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A Topological Study of The Grothendieck-Teichmüller Group

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1 Introduction

The goal of this report is to give an accessible answer to the question "what is the Grothendieck-Teichmüller group?" while providing an overview of the background materials involved. We start by briefly discussing the mathematical context in which the Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$ arises before introducing the geometric stage where it resides. We then conclude with the formal definition of the group $\widehat{\mathrm{GT}}$ offered by Drinfeld in 1991.

1.1 Convention and Notation

In this report, whenever we say a surface we will always mean a closed orientable surface, possibly with finitely many points or disjoint open discs removed. When a surface X has n points removed we either say X is a surface with n punctures or n marked points, and when X has b disjoint open disc removed we say that X has b boundary components. We often will write $\Sigma_{g,b}^n$ to denote any surface of genus g with b boundary components and n punctures, and we denote their fundamental group simply as $\pi_{g,b}^n$. When either b or n are zero, we suppress them from the notation and simply write $\Sigma_{g,b}$ or π_g^n. We will also denote the closed unit interval by $I = [0, 1]$.

1.2 Motivation

We begin by discussing the mathematical context in which the Grothendieck-Tecihmüller group arises. It is known that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is of prime interest to the number theorists. Indeed, the history of the Grothendieck-Teichmüller group begins with the remarkable theorem called Belyi's theorem which states that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ injects into the group $\widehat{\pi}_{0,3}$, the profinite completion of the fundamental group of the genus zero surface with three punctures. Since the sphere with three punctures is homotopic to the figure eight space, note that the group $\widehat{\pi}_{0,3}$ is the profinite completion of the free group on two generators. Unfortunately, this group is too big to have any hope to be isomorphic to the Galois group. What we want is some subgroup K of $\widehat{\pi}_{0,3}$ that sits in between the Galois group and the full profinite group.

$$\begin{tikzcd} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[rr, hook] \arrow[dr, hook] & & \widehat{\pi}_{0,3} \\ & K \arrow[ur, hook] & \end{tikzcd}$$

Ideally we want the group K to be isomorphic to the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ while admitting a sufficiently geometric description as a subgroup of $\widehat{\pi}_{0,3}$. This would allow us to work with the Galois group with all its number-theoretic properties purely geometrically by working with the subgroup K. In this sense, the Grothendieck-Teichmüller group is a candidate for such a subgroup.

2 Sketch of a Programme

The Grothendieck-Teichmüller group begins with Grothendieck's "Esquisse d'un Programme" (Sketch of a Programme [Gro97]) where he proposes a Lego game of surfaces. Suppose two surfaces with non-zero boundary components each. By gluing along a common boundary component, we may glue these surfaces together as if they are pieces of Lego bricks.

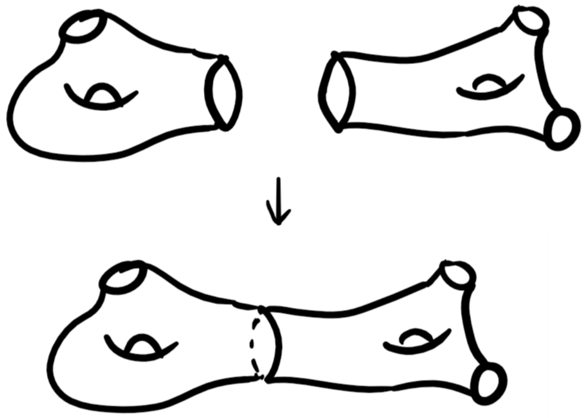

Figure 1: Lego Tecihmüller

It turns out that in this game of Lego surfaces, the basic building blocks are given by the pants surface and the disc surface. To be more specific, it is clear that class of surfaces that we consider can be constructed from gluing together several copies of the pants surface and the disc surface. In fact, when the Euler characteristics of the surface is negative then we only need copies of pants surface. Note that the pants surface is homotopic to the sphere with three punctures.

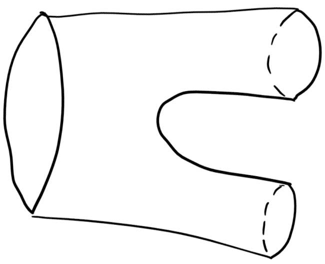

Figure 2: pants surface

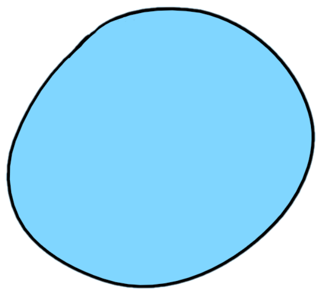

Figure 3: disc surface

The following figure illustrates the construction of the surface $\Sigma_{1,3}$ using three pairs of pants surfaces.

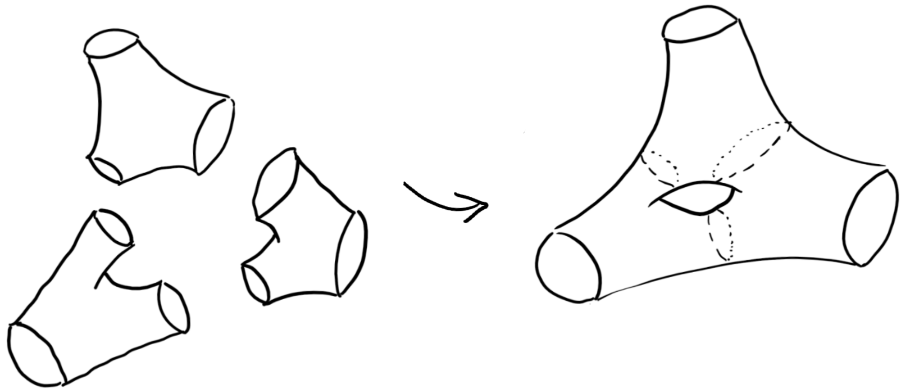

Figure 4: Lego construction

2.1 Mapping Class Groups

It should be noted that we are primarily interested in working with the groups associated to each surfaces called its mapping class groups. In this section we define isotopies and the mapping class groups. We will also see that the mapping class groups are compatible with our game of Lego surfaces. For a thorough and accessible introduction to mapping class groups, see the book [FM11].

Let X be a surface with boundary component ∂X. Suppose that $f, g\colon X \to X$ are two homeomorphisms fixing the boundary components. Recall that a homotopy is a continuous map $H\colon X \times I \to X$ such that at the end points of I, we have $H(-,0) = f(-)$ and $H(-,1) = g(-)$.

Definition 2.1. An *isotopy* from f to g relative to the boundary components ∂X is thus defined as a homotopy $H\colon X \times I \to X$ satisfying the extra condition that at every t in I the map $H(-,t)$ remains a homeomorphism fixing the boundary components ∂X. Just like homotopy, isotopy defines an equivalence relation on orientation preserving homeomorphisms relative to ∂X.

Suppose we have a surface X and $\mathrm{Homeo}(X)$ the group of homeomorphisms on X, equipped with the compact-open topology. Write $\mathrm{Homeo}^{+}(X)$ to denote the subgroup of orientation preserving homeomorphisms of X and $\mathrm{Homeo}^{+}(X, \partial X)$ to be the stabilizer subgroup of the boundary components ∂X.

Definition 2.2. The *mapping class group* of a surface X is the group of isotopy classes of $\mathrm{Homeo}^{+}(X, \partial X)$ denoted by $\Gamma(X) = \pi_0\left(\mathrm{Homeo}^{+}(X, \partial X)\right)$.

We will now describe some examples of (non-trivial) elements of the mapping class groups. Suppose We have a cylinder $X = S^1 \times I$ (i.e. a surface of type Σ_0^2) and $\alpha : S^1 \to X$ is a simple closed curve at the equator. We may twist X along the oriented curve α to obtain a homeomorphism of X called a *Dehn twist* along α. The following figure illustrates the twist acting on the blue straight line on the cylinder.

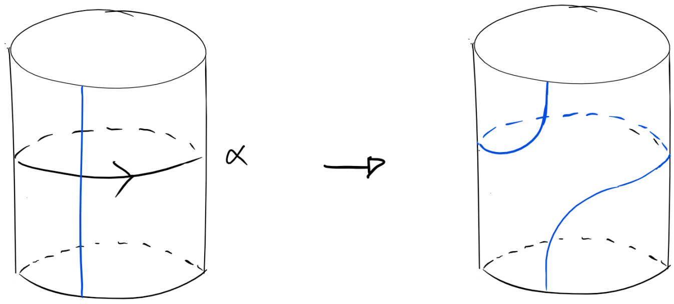

Figure 5: a Dehn twist

Dehn twists forms an important class of elements of mapping class groups of surfaces. These groups elements are represented by isotopy classes of closed curves on your surfaces and their action are relatively simple. It turns out that only finitely many Dehn twists are required to generate the mapping class groups of closed surfaces, while for more general surfaces they still generate the whole mapping class groups with the help of finitely many non-Dehn twists. Dehn twists along curves on more general surfaces are define similarly.

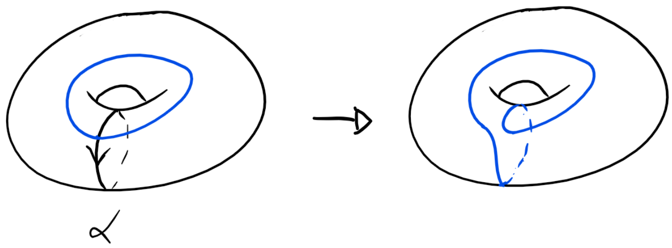

Figure 6: a Dehn twist on a torus

2.2 Lego Teichmüller

Recall that in our game of Lego surfaces, we are interested in gluing surfaces together along a common boundary component. Suppose we have some gluing construction $X \amalg X' \to X \amalg X'/{\sim} = Y$, we can also see this as a subsurface inclusion of X to the bigger surface Y.

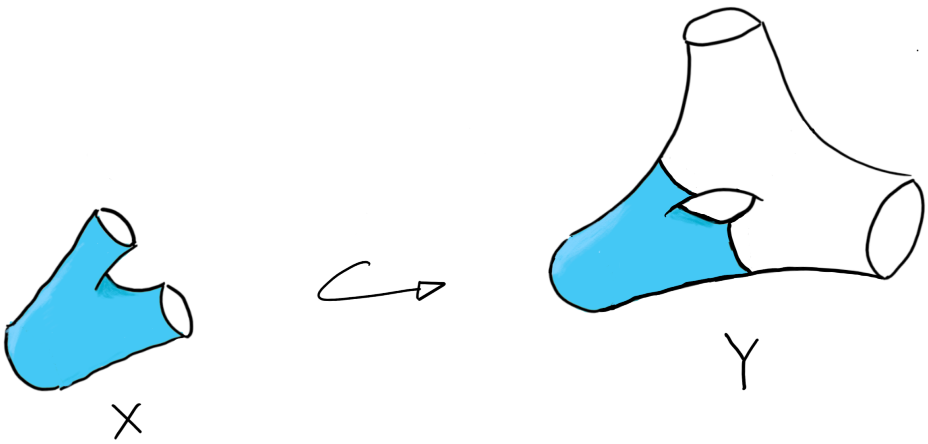

Figure 7: subsurface inclusion

Given a subsurface inclusion $i\colon X \to Y$ as above, there is a natural map from $\Gamma(X) \to \Gamma(Y)$ given by extending a homeomorphism of X to all of Y by the identity. To be more precise, let $f\colon X \to X$ be a representative homeomorphism of some isotopy class in $\Gamma(X)$. We define a group homomorphism $i_*\colon \Gamma(X) \to \Gamma(Y)$ as the map that sends f to to the map $i_*(f)$ given by

$$i_*(f)(x) = \begin{cases} i \circ f \circ i^{-1}(x) & \text{if } x \in i(X) \\ \mathrm{id}_Y(x) & \text{if } x \notin i(X). \end{cases}$$

The *Teichmüller tower* is defined as the collection of all mapping class groups of surfaces $\{\Gamma(X)\}$ equipped with all the natural homomorphisms $i_{X,Y}\colon \Gamma(X) \to \Gamma(Y)$ induced by subsurface inclusions. Grothendieck proposed that we study the automorphism group of an analogous tower with each mapping class groups $\Gamma(X)$ replaced by their profinte completion $\widehat{\Gamma}(X)$. We will elaborate on what this means in the next section. This tower is often called the *profinite Teichmüller tower* and its corresponding automorphism group became what is known as the Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$. With this definition, an element of the Grothendieck-Teichmüller group consits of automorphisms on each objects $\hat{f}_X\colon \widehat{\Gamma}(X) \to \widehat{\Gamma}(X)$, compatible with every natural homomorphisms $\hat{i}_{X,Y}\colon \widehat{\Gamma}(X) \to \widehat{\Gamma}(Y)$ in the sense that the following diagram commutes.

$$\begin{tikzcd} \widehat{\Gamma}(Y) \arrow[r, "\hat{f}_Y"] & \widehat{\Gamma}(Y) \\ \widehat{\Gamma}(X) \arrow[u, "\hat{i}_{X,Y}"] \arrow[r, "\hat{f}_X"'] & \widehat{\Gamma}(X) \arrow[u, "\hat{i}_{X,Y}"'] \end{tikzcd}$$

2.3 Profinite Completions

Let G be a group equipped with the discrete topology. We present two definition of a profinite completion of a group $\widehat{G}$. The first definition is given in a categorical language while in the second definition we will give a more concrete description of the group $\widehat{G}$. We write $\mathcal{F}_G$ to denote the collection of normal subgroups N of G such that G/N is a finite group. See [Oss] for a short note discussing some basic properties of profinite groups.

Definition 2.3. Consider the diagram consisting of objects G/N for every normal subgroups $N \in \mathcal{F}_G$ with morphisms given by quotient maps $\varphi_{N,M}\colon G/N \to G/M$ whenever $N \subseteq M$. The profinite completion $\widehat{G}$ of the group G is defined as the limit over this diagram in the category of topological groups.

$$\widehat{G} = \varprojlim_{N \in \mathcal{F}_G} G/N$$

We say that a group G is a profinite group if G can be obtained as the limit of a diagram of finite groups equipped with the discrete topology.

Definition 2.4. The profinite completion of the group G is the subgroup $\widehat{G}$ of $\prod_{N\in\mathcal{F}_G} G/N$ consisting of elements $(g_N)_{N\in\mathcal{F}_G}$ in $\prod_{N\in\mathcal{F}_G} G/N$ such that whenever $N \subseteq M$, the quotient map $\varphi_{N,M}\colon G/N \to G/M$ sends the element g_N to the element $\varphi_{N,M}(g_N) = g_M$. By regarding each factor G/N as a discrete topological group, we equip this group with the subspace topology of the product topology in $\prod_{N\in\mathcal{F}_G} G/N$.

Writing the projection maps as $\pi_N\colon \widehat{G} \to G/N$, the condition given above is equivalent to the requirement that the following diagram commutes.

$$\begin{tikzcd} & \widehat{G} \arrow[dl, "\pi_N"'] \arrow[dr, "\pi_M"] & \\ G/N \arrow[rr, "\varphi_{N,M}"'] & & G/M \end{tikzcd}$$

We briefly mention the following basic result on profinite groups. For proof of statement and discussion on profinite groups, consult [Oss].

Definition 2.5. Let G be a topological group. A subset $S \subseteq G$ topologically generates G if the only closed subgroup containing S is G itself.

Theorem 2.6. *Let G be a group and $\widehat{G}$ be its profinite completion. There exists a natural map $G \to \widehat{G}$ given by $g \mapsto (gN)_{N\in\mathcal{F}_G}$. Moreover, the image of the group G topologically generates $\widehat{G}$.*

Example 2.7. If G is a finite group then its profinite completion $\widehat{G}$ is isomorphic to G. This is because the trivial subgroup e is a finite rank normal subgroup when G is finite, and hence G/e is one of the factors in the product of groups $\prod_{N\in\mathcal{F}_G} G/N$. The map $G \to \widehat{G}$ is clearly an injective homomorphism and the inverse map is given by the projection map $\pi_e\colon \widehat{G} \to G/e \simeq G$.

Example 2.8. The profinite completion of the integer $\mathbb{Z}$ is given by the product of the p-adic integers $\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p$ where p is prime.

3 Drinfeld's Construction

Although it was Grothendieck who proposed the study of the Teichmüller tower, it was not until 1991 when Drinfeld constructed the Grothendieck-Teichmüller group. In [Dri91], the group $\widehat{\mathrm{GT}}$ was defined a certain subset of the profinite completion of the free group on two generators. The following definition of $\widehat{\mathrm{GT}}$ was due to Drinfeld but this formulation can be found in [Sch97].

We begin by introducing some notation following [Sch97]. Let F_2 be the free group on two generators x and y. Given its profinite completion $\widehat{F}_2$, let $\widehat{F}_2'$ denote the commutator subgroup $[\widehat{F}_2, \widehat{F}_2]$. Let $\widehat{\mathbb{Z}}$ be the profinite integer and $\widehat{\mathbb{Z}}^*$ be its group of units. Suppose G is a profinite group and $\widehat{F}_2 \to G$ is a group homomorphism given by $x \mapsto a$ and $y \mapsto b$. Given an element $f \in \widehat{F}_2$, let $f(a, b)$ denote the image of f under this homomorphism.

We are also interested in particular elements of the mapping class group $\Gamma\left(\Sigma_0^5\right)$ of the sphere with five boundary components. Suppose we fix a labeling on the boundary components of Σ_0^5 from one to five. Let x_{ij} be the Dehn twist in the mapping class group $\Gamma\left(\Sigma_0^5\right)$ represented by the closed curves containing boundaries i and j.

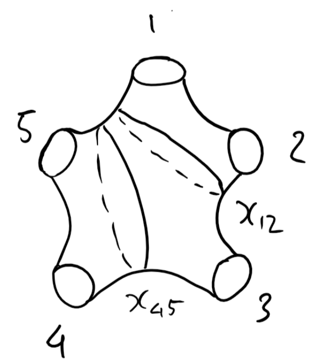

Figure 8: curves representing x_{12} and x_{45}

Definition 3.1. The Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$ are given by pairs $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$ such that the map $x \mapsto x^\lambda$ and $y \mapsto f^{-1} y^\lambda f$ induces an automorphism of $\widehat{F}_2$ which further satisfies the following three relations

(I) $f(x, y) f(y, x) = 1$ where x, y are the generators of $\widehat{F}_2$;

(II) $f(a, b) a^m f(c, a) c^m f(b, c) b^m = 1$ whenever $abc = 1$ in $\widehat{F}_2$ and $m = \frac{\lambda - 1}{2}$

(III) $f(x_{12}, x_{23}) f(x_{34}, x_{45}) f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{51}) = 1$ where x_{ij} are elements in $\Gamma\left(\Sigma_0^5\right)$ represented by closed curves containing i-th and j-th boundary components.

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