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A Laplace Transform Approach to Pricing Convertible Bonds

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Abstract

The aim of this project is to understand various methods used to price Convertible Bonds (CBs) and to derive an analytical approximation pricing formula for the optimal conversion exercise price of CBs using the Laplace transform technique. The pseudo-steady-state approximation is then used in the performance of the Laplace transform for treatment of the moving boundary. A formula for the optimal exercise boundary in transform space is found and an inversion technique using complex analysis is investigated.

1 Introduction

1.1 Financial background

Stocks and bonds represent two different ways of raising money for an entity to finance or expand its operations:

- Stock refers to share in the ownership of a company. Because of their volatility and the risk that the investor could lose money in the short term, stocks tend to be on the riskier end of the investment spectrum, despite its superior long-term returns potential.
- Bond is a fixed income instrument, which is a form of debt borrowing and must be repaid over time. Despite the inferior long-term earning capacity compared to stocks, these securities are preferred by investors with income priority.

Convertible bonds may offer a combination of reliable income and potential capital appreciation, which are becoming more popular among investors looking for safer alternatives in today's uncertain financial market. Convertible bonds are corporate bonds, in which investors have the option to convert to a certain number of shares of the issuing company's common stocks. The conversion can take place at any time prior to maturity if the bond is of American style, while European style only allows conversion at the maturity date. This feature gives convertible bonds greater upside than regular bonds, and for that reason, most investors treat convertible bonds as a hybrid between a fixed-income investment and a stock investment. As a result of this potential for growth, the coupon might be relatively lower than that offered for a similar non-convertible bond. Conversion price refers to the stock price at which a convertible bond can be exchanged for shares of common stock is called. The number of shares of common stock for which a CB may be exchanged is called conversion ratio, which can be found by dividing the par value of the bond by the conversion price.

The most common type of convertible bonds are vanilla convertible bonds, in which investors are given the right to convert their bonds to a certain number of shares at a predetermined conversion price and conversion rate at the maturity date (see Fig. 1)(CFI (2019)). Coupon payments are made during the life of the bonds and bonds come with a fixed maturity date at which the investors are entitled to the nominal value of the bond.

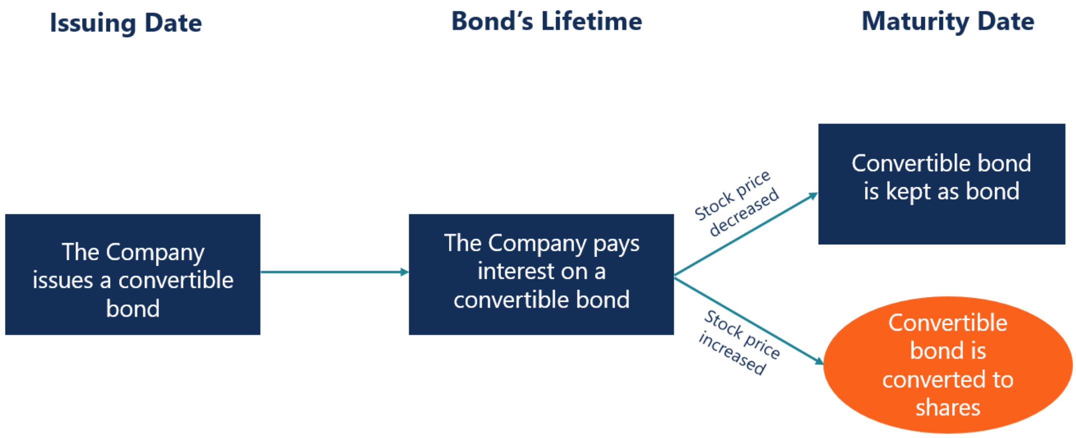

Figure 1: Vanilla Convertible Bonds Illustration (CFI (2019))

1.2 Why Convertible Bonds?

There are certain advantages corporations might want to consider by issuing convertible bonds. As mentioned, with a lower interest rate, this would reduce the amount issuers must pay out to investors. Additionally, the conversion of these bonds means that the issuer no longer has to make fixed interest payments, which also means reduction in debt. Issuing convertible bonds does not instantly increase the issuer's number of outstanding shares, which makes it more advantageous as oppose to common stock, where doing so would decrease earnings per share (CFI (2019)).

Being a fixed income and equity securities hybrid, convertible bonds also offer investors some unique benefits. For instance, CB holders maintain a precedence over common stock holders in terms of liquidation if the bonds are not yet to be converted. If the security will not be converted, CB behaves exactly like regular bond with fixed interest payments and returned par value at maturity. Typically, these interest payments maintain a precedence in terms of payment priority over dividend payments, to which holders of common stock may be entitled to. There is no official tax obligation associated with converting the bond to common stocks, as such a conversion of bond to stock is not deemed to be buying and selling from a taxation perspective (CFI (2019)).

1.3 Why Laplace transform?

Because of the hybrid nature, pricing convertible bonds can be much more complicated than that of simple options, especially with additional complexity such as callability and putability or the issuer's default risk. This report focuses on the valuation of CBs with American-style conversion, since most traded CBs in the financial market are of American style. The CB holder's right to initiate conversion any time prior to expiry contributes significantly to the complexity of the problem, as by doing that, there exists an optimal value of the underlying asset, at which the CB holder should convert the bond into the underlying asset. Mathematically, similar to the problem of valuing American options (Zhu (2006b)), this constitutes a free boundary value problem, in which the boundary location itself is part of the solution of the problem.

The Laplace transform will be used to solve the problem since it can be a powerful tool in solving partial differential equations (PDE), dealing with the boundary conditions of a PDE on an infinite domain. Pseudo-steady-state approximation will also be used in the application of the Laplace transform on the moving boundary, to convert the systems of PDEs to systems of ordinary differential equations in the Laplace space. This treatment of the moving boundary was introduced in the American put problem (Zhu (2006b)), upon which this paper heavily was based to attempt to derive an analytical approximation formula for the optimal conversion exercise price for CB.

2 Formulation of the problem

The formula for the CB will be derived using Black-Scholes framework, which makes use of "ideal conditions" assumptions in the market for the stock and the CB, such as constant short-term interest rate over time; random walk in stock price; no dividend payments; no transaction costs; short-selling permitted, etc.(Black and Scholes, (1973)).

If we let $V(S,t)$ be the value of a convertible bond, S be the price of the underlying asset and t be the current time. Then, under the Black-Scholes framework (Black and Scholes, (1973)), the value of a convertible bond V should satisfy the following partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where σ is the volatility of the underlying asset price and D_0 is the rate of continuous dividend paid to the underlying asset. As mentioned, r and σ are assumed to be constant. This equation needs to be solved together with a set of appropriate boundary conditions and the final condition.

At the expiry time T when the CB has been redeemed by the issuer, the investor would technically get the principal Z. Thus, the final condition is

$$V(S,T) = Z. \tag{2}$$

However, there is an option to convert the bond if there is an increase in stock price and thus for maximisation of profit, the condition

$$V(S,T) = \max\{nS, Z\} \tag{3}$$

should always be used to value a CB for any time prior to its expiry. Here, n is the conversion ratio. In other words, the valuation problem of CBs can be mathematically conducted in two time regions: the first region includes all the time up to but not including the expiry time and the second region has one single point on the time axis with $t = T$. In the second region we already know the value of the CB through Eq. (2) and what is being evaluated here is the value the CB in the first region in Eq. (3) (Zhu (2006a)).

When default issues are absent from the valuation task, the boundary condition at $S = 0$ for convertible bonds is expressed as follows:

$$V(0,t) = Ze^{-r(T-t)} + \sum_i K_i e^{-r(t_i - t)} \tag{4}$$

where K_i represents discrete coupon payments to the CBs holder by the issuer and t_i is the time at which the ith coupon will be paid ($t_i > t$). The boundary condition financially implies the similarity of value between the CB and a regular bond when the stock price is very low, which is normally true, except that when the issuers credit risk is added to the valuation, the behaviour of the bond can be seen as a risky bond. When risky bonds default, their value becomes zero. Therefore, this boundary condition has to be altered if we take into consideration this default possibility of risky bonds(Zhu (2006a)). Additionally, since the presence of discrete coupon payments introduces no additional difficulty other than making the solution process a little bit more tedious, we concentrate only on the case with zero coupon payments, which means all K_is in the above equation will be set to zero from now on.

For European style CBs, i.e the bond can only be converted at expiry, the boundary condition for large underlying asset values must be placed at infinity just like that in the problem of European options (Zhu (2006a)).

$$\lim_{S \to \infty} \frac{V(S,T)}{S} = n \tag{5}$$

or $V(S,t) \sim nS$ as $S \to \infty$.

However, most of the convertible bonds issued are of American style, i.e conversion is allowed at any time prior to the expiry of the CB, just like American options. For these American-style CBs, the boundary condition at infinity should be replaced by a set of two conditions, because of the introduction of an unknown moving boundary.

$$\begin{cases} V(S_f(t),t) = nS_f(t) \\ \frac{\partial V}{\partial S}(S_f(t),t) = n \end{cases} \tag{6}$$

The first condition is often called the value-matching condition, while the second one is called the smooth-pasting condition. Here, $S_f(t)$ is a moving boundary which needs to be found as part of the solution.

The CB valuation problem is now completely defined by a differential system composed of Eqs. (1), (3), (4) and (6). To solve this system more efficiently and consistently, we first normalise the system by introducing dimensionless variables as follows (Zhu (2006a)):

$$V' = \frac{V}{Z},\ S' = \frac{S}{Z},\ \tau' = \tau\frac{\sigma^2}{2} = (T-t)\frac{\sigma^2}{2}.$$

and then upon omitting all primes for simplicity, the normalised system can be easily derived as follow:

$$\begin{cases} -\frac{\partial V}{\partial \tau} + S^2\frac{\partial^2 V}{\partial S^2} + (\gamma-\beta)S\frac{\partial V}{\partial S} - \gamma V = 0, \\ V(0,\tau) = e^{-\gamma\tau}, \\ V(S_f(\tau),\tau) = nS_f(\tau), \\ \frac{\partial V}{\partial S}(S_f(\tau),\tau) = n, \\ V(S,0) = \max\{nS, 1\} \end{cases} \tag{7}$$

in which $\gamma = \frac{2r}{\sigma^2}$ and $\beta = \frac{2D_0}{\sigma^2}$ are the risk-free interest rate relative to the volatility of the underlying asset price and the dividend yield rate relative to the volatility of the underlying asset price respectively. There are certain advantages to the normalisation of this system, such as mathematical efficiency in the solution procedure and some financial advantage involving the comparison among CBs of different face values and under different currencies (Zhu (2006a)). Reading from the nondimensional differential system (7), the solution will depend on four parameters: the relative interest rate γ, the conversion ratio n, the relative dividend payment rate β and the dimensionless time to expiry $\tau_{\text{exp}} = T.\frac{\sigma^2}{2}$. The introduction of time to expiration τ as the difference between the expiration time T and the current time t results in the change of the terminal condition (3) to an initial condition in (7).

3 Analytical approximation of optimal conversion boundary using Laplace transform

To make the boundary conditions at the moving boundary $S = S_f(\tau)$ homogeneous to facilitate the solution procedure, we define a new function U(S,τ) as follow.

$$U(S,\tau) = \begin{cases} V(S,\tau) - nS, \text{ if } S > \frac{1}{n}, \\ V(S,\tau) - 1, \text{ if } S \leq \frac{1}{n} \end{cases} \tag{8}$$

Thus, the differential system (7) can be written as the following two sets of equations and boundary conditions

$$\begin{cases} -\frac{\partial U}{\partial \tau} + S^2 \frac{\partial^2 U}{\partial S^2} + (\gamma - \beta) S \frac{\partial U}{\partial S} - \gamma U = n\beta S, \\ U(S_f(\tau),\tau) = 0, \\ \frac{\partial U}{\partial S}(S_f(\tau),\tau) = 0, \\ U(S,0) = 0 \end{cases} \quad \text{if } S > \frac{1}{n} \tag{9}$$

$$\begin{cases} -\frac{\partial U}{\partial \tau} + S^2 \frac{\partial^2 U}{\partial S^2} + (\gamma - \beta) S \frac{\partial U}{\partial S} - \gamma U = \gamma, \\ U(0,\tau) = e^{-\gamma\tau} - 1, \\ U(S,0) = 0 \end{cases} \quad \text{if } S \leq \frac{1}{n} \tag{10}$$

To guarantee U being a continuous function of S, the continuity of the unknown function U and its derivatives are introduced on the boundary $S = \frac{1}{n}$, which results in the following interfacial matching conditions

$$\begin{cases} \lim_{S \to \frac{1}{n}^-} U(S,\tau) = \lim_{S \to \frac{1}{n}^+} U(S,\tau) \\ \lim_{S \to \frac{1}{n}^-} \frac{\partial U}{\partial S}(S,\tau) = \lim_{S \to \frac{1}{n}^+} \frac{\partial U}{\partial S}(S,\tau) - n \end{cases} \tag{11}$$

where $\frac{1}{n}^-$ indicates S approaching $\frac{1}{n}$ from the left and $\frac{1}{n}^+$ indicates S approaching $\frac{1}{n}$ from the right.

Laplace transform is then performed on the system (9)-(11). For the CB price U and the optimal conversion exercise price S_f, all three conditions for the existence of the Laplace transform (Hildebrand (1976)) are satisfied and all variables in the Laplace space are denoted with bars. For instance,

$$\mathscr{L}U(S,\tau) = \int_0^\infty e^{-p\tau}U(S,\tau)d\tau = \overline{U}(S,p)$$
$$\mathscr{L}S_f(\tau) = \int_0^\infty e^{-p\tau}S_f(\tau)d\tau = \overline{S_f}(p)$$

Under the Laplace transform after the initial conditions have been substituted in, the systems (9)-(11) become the following ordinary differential equation systems in terms of parameter p. The derivation of the differential equation under Laplace transform and the interfacial matching conditions should be straightforward.

$$\begin{cases} -[p\overline{U} - 0] + S^2\frac{d^2\overline{U}}{dS^2} + (\gamma - \beta)S\frac{d\overline{U}}{dS} - \gamma\overline{U} = \frac{n\beta S}{p}, \\ \overline{U}(p\overline{S}_f(p), p) = 0, \\ \frac{d\overline{U}}{dS}(p\overline{S}_f(p), p) = 0, \end{cases} \tag{12}$$

$$\begin{cases} -[p\overline{U} - 0] + S^2\frac{d^2\overline{U}}{dS^2} + (\gamma - \beta)S\frac{d\overline{U}}{dS} - \gamma\overline{U} = \frac{\gamma}{p}, \\ \overline{U}(0, p) = \frac{-\gamma}{p(p+\gamma)}, \end{cases} \tag{13}$$

$$\begin{cases} \overline{U}(\frac{1}{n}^-, p) =, \overline{U}(\frac{1}{n}^+, p) \\ \frac{d\overline{U}}{dS}(\frac{1}{n}^-, p) = \frac{d\overline{U}}{dS}(\frac{1}{n}^+, p) - \frac{n}{p} \end{cases} \tag{14}$$

However, the treatment of the two nonlinear moving boundary conditions requires an approximation based on the pseudo-steady-state approximation used in the American put option problem (Zhu (2006b)), where approximate solutions have been able to be worked out with good accuracy. Based on the pseudo-steady-state approximation, the optimal conversion exercise boundary is assumed to move slowly compared to the diffusion of the option price ($\mathscr{L}S = \mathscr{L}S_f(\tau) \to \frac{S}{p} = \overline{S}_f$), we can hold S as a constant during the Laplace transform, which will then be replaced by the Laplace transform performed on the interfacial condition

$S = S_f(\tau)$, with S being held as a constant as well. That means the moving boundary condition $U(S_f(\tau), \tau) = 0$ in the original time space can be approximated by the boundary condition $\overline{U}(S, p) = 0$, with $S = p\overline{S}_f$ in the Laplace space. Similarly, we have the same argument for the second moving boundary condition in (12).

The solution of the differential system (12)-(14) is expressed as follow

$$\overline{U}(S,p) = \begin{cases} D_1 S^{q_1} + D_2 S^{q_2} - \frac{n\beta S}{p(p+\beta)} \text{ for } S > \frac{1}{n} \\ \\ D_3 S^{q_1} + D_4 S^{q_2} - \frac{\gamma}{p(p+\gamma)} \text{ for } S \le \frac{1}{n} \end{cases} \tag{15}$$

where $q_{1,2}$ are roots of the characteristic equation of the homogeneous part of the corresponding equation (i.e, the quadratic equation $q^2 + (\gamma - \beta - 1)q - (p + \gamma) = 0$)

$$q_{1,2} = -\alpha \pm \sqrt{\alpha^2 + (p+\gamma)}, \text{ where } \alpha = \frac{\gamma - \beta - 1}{2}; \tag{16}$$

and D_1, D_2, D_3 and D_4 are four arbitrary complex constants to be determined in order to satisfy all boundary conditions. It can also be shown that q_1 is always positive and q_2 is always negative and the boundary condition in Eq. (13) implies D_4 to be set to 0 in order for the solution to be bounded.

The satisfaction of the remaining boundary conditions as well as the interface conditions leads to a set of algebraic equations:

$$\begin{cases} D_1(p\overline{S}_f)^{q_1} + D_2(p\overline{S}_f)^{q_2} = \frac{n\beta\overline{S}_f}{p+\beta} \\ \\ D_1 q_1 (p\overline{S}_f)^{q_1-1} + D_2 q_2 (p\overline{S}_f)^{q_2-1} = \frac{n\beta}{p(p+\beta)} \\ \\ D_1(\frac{1}{n})^{q_1} + D_2(\frac{1}{n})^{q_2} - \frac{\beta}{p(p+\beta)} = D_3(\frac{1}{n})^{q_1} - \frac{\gamma}{p(p+\gamma)} \\ \\ D_1 q_1(\frac{1}{n})^{q_1-1} + D_2 q_2(\frac{1}{n})^{q_2-1} - \frac{n\beta}{p(p+\beta)} = D_3 q_1(\frac{1}{n})^{q_1-1} - \frac{n}{p} \end{cases} \tag{17}$$

The system of equations is solved by solving for D_1 and D_2 using the first two equations in terms of $\overline{S_f}$. Then upon equating D_1 and D_2 obtaining from the last two equations, a formula for the optimal conversion exercise price in the Laplace space is obtained as follows:

$$\overline{S}_f(p) = \frac{1}{np}\left[\frac{-\beta q_2}{p}\right]^{\frac{1}{q_2-1}} \tag{18}$$

as well as three coefficients from which the option price $U(S,\tau)$ will be determined.

The formula is unfortunately still in terms of the Laplace parameter p, with q_2 being dependent on p as well. In order to obtain an analytical formula for the optimal conversion price, the Laplace inverse transform still need to be carried out using complex inversion formula. Once the optimal conversion exercise price is found, the price of the CB can also be found analytically using the same inversion technique.

By definition, the inversion of Eq. (18) should lead to the optimal exercise price $S_f(\tau)$ in the time domain. Using the complex inversion formula, we obtain

$$\begin{aligned} S_f(\tau) &= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{e^{p\tau}}{np}.\left[\frac{-\beta q_2}{p}\right]^{\frac{1}{q_2-1}}dp \\ &= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{e^{p\tau}}{np}.\exp\left\{\frac{\log(\frac{\beta(\alpha+\sqrt{\alpha^2+p+\gamma})}{p})}{-\alpha-1-\sqrt{\alpha^2+p+\gamma}}\right\}dp \end{aligned} \tag{19}$$

The integration in (19) is to be performed along a straight line $\text{Re}(p) = \mu$ in the complex plane, where μ is a real number to be chosen so that $\overline{S}_f$ is analytic to the right of all the singularities.

The inversion of (19) is complex due to the fact that the integrand appears to have two branch cuts, one from $p = -\infty$ to $p = -\alpha^2 - \gamma$ and another one associated with the logarithm function. Zhu(2006b) introduced a conformal mapping to prove that there was no need to deal with both branch cuts, since the branch cut of the logarithm function would never be reached. We believe that similar reasoning could be applied here, and so we can construct a modification of Bromwich contour in case of one branch point to evaluate the inverse Laplace transform.

A reproduction of Zhu's paper (Zhu (2006)) for the case of American put was conducted to thoroughly understand the inversion technique using complex analysis. A preliminary formula was also obtained for the optimal conversion exercise price S_f using complex analysis. However, further analysis is still needed to check the tedious algebra of this formula and it will be reported separately once finished.

4 Discussions

Talbot's method, a numerical Laplace inversion technique is used to numerically invert $\overline{S}_f$ in MATLAB, which will the be compared to the results obtained using homotopy analysis method. The Talbot algorithm is based on a deformation of the Bromwich contour. It replaces the contour with one that opens towards the negative real axis, which results in the damping of highly oscillatory terms. The key issue in Talbot's method is the choice of the parameters. The method may still be applicable in this case, but with the restriction of time τ (Murli and Rizzardi (1989)).

A similar example of a basic CB to Zhu (2006b) was used for comparison purpose, with conversion being allowed any time prior to expiry. All results was converted back to dimensional quantities before they are graphed and presented and the bond's parameters are as follow:

- Strike price X = \$100
- Risk-free interest rate r = 10%;
- Continuous dividend rate D_0 = 7%;
- Volatility $\sigma = 0.4$;
- Time to maturity T = 1 (year).

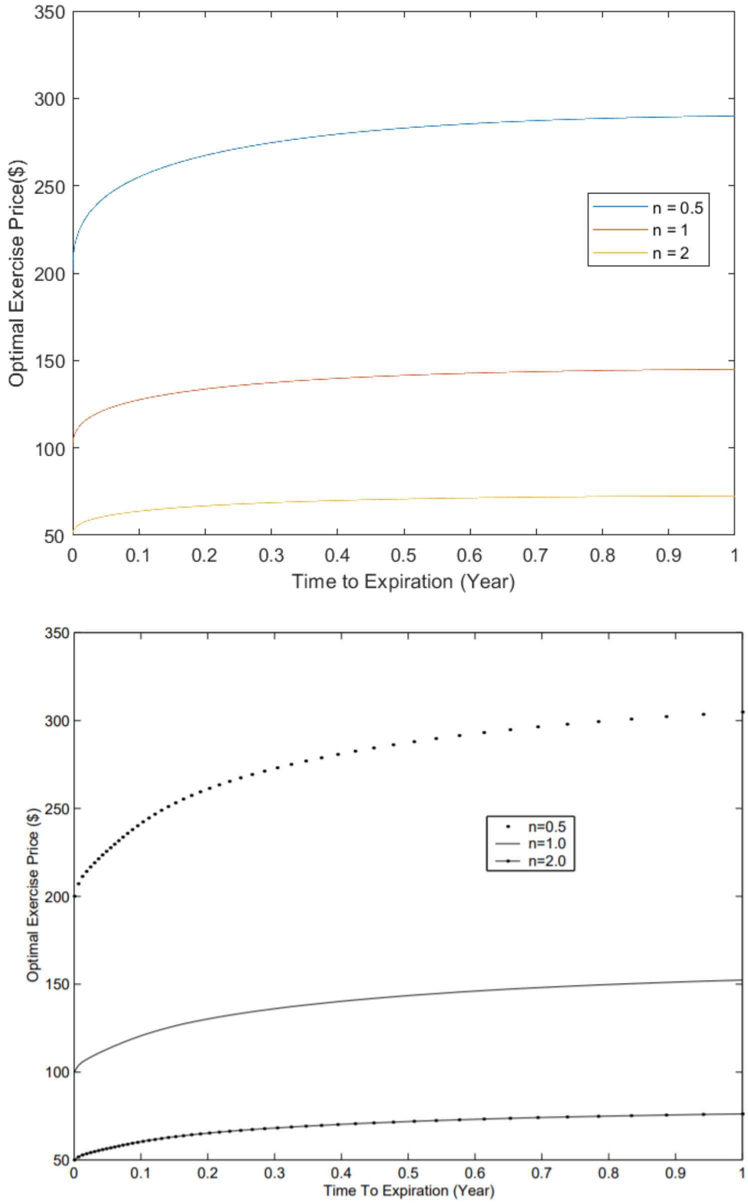

Figure 2: Numerical inversion results vs Zhu (2006b)

As expected, all optimal exercise prices increase monotonically with time to expiry, $\tau = T - t$. Additionally, the curve becomes flatter as the conversion ratio becomes larger, which could also be seen in the results obtained from Zhu (2006b). However, there still appears to be a margin of errors as this could be due to truncation errors when evaluating numerically using Talbot's method.

5 Conclusion

The aim of this research was to look into Laplace transform approach, which was used to obtain an analytical approximation formula for the optimal excise price of the convertible bond with constant interest rate and volatility. Under the Black-Scholes framework, the formula was derived by solving the Black-Scholes partial differential equation in the Laplace space using a treatment of the moving boundary conditions through an approximation made in the Laplace transform of the moving boundary conditions. This formula was inverted numerically using Talbot algorithm, which was then compared to the results obtained using homotopy analysis method, which appeared to be closely similar with a reasonable margin of errors. The proposed approach can be further extended to find an approximation formula for the convertible bond price using inversion technique using complex analysis. Further research on different numerical techniques could also be done to better approximate the optimal exercise price S_f.

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References

[1] Corporate Finance Institue (CFI) 2019, 'Convertible Bond', [Online] Available at: corporatefinanceinstitute.com/resources/knowledge/trading-investing/convertible-bond/. [Accessed 4 February 2019].

[2] F. Black and M. Scholes 1973, 'The Pricing of Options and Corporate Liabilities', J. of Political Economy, vol. 81, no. 3, pp. 637-654.

[3] J. W. Brown and R. V. Churchill 1996, 'Complex Variables and Applications', 6th ed., McGraw-Hill Inc., New York, pp. 183, 213 and 235.

[4] F. B. Hildebrand 1976, 'Advanced Calculus for Applications', Prentice-Hall, Englewood Cliffs, N.J., pp. 57.

[5] A. Murli and M. Rizzardi 1989, 'ALGORITHM 682-Talbots Method for the Laplace Inversion Problem', ACM Transactions on Mathematical Software, Vol. 16, No. 2, pp. 158-168.

[6] M. R. Spiegel 1965, 'Schaum's Outline of Laplace Transforms', McGraw-Hill Education - Europe, New York, pp. 201-210.

[7] P. Wilmott, S. Howison and J. Dewynne 1995, 'The Mathematics of Financial Derivatives', Cambridge University Press, Oxford, pp. 286-294.

[8] S.P. Zhu 2006a, 'A closed-form analytical solution for the valuation of convertible bonds with constant dividend yield', ANIZAM J., 47, pp. 447-494.

[9] S.P. Zhu 2006b, 'A new analytical approximation formula for the optimal exercise boundary of American put options', International Journal of Theoretical and Applied Finance, 9 (7), pp. 1141-77.

[10] X. Lu and E.R.M. Putri (2015). 'Semi-analytic valuation of stock loans with finite maturity', Communications in Nonlinear Science and Numerical Simulation Journal, 27, pp. 206-215.