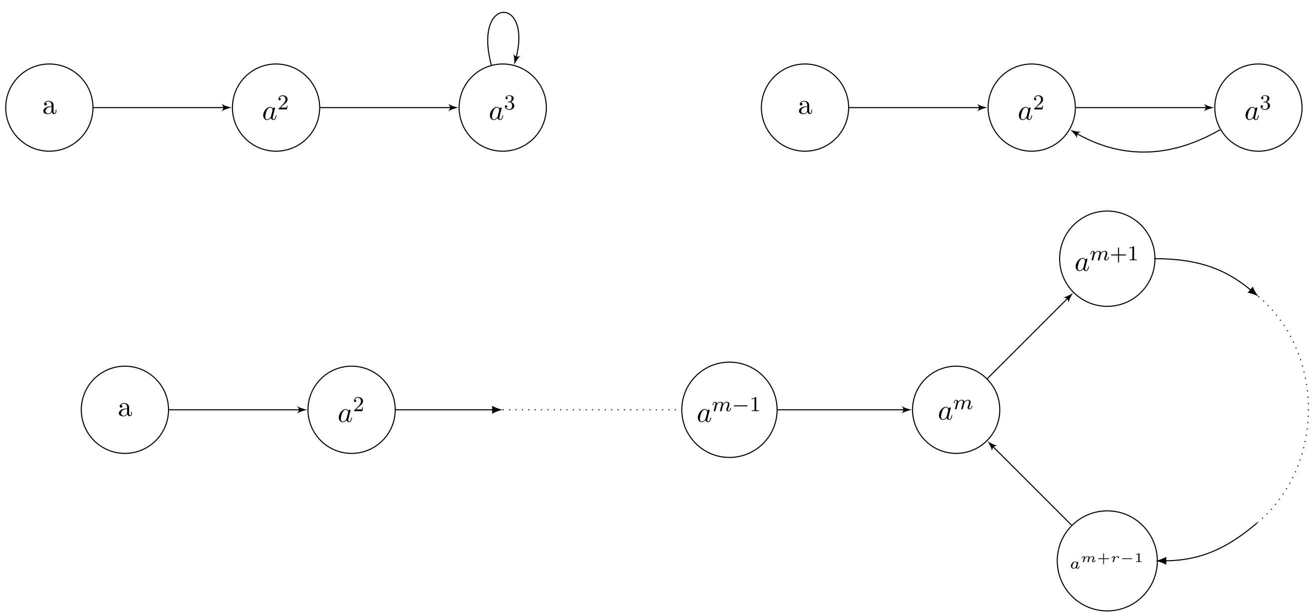

Figure 4: The Cayley graphs of the semigroups M(3,1) (left) and M(2,2) (right) and M(m,r) (below) with $\{a\}$ as the set of multipliers

[1]It is possible that $a^m = a^n \Rightarrow m = n$ $(m, n \in \mathbb{N})$. In such case, $\langle a \rangle$ is isomorphic to the semigroup $\mathbb{N}$ under addition.

Proposition 3. *Let $S = M(m, r)$ have a maximal subgroup G. Then for every a^i in S there exists a^j in G such that a subgraph of Cay(a^i, S) is the same as a subgraph of Cay(a^j, S).*

Proof. To show two Cayley graphs of the same semigroup are the same we need only show that they have the same edge-set. Let $j \equiv i(\text{mod r})$ then $a^i s = a^j s$ if and only if $a^i s \in G$. So the subgraph with vertices $a^k, \cdots, a^{m+r-1}$ for $k = m - i$ of Cay(a^i,S) is equal to Cay(a^j,S). □

Essentially, since G is a cyclic subgroup, then if powers of a in S are equivalent modulo the size of the subgroup, they will produce the same graph if and only if they are in the subgroup. So if, for example, we have an element that is two elements along the tail (out of the subgroup) we need to remove one element from the end of the tail so that their graphs are isomorphic.

2.2 Semigroups of Higher Order

We have already highlighted some properties in smaller semigroups that can extend to arbitrarily large semigroups. In this section we will continue to look at the Cayley graphs of larger semigroups. Since there are many more semigroups of order 4 and above than order 3, with limits on time we can only pick interesting examples. An exhaustive list of all semigroups up to order 8 is contained in the GAP system: https://www.gap-system.org/.

We now move on to studying the graph theoretic properties of a special sort of band, called a rectangular band. A rectangular band is the direct product of two non-empty sets X and Y such that $(x_1, y_1)(x_2, y_2) = (x_1, y_2)$ ($x_1, x_2 \in X$ and $y_1, y_2 \in Y$) i.e. the direct product of a left-zero and right-zero semigroup. So far, we have mentioned that a commutative band is a semilattice, however, the same is not true of a non-commutative band (we will look at the semilattice decompositions of bands in general). A rectangular band is an example of a non-commutative band.

Proposition 4. *Let S be a rectangular band that is neither a left-zero nor a right-zero semigroup. Cay(S,E) is then k-partite where k is the number of $\mathscr{R}$ classes.*

Proof. We first note that a rectangular band is D-simple. A semigroup is D-simple if it only has one $\mathscr{D}$-class [4]. Also, a k-partite graph is a graph whose vertices can be partitioned into k sets where no two vertices in the same set have an edge between them.

Suppose S is the direct product of X and Y where $\mid X \mid, \mid Y \mid > 1$. We show that there are no such elements $a, b, c \in S$ such that $a \mathscr{R} b$ and $ca = b$ or $cb = a$. For the sake of contradiction, assume such an $a, b, c \in S$ do exist and $ca = b$. Since S is a rectangular band, $a = (x_1, y_1), c = (x_2, y_2)$ for some $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then, $ca = b \Rightarrow b = (x_2, y_1)$ but $a \mathscr{R} b$ so $as = b$ and $b = (x_1, y_s)$ for some

$y_s \in S$ and $x_2 = x_1$. Therefore $b = a$. We conclude by using the fact that every $\mathscr{L}$-class of S generates a complete subgraph of Cay(S,S). Therefore each $\mathscr{R}$-class partitions the graph Cay(S,E) into disjoint sets and so Cay(S,E) is k-partite. □

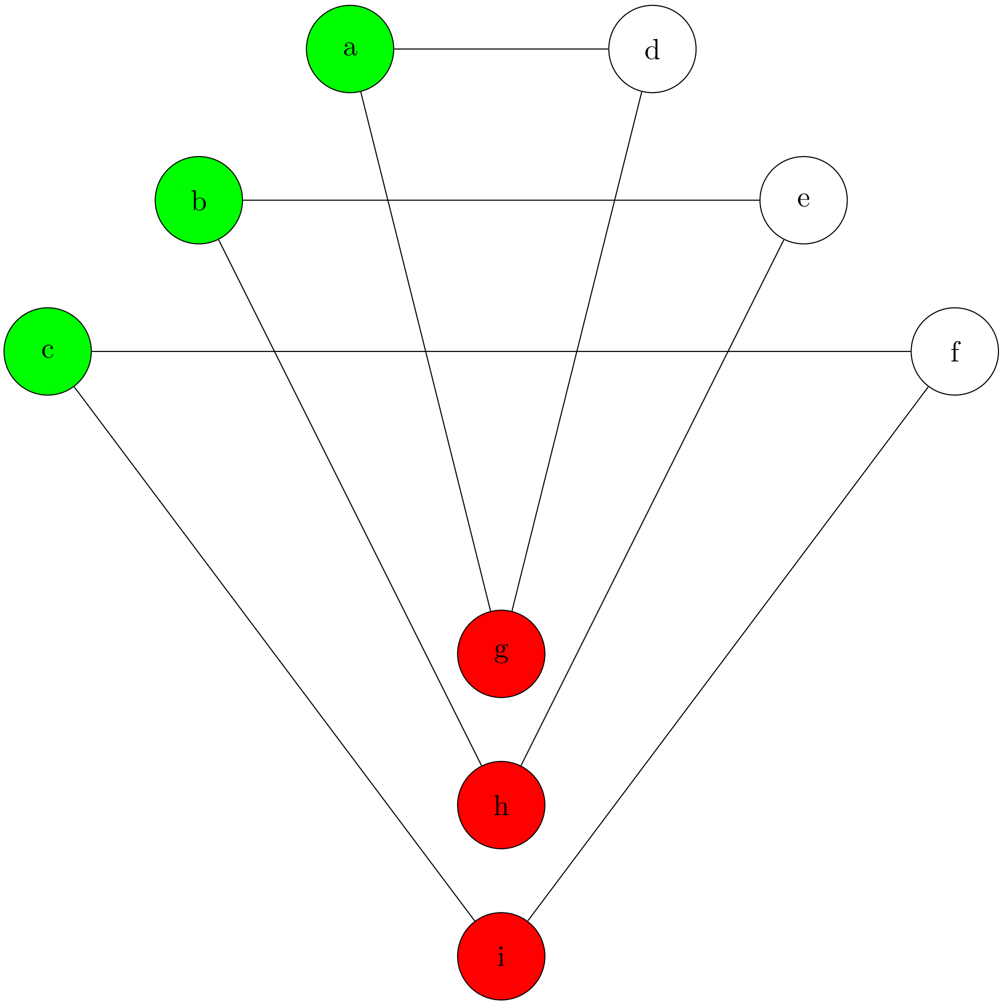

Figure 5: The Cayley graph of the 9 element rectangular band (the direct product of a 3 element left-zero semigroup and 3 element right-zero semigroup). The partite sets are coloured for clarity.

We can combine any two semigroups to create another semigroup by taking their direct products. Suppose S, T are semigroups then the operation of the direct product of these semigroups, $S \times T$, is defined as:

$$(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2) \ (s_1, s_2 \in S, t_1, t_2 \in T)$$

It is easily verified that $S \times T$ is a semigroup

Proposition 5. *Let S be a semigroup, G be a group and L be a left-zero semigroup where $\mid G \mid = \mid L \mid$. Let $T = S \times G$ and $U = S \times L$ then Cay(T,T) is isomorphic to Cay(U,U).*

Proof. Let S be a semigroup, G be a group and L be a left-zero semigroup where $\mid G \mid = \mid L \mid$. Let $T = S \times G$ and $U = S \times L$. Suppose $(a_1, g_1) \in T$ and $(a_1, l_1) \in U$. Then

$$T(a_1, g_1) = \{(a_1 a_1, g_1 g_1), (a_1 a_1, g_2 g_1), ..., (a_1 a_1, g_n g_1), ..., (a_n a_1, g_n g_1)\}$$

but $Gg_1 = G$ so,

$$T(a_1, g_1) = \{(a_1 a_1, g_1), (a_2 a_1, g_1), ..., (a_n a_1, g_1), ..., (a_n a_1, g_n)\}.$$

Also,

$$U(a_1, l_1) = \{(a_1 a_1, l_1), (a_2 a_1, l_1), ..., (a_n a_1, l_1), ..., (a_n a_1, l_n)\}$$

Therefore, $\mid S \times G \mid = \mid S \times L \mid$.

Since we can always find a g_i such that $g_k = g_i g$ for any $g \in G$ and $l_i = l_i l$ for any $l \in L$, there is a bijection between the edge sets of Cay(T,T) and Cay(U,U) and so Cay(T,T) is isomorphic to Cay(U,U). □

2.3 Semilattice Decompositions

A (meet) semilattice is a partially ordered set which has a meet for every pair of nonempty finite subsets. It can be shown that this is equivalent to a commutative band under the natural partial order ($a \leq b \iff ab = ba = a$). [4]. Yamada showed that we can find a congruence on a semigroup S, η, such that S/η is a semilattice. If $a \in S_a, b \in S_b$ and $S_a, S_b \in S/\eta$ are subsemigroups of S, then $S_a S_b \subseteq S_{ab}$.

We call a semigroup a strong semilattice of subsemigroups of type $\mathscr{C}$ if we can construct it in the following way: Let Y be a semilattice and let $\{S_\alpha\}_{\alpha \in Y}$ be a family of disjoint semigroups of type $\mathscr{C}$. If $\alpha, \beta \in Y$ and $\alpha \geq \beta$ let $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism. Suppose $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$

$(\alpha \geq \beta \geq \gamma)$. We construct S by letting $S = \cup\{S_\alpha\}_{\alpha \in Y}$ and the operation $*$ on S by $a \in S_\alpha, b \in S_\beta$, $a * b = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})$.

If all $\mathscr{H}$-classes of a semigroup S are groups we call S a Clifford Semigroup. Every Clifford semigroup is a semilattice of completely simple semigroups, in which $\mathscr{D} = \mathscr{J} = \eta$. [7]

Proposition 6. *Every band is a semilattice of rectangular bands.*

Proof. Since each $\mathscr{H}$-class of a semigroup can contain only one idempotent, it follows that the $\mathscr{H}$-classes of a band are single idempotent sets and so are trivial groups. Therefore any band is a Clifford semigroup and $\mathscr{D} = \eta$.
Let S be a band and $a, b \in D$ where D is a $\mathscr{D} - class$. Then $ab \in R_a \cap L_b$ since $R_b \cap L_a$ contains an idempotent. So $aba \in R_a \cap L_a$ but only $a \in H_a$ and so $aba = a$ for all $a, b \in D$. Therefore every $\mathscr{D}$-class of a band is a rectangular band and a band is a semilattice of rectangular bands. □

We call a band B normal if $\forall a, b, c \in B, abca = acba$. A band is normal if and only if it is a strong semilattice of rectangular bands. [8]

Proposition 7. *If L_1 and L_2 are $\mathscr{L}$-classes in the same $\mathscr{D}$-class of a normal band S then Cay(L_1,S) is the same graph as Cay(L_2,S).*

Proof. Let S be a normal band and let $\{S_\alpha : \alpha \in Y\}$ be the semilattice decomposition of S into a strong semilattice of rectangular bands. Suppose $r_1 \mathscr{R} r_2$ $(r_1, r_2 \in D_r)$ and $(a, b) \in Cay(r_1, S)$.

case 1. *If $a, b \in D_r$*

$$r_1 a = b \in R_{r_1} \cap L_a = R_{r_2} \cap L_a \Longrightarrow (a, b) \in Cay(r_2, S)$$

case 2. *If $a, b \notin D_r$*

There exists homomorphisms $\phi_{r_1,r_1a} : S_{r_1} \longrightarrow S_{r_1a}$ and $\phi_{a,r_1a} : S_a \longrightarrow S_{r_1a}$ such that

$$(r_1\phi_{r_1,r_1a})(a\phi_{a,r_1a}) = b \in R_{r_1\phi_{r_1,r_1a}} \cap L_{a\phi_{a,r_1a}}$$

but for all $s \in S$, $(r_1\phi_{r_1,s})\mathscr{R}(r_2\phi_{r_2,s})$, so

$$b \in R_{r_1\phi_{r_1,r_1a}} \cap L_{a\phi_{a,r_1a}} = R_{r_2\phi_{r_2,r_2a}} \cap L_{a\phi_{a,r_2a}} \Longrightarrow (a, b) \in Cay(r_2, S)$$

It follows that for any two $\mathscr{L}$-classes L_1, L_2 in the same $\mathscr{D}$-class, Cay(L_1,S) is isomorphic to Cay(L_2,S). □

It is not necessary for a band to be a strong semilattice to have this property. Take a band that is a strong semilattice and attach an identity. The band is (in general) no longer a strong semilattice. Then the Cayley graphs relative to two $\mathscr{L}$-classes are isomorphic since we can put the edge (e, a) in $Cay(L_1, S)$ $(a \in L_1)$ in a one-to-one correspondence with (e, b) in $Cay(L_2, S)$ $(b \in L_2)$. Obviously any two $\mathscr{L}$-classes of the same $\mathscr{D}$-class are the same size and so we can find an isomorphism between $Cay(L_1, S)$ and $Cay(L_2, S)$.

For example, in figure 6 we see a strong semilattice of rectangular bands. The rectangular bands are $S_a = \{a, b, c, d, e, f\}$, $S_g = \{g, h\}$ (a right-zero semigroup) and $S_0 = \{0\}$. In figure 7 we see the Cayley graphs of S with multiplier sets L_1 and L_2 being the two L-classes of S_a. We have only included one graph, since as we just proved Cay($L_1 = \{a, b, c\}, S$)=Cay($L_2 = \{d, e, f\}, S$).

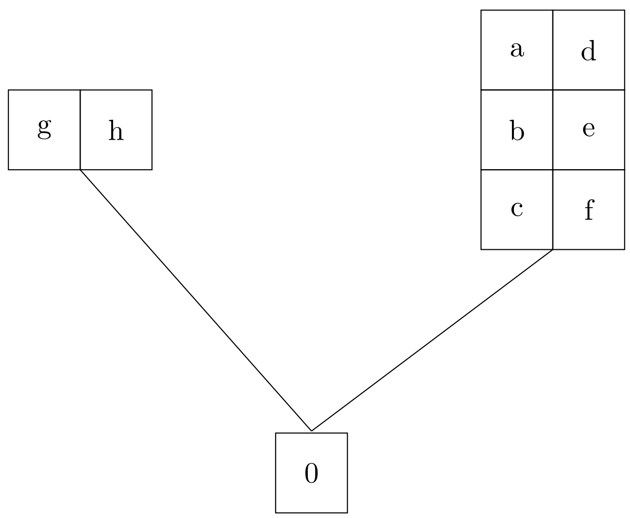

Figure 6: The strong-seimlattice of a 9 element normal band.

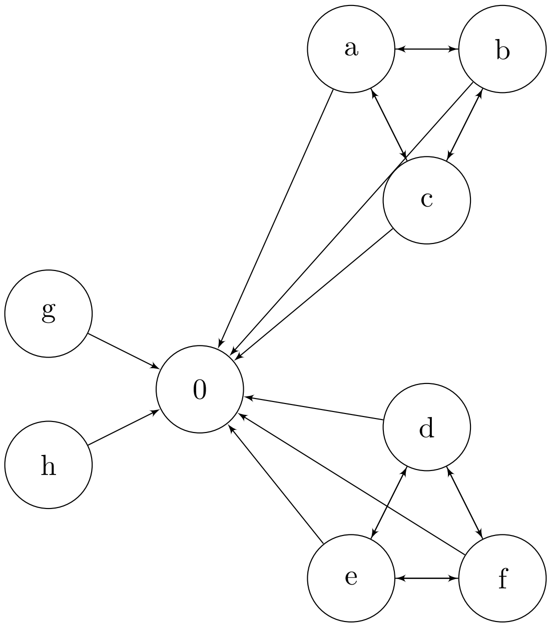

Figure 7: The Cayley graph of the above normal band showing Cay($L_1 = \{a, b, c\}, S$)=Cay($L_2 = \{d, e, f\}, S$).

3 Conclusion

We have seen many interesting examples of the Cayley graphs of finite order. The results and examples described above suggest an interesting interplay between the algebraic properties of semigroups and the graph-theoretic properties of their Cayley graphs. There is still much to be investigated in relation to this interplay. Looking at a wider class of semigroups, semigroups of larger size, and comparing the left and right Cayley graphs of semigroups are just a few areas where more investigation is required.

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