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Drawings of the complete graphs K_5 and K_6, and the complete bipartite graph $K_{3,3}$.

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Abstract

It is shown that for K_5 (resp. $K_{3,3}$) there is a drawing with i independent crossings, and no pair of independent edges cross more than once, provided i is odd with $1 \leq i \leq 15$ (resp. $1 \leq i \leq 17$). Conversely, using the deleted product cohomology, it is shown that for K_5 and $K_{3,3}$, if A is any set of pairs of independent edges, and A has odd cardinality, then there is a drawing in the plane for which each element in A cross an odd number of times, while each pair of independent edges not in A cross an even number of times. For K_6 it is shown that there is a drawing with i independent crossings, and no pair of independent edges cross more than once, if and only if $3 \leq i \leq 40$.

1 Introduction

Topological graph theory is the study of graph drawings, involving various mathematical techniques from algebraic topology, group theory, enumerative combinatorics and the analysis of algorithms (Gross and Tucker, 2001).

For a graph which does not admit to drawings in the plane with no edge crossing, it is natural to question the structure and the cardinality of the set of pairs of crossing edges. This report presents drawings of the complete graphs K_5, K_6, and the complete bipartite graph $K_{3,3}$ in the plane (see Figure 1), and a discussion on what the outcomes gave in the pursuit of answering this question for certain properties of the graphs listed. Results given directly from the drawings concerns the existence of tolerable drawings having a certain number of independent crossings (see Definition 1). The approach taken is somewhat non-traditional, as so-called *good* drawings are not the main focus.

For context, finite simple graphs in the plane were considered, exhibiting no multiple edges or loops, where vertices are represented as points and edges as smooth arcs joining them. Vertices are distinct, and edges do not self-intersect or pass through any vertex. The distinct edges only meet at common vertex endpoints or at transverse crossings. From Guy (1972), recall that two edges are said to be *independent* if they are distinct and not adjacent, where adjacent means that two edges share a common vertex. Also, recall that a drawing of a graph is *good* if no pair of adjacent edges cross, and each pair of independent edges cross at most once. Lastly, when two independent edges cross this is called an *independent crossing*.

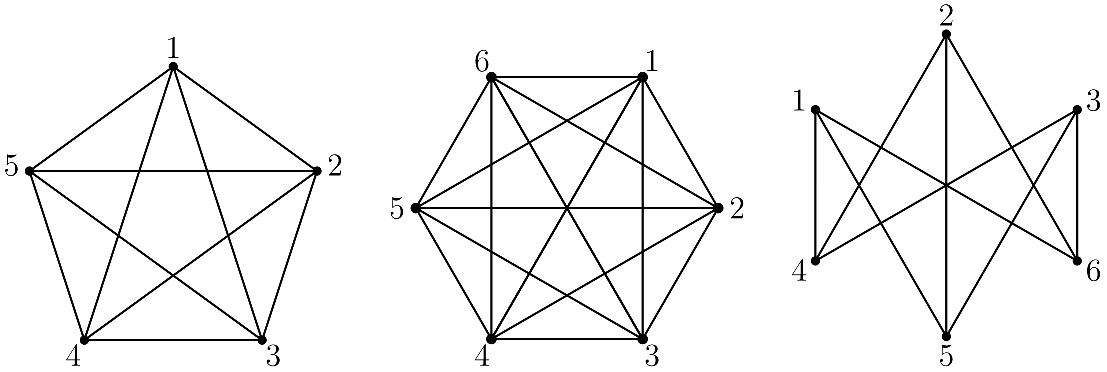

Figure 1: The complete graphs K_5, K_6, and the complete bipartite graph $K_{3,3}$.

Definition 1 *We say that a graph drawing is* bad *if it is not good, but that it is* tolerable *if no pair of independent edges cross more than once.*

Undertaking the journey of producing tolerable drawings for each of the graphs, the following theorem was proved.

Theorem 1

(a) For each odd integer i with $1 \leq i \leq 15$, there is a tolerable drawing of K_5 with i independent crossings.

(b) For each odd integer i with $1 \leq i \leq 17$, there is a tolerable drawing of $K_{3,3}$ with i independent crossings.

(c) For each integer i with $3 \leq i \leq 40$, there is a tolerable drawing of K_6 with i independent crossings.

Drawings that satisfy the conditions described in this theorem above are presented explicitly in Section 2.

In conjunction with these findings, a second lot of results demonstrates that for each of these graphs, there are no tolerable drawings having a number of independent crossings other than those indicated in Theorem 1. For this part, the deleted product cohomology was used. Terminology required for this is given in Definition 2. Section 3 gives a brief insight on the results.

Definition 2 *For a graph G, let PG denote the set of pairs of independent edges of G. We will say that a subset A of PG is* 2-realisable *if there is a drawing of G in the plane for which each element in A cross an odd number of times, while each element of $PG \backslash A$ cross an even number of times. Further, we say that such a drawing* 2-realises A.

The definition above does not confirm if such a drawing is tolerable. So, for a given graph G and given subset A of PG, it is natural to ask whether A is 2-realisable, and if so, is there a tolerable, or even good, drawing that realises A.

Section 3.1 shows that if G is K_5 or $K_{3,3}$ and $A \subset PG$ is 2-realisable, then the cardinality of A satisfies the corresponding condition in Theorem 1. This is a direct consequence of Kleitman's theorem. Here is the stronger statement (see Theorem 4):

Theorem 2 *If G is K_5 or $K_{3,3}$ and $A \subset PG$, then A is 2-realisable if and only if its cardinality satisfies the corresponding condition of Theorem 1(a) or (b).*

Now, if G is K_6 and $A \subset PG$ is 2-realisable, then the cardinality of A satisfies the corresponding condition in Theorem 1. This is similar to K_5 or $K_{3,3}$, however, the results slightly differ as there are 3-element subsets of PK_6 that are 2-realisable and there are 3-element subsets of PK_6 that are not 2-realisable (see Section 3.2). Yet, one does have:

Theorem 3 *If $A \subset PK_6$ is 2-realisable, then the cardinality of A satisfies the condition in Theorem 1(c).*

The drawings of the graphs in the plane were produced by hand and then drawn in LaTeX. Efforts of producing these entailed problem solving strategies alongside some trial and error. Computer assisted methods in Mathematica were used for the second part of the results.

Necessary background information on the deleted product space and cohomology of the graphs can be found in (Abrams and Ghrist, 2002; Barnett and Farber, 2009; Copeland, 1965; Copeland and Patty, 1970; Farber and Hanbury, 2010; Patty, 1962; Ummel, 1972).

Note that the computations involving the deleted product space cohomology (Section 3) were established by my supervisors.

2 Tolerable drawings for the graphs $K_5, K_{3,3}$ and K_6

The complete graph K_5 has 10 edges and 15 pairs of independent edges. Up to relabelling, only 5 of these subsets of PK_5 can be realised by good drawings; see Figure 3.1 of Rafla (1988) or Figure 1.7 of Schaefer (2018). They each have 1, 3 or 5 crossings. The first 3 drawings of Figure 2 are good, and have 1, 3 and 5 crossings respectively. The remaining drawings of Figure 2 are tolerable and have 7, 9, 11, 13 and 15 independent crossings respectively. An interesting feature of the last drawing with 15 independent crossings is that it has 5-fold symmetry.

The complete bipartite graph $K_{3,3}$ has 9 edges and 18 pairs of independent edges. Harborth (Harborth, 1976) determined that there are 102 good drawings of $K_{3,3}$ up to isomorphism; there are 1, 9, 33, 48, and 11 good drawings with 1, 3, 5, 7, and 9 crossings, respectively. In Figure 3, the first 5 drawings are good. The remaining drawings are tolerable and have 11, 13, 15 and 17 independent crossings, respectively.

The complete graph K_6 has 15 edges and 45 pairs of independent edges. It is known that K_6 only has good drawings for i independent crossings if and only if either $3 \leq i \leq 12$ or $i = 15$; see (Rafla, 1988). Figure 4 gives examples of such good drawings. Figures 5 through 8 give tolerable drawings having i independent crossings, for $i = 13, 14$ and $16 \leq i \leq 40$, respectively. Note that in the drawings in Figure 8, the idea is that one extends the red lines out and connects them up to a 6th vertex (at infinity, if one likes). It is perhaps easiest to keep track of the independent crossings in these diagrams by comparing each drawing having i independent crossings with the drawing having $i+3$ independent crossings. Also, note that the last drawing in Figure 8 has 5-fold symmetry and is easy to understand; in this drawing the blue and black edges give a tolerable drawing of K_5 with 15 independent crossings. Whereas, each red line contributes 5 independent crossings, that is 3 independent crossings with blue edges and 2 independent crossings with black edges. Consequently, adding up to 40 independent crossings.

3 2-realisable subsets of $PK_5, PK_{3,3}$ and PK_6.

3.1 Drawings of K_5 and $K_{3,3}$

The graphs K_5 and $K_{3,3}$ have deleted product spaces that are closed surfaces, given by Abrams (2002) and proved by Abrams' thesis; see Theorem 5.1 in Abrams (2000). The following is a necessary and sufficient condition:

($*$) for each edge e in G, the graph $G - e$ obtained by deleting e and all adjacent edges, is a union of disjoint cycles.

Now, if a graph G satisfies ($*$), it must have at least 5 vertices. It is easy to verify that if G has 5 vertices and satisfies ($*$), then it is K_5, and if G has 6 vertices and satisfies ($*$), then it is $K_{3,3}$.

Kleitman proved that for odd m, n, any two drawings of $K_{m,n}$ (or of K_n) have equal numbers of independent crossings, mod 2 (Kleitman, 1976). In particular, all drawings of K_5 and $K_{3,3}$ have an odd number of independent crossings. The following result, which is a rewording of Theorem 2, is a converse to Kleitman's Theorem:

Theorem 4 *For G equal to K_5 or $K_{3,3}$, every odd subset A of PG is 2-realisable.*

3.2 Drawings of K_6

The drawings of K_6 gave that there are no 2-realisable crossing set for K_6 with cardinality in $\{0, 1, 2, 41, 42, 43, 44, 45\}$. This establishes Theorem 3. Furthermore, observing the 2-realisable crossing set of K_6, one notices that there are tolerable drawings of K_6 with just 3 independent crossings, but not all subsets $A \subseteq PK_6$ with 3 elements are 2-realisable.

4 Summary and Afterthoughts

The work put into finding tolerable drawings of each graph is indeed relevant to studies within topological graph theory. Many researchers in this field previously have been focused on just good drawings, almost not seeing the interesting side to bad drawings. This work carries through how tolerable drawings significantly add to the well understood drawings of $K_5, K_{3,3}$ and K_6 found in literature (as discussed in Section 2).

The tolerable drawings determined for each graph ($K_5, K_{3,3}$ and K_6) occurring at particular numbers of independent crossings is what provided Theorem 1 with the inequalities. As Theorem 1 states, for each odd integer i with $1 \leq i \leq 15$, there is a tolerable drawing of K_5 with i independent crossings, and for each odd integer i with $1 \leq i \leq 17$, there is a tolerable drawing of $K_{3,3}$ with i independent crossings, and for each integer i with $3 \leq i \leq 40$, there is a tolerable drawing of K_6 with i independent crossings.

Section 3.1 presented Theorem 4 (a rewording of Theorem 2) which states, for G equal to K_5 or $K_{3,3}$, every odd subset A of PG is 2-realisable. Furthermore, Section 3.2 presented that the graph K_6 has no 2-realisable crossing set with cardinality in $\{0, 1, 2, 41, 42, 43, 44, 45\}$, establishing Theorem 3. Additionally, discussed that there are 3-element subsets if PK_6 that are 2-realisable, and 3-element subsets of PK_6 that are not 2-realisable.

Considering the case of the complete graph K_6, another similar candidate to look into would be the complete bipartite graph $K_{3,4}$ (see Figure 9). Reasoning for this is because $K_{3,4}$ would neatly fit into the research on tolerable drawings, adding to the previously explored results. In more detail, $K_{3,4}$ has 12 edges and 36 pairs of independent edges. A natural question to ask is if tolerable drawings can be realised for $K_{3,4}$, with i independent crossings for all i with $2 \leq i \leq 34$ (Richter and Thomassen, 1997). Due to this large number of possible tolerable drawings to be found, it ties in nicely with K_6. Hence, investigation of tolerable drawings of $K_{3,4}$ would be the next step in contributing to this work, building upon the literature of good drawings once again.

5 Acknowledgements

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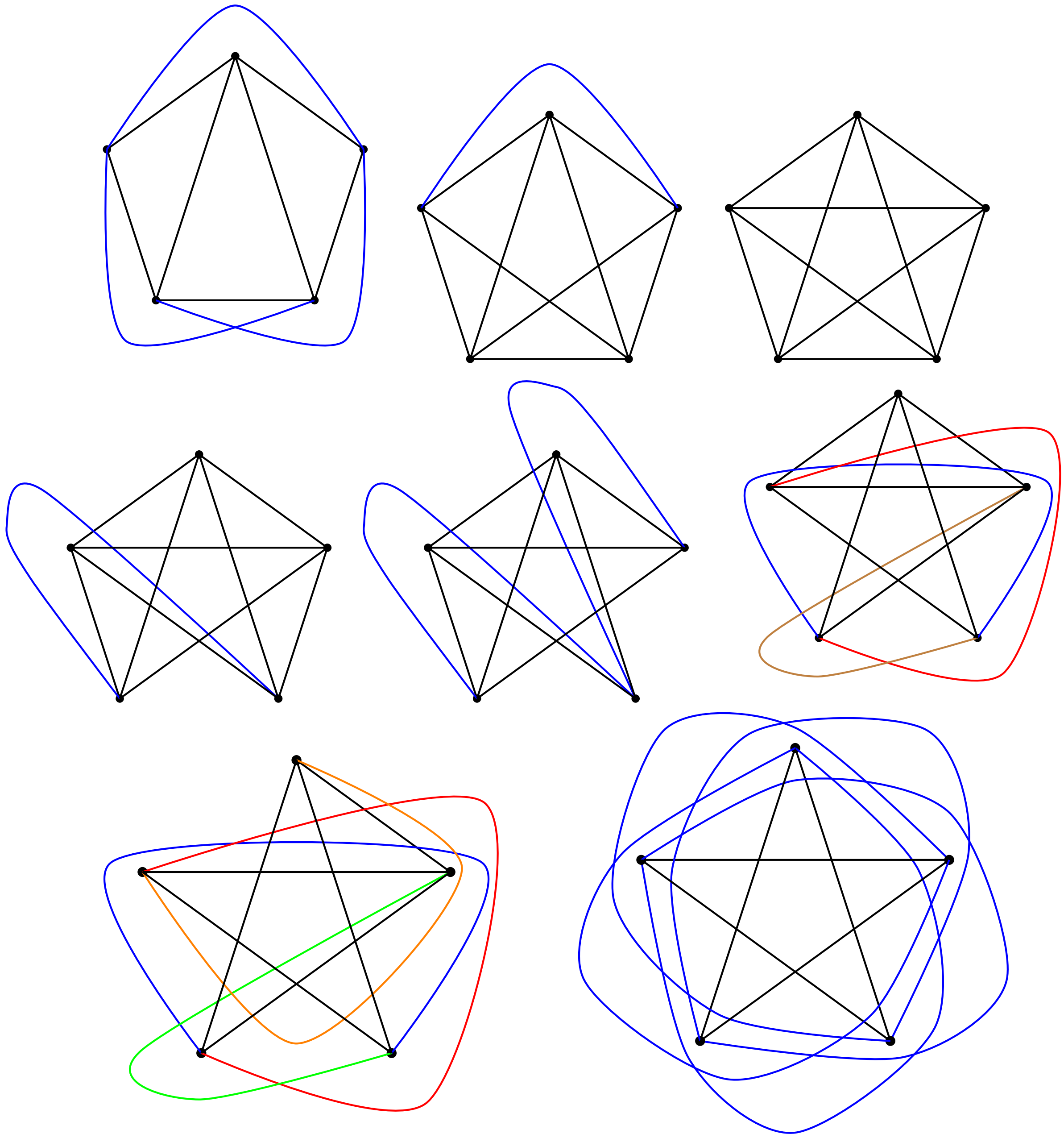


Figure 2: Tolerable drawings of K_5.

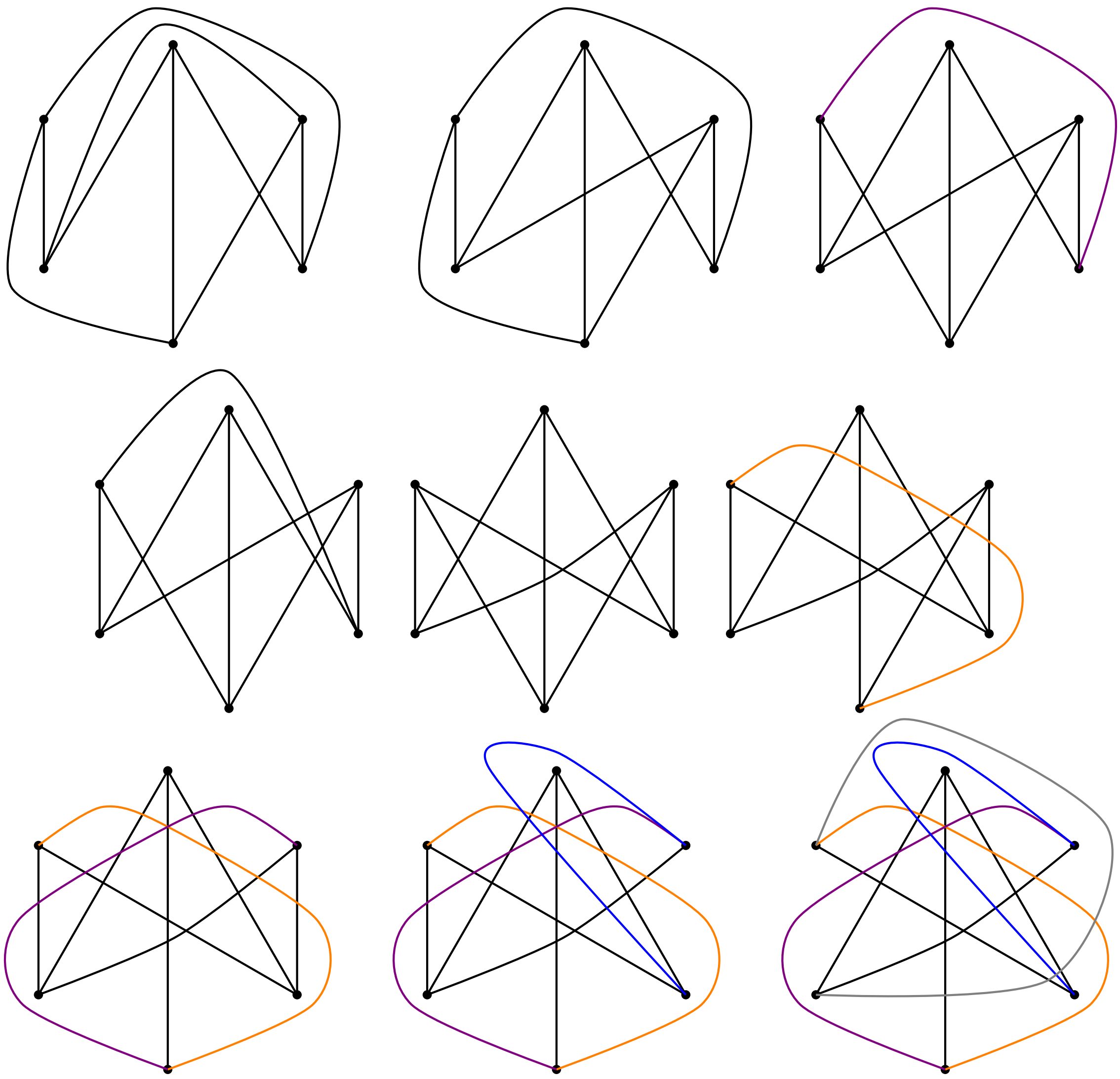


Figure 3: Tolerable drawings of $K_{3,3}$.

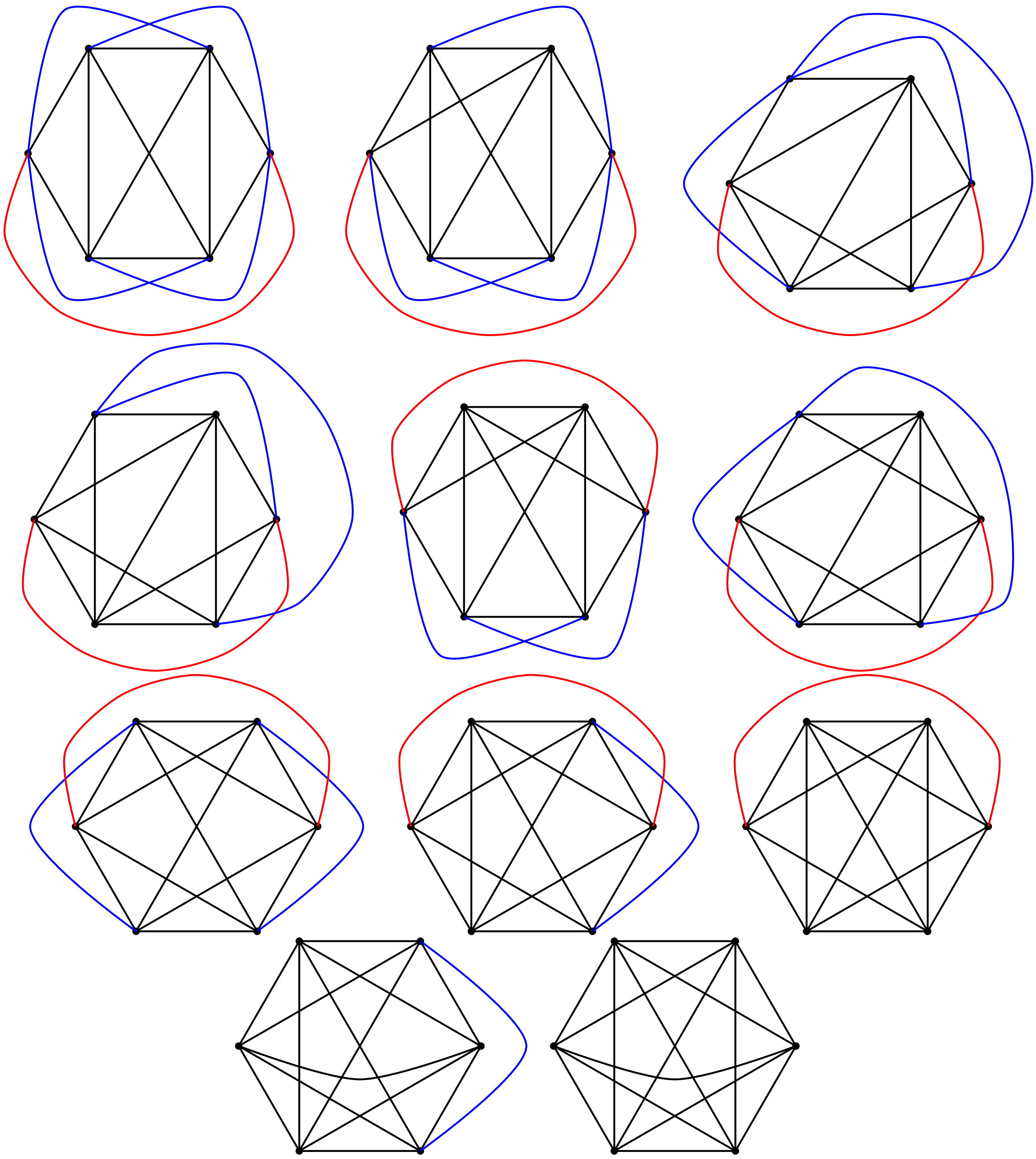


Figure 4: Good drawings of K_6 with 3 to 12 and 15 crossings.

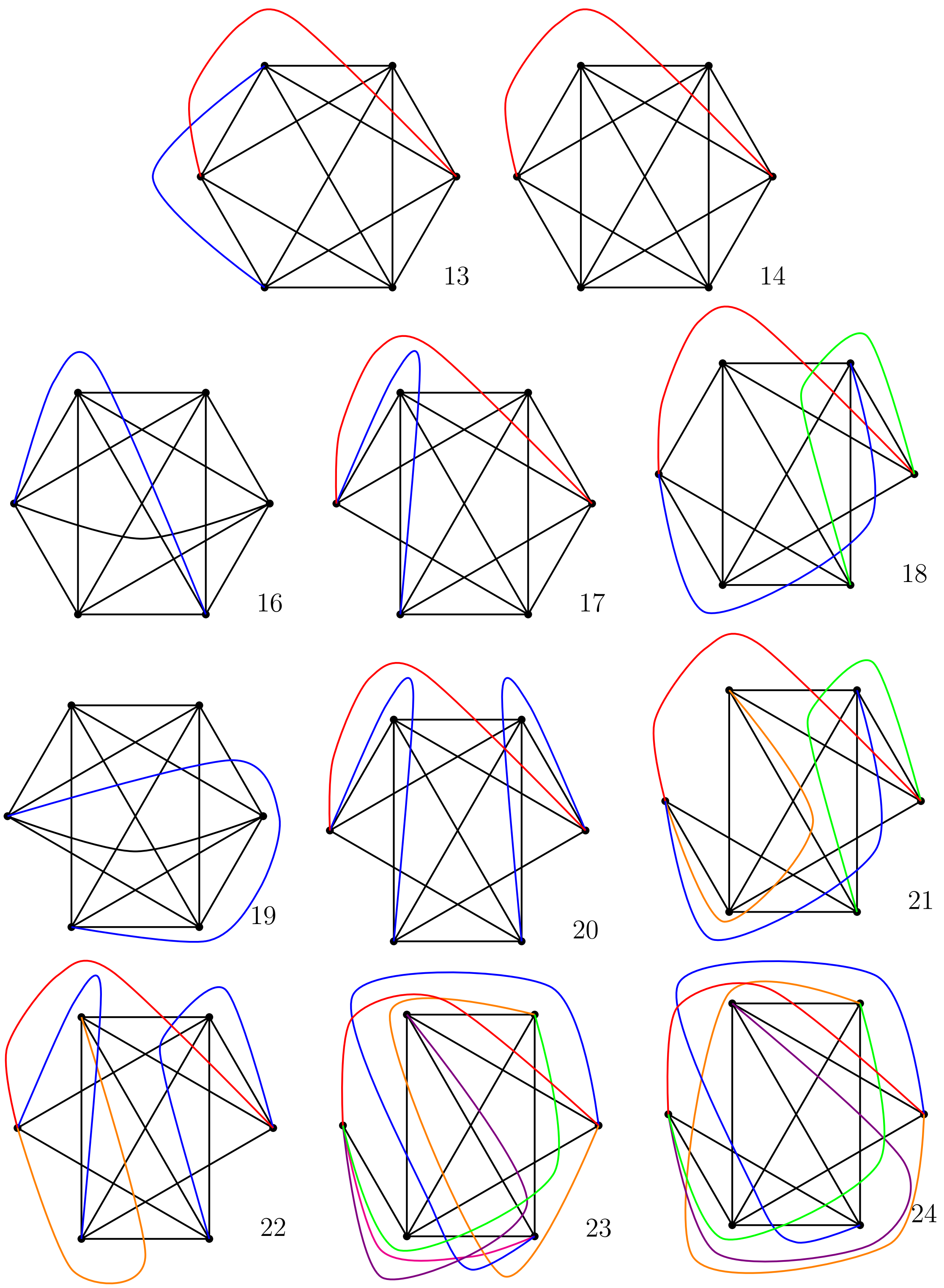


Figure 5: Tolerable drawings of K_6 with 13,14 and 16 to 24 independent crossings.

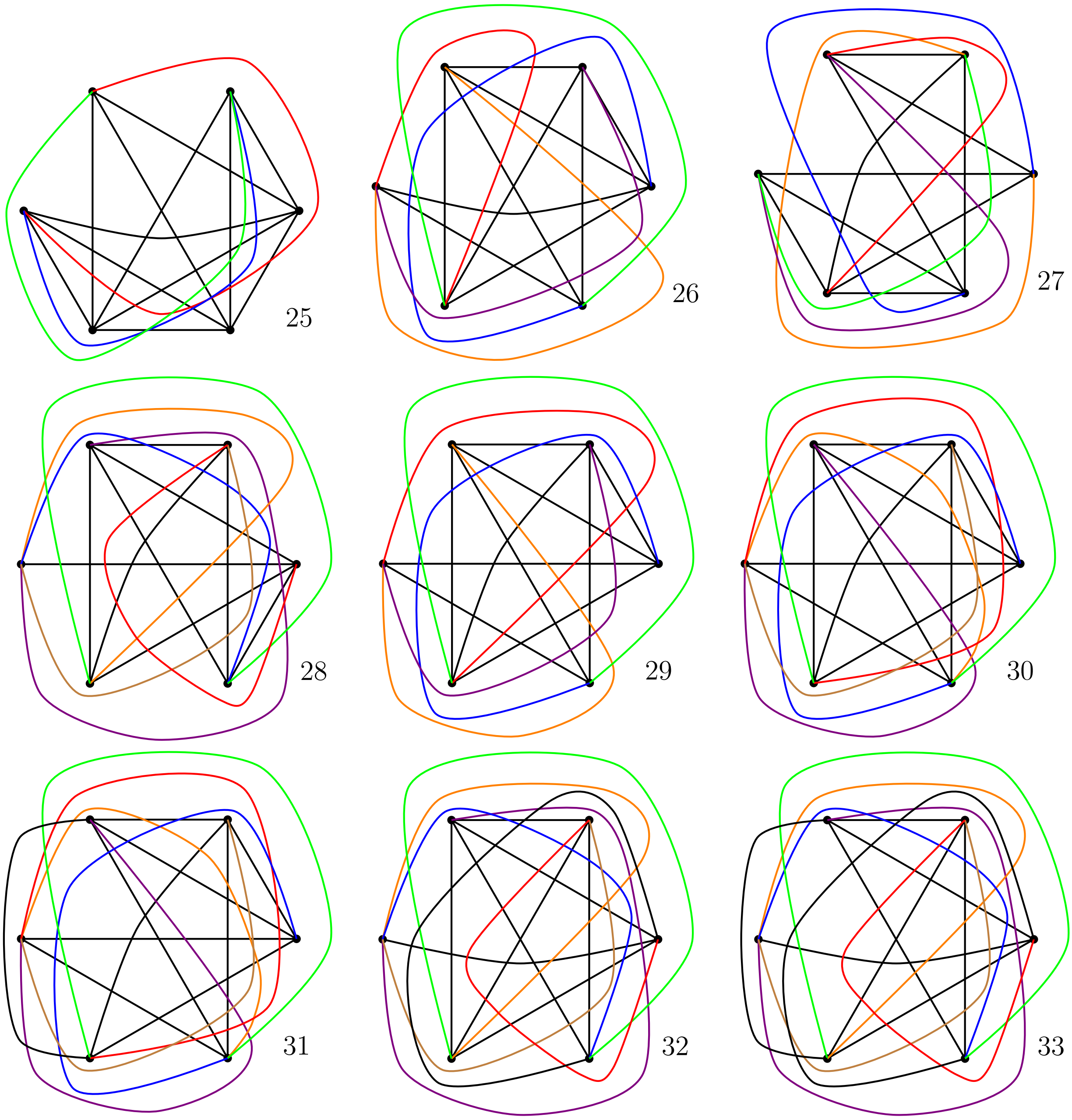


Figure 6: Tolerable drawings of K_6 with 25 to 33 independent crossings.

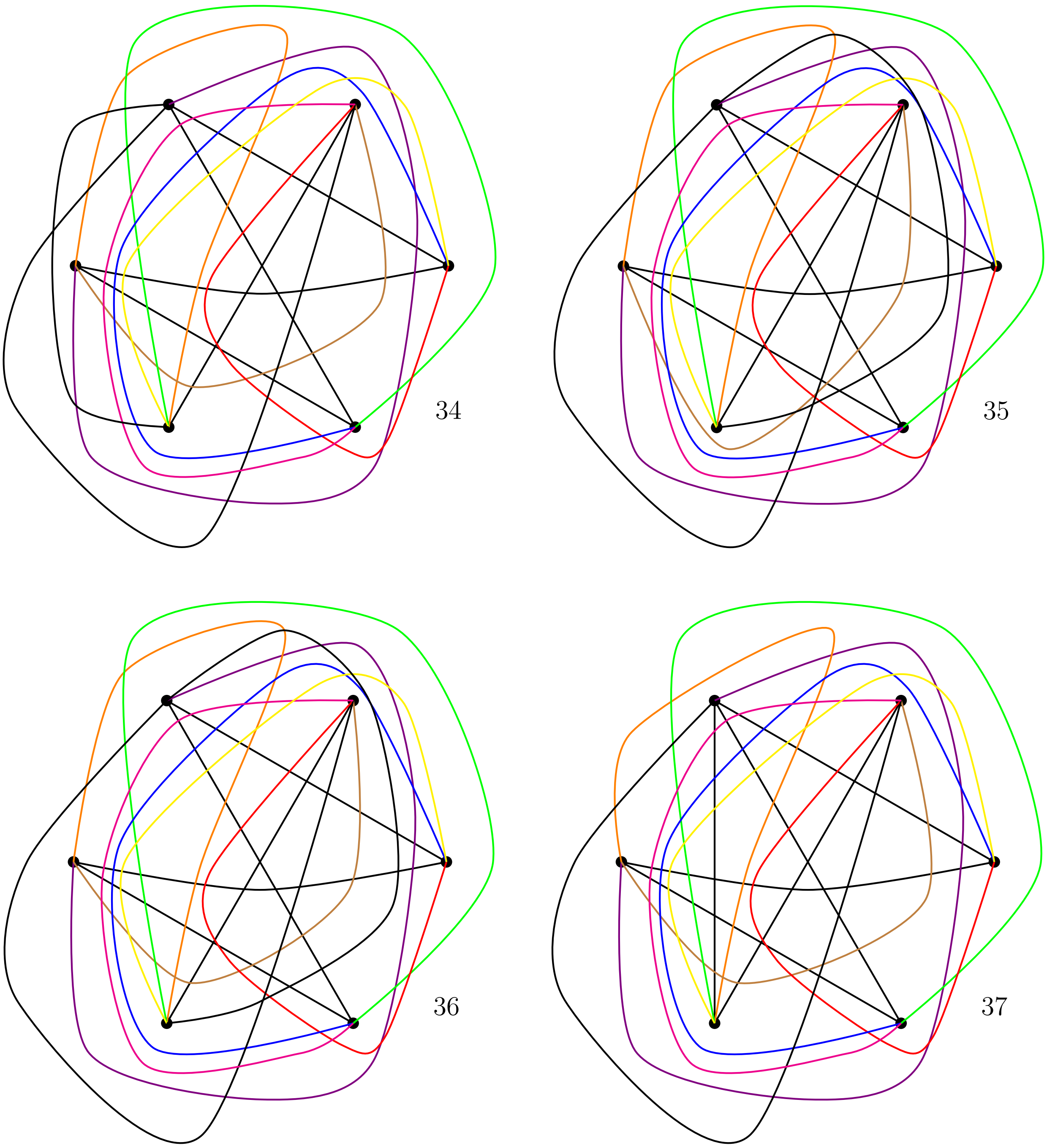
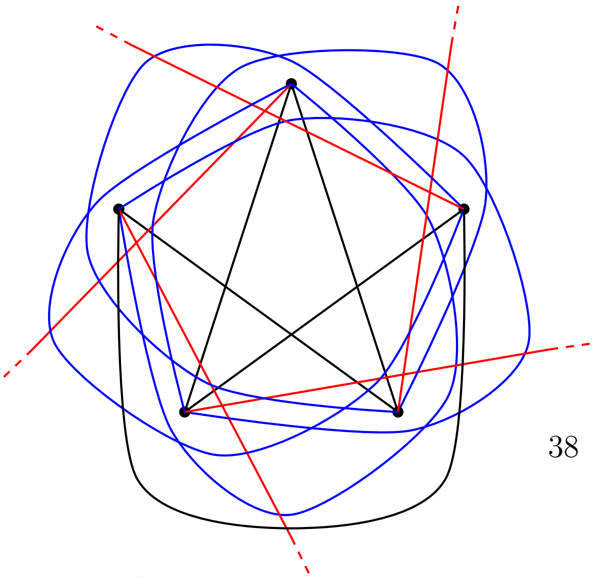
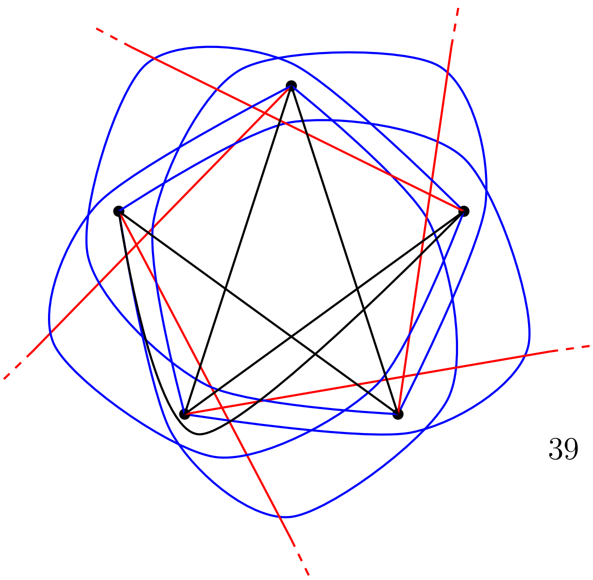


Figure 7: Tolerable drawings of K_6 with 34 to 37 independent crossings.





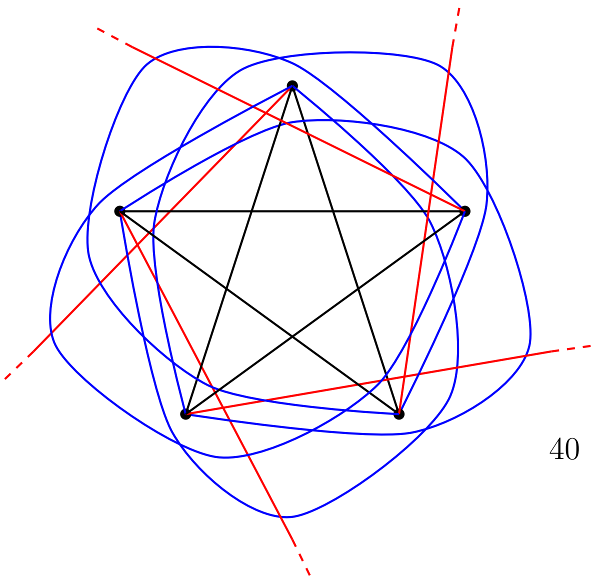


Figure 8: Tolerable drawings of K_6 with 38 to 40 independent crossings.

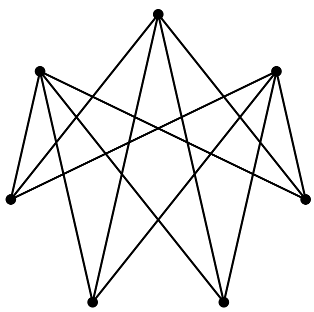


Figure 9: The complete bipartite graph $K_{3,4}$.