AMSI VACATION RESEARCH SCHOLARSHIPS 2018-2019

Fourier Optics, Hermite Functions, and Prolates

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Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

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Abstract

In this paper we outline the relationships between the fractional Fourier transform and propagation of light in quadratic-phase optical systems, specifically quadratic graded-index media. The propagation of light in quadratic graded-index media with periodic apertures is analysed and a matrix operator computed using the Legendre polynomials and Hermite functions. We reconstruct the eigenfunctions of the operator from the eigenvectors and compare them to the prolate spheroidal wave functions.

1 Introduction

The fractional Fourier transform has rich applications in numerous fields, including optics. In the first subsection of this introduction we provide the necessary background of Fourier analysis. In the second subsection we cover the relevant background and other related topics in optics. In the third subsection we tie these concepts together with Fourier optics. In the second section of the paper we present the problem to be analysed and our theoretical derivations and results. In the third section we describe our MATLAB simulation and present our results. Most of the theoretical background behind this project was sourced from the book *"The Fractional Fourier Transform with Applications in Optics and Signal Processing"* by Ozaktas, Zalevsky, and Kutay [1].

1.1 Fourier Analysis

Definition 1 (Lebesgue Spaces) *The* Lebesgue space $L^p(\mathbb{R})$ *is the set of measurable functions* $f(t)$ *such that the pth* Lebesgue norm *is finite:*

$$||f(t)||_p := \left(\int_D |f(t)|^p dt\right)^{\frac{1}{p}} < \infty \tag{1.1.1}$$

for $1 \leq p < \infty$. *The 2nd Lebesgue norm squared,* $||f(t)||_2^2$, *is called* energy.

Reference: [1]

Definition 2 (Integral Transforms) *An* integral transform *takes functions over some domain* $D \subset \mathbb{R}$ *to*

$$T[f](x) = \int_D K(x, x')f(x')dx' \tag{1.1.2}$$

where $x \in D$. *The function* $K(x, x')$ *characterises the transform and is called the* kernel *of the transform.*

Reference: [1]

Definition 3 (Linear Canonical Transforms) Linear Canonical Transforms *(LCTs) are a subclass of integral transforms parameterised by a matrix M of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1.3}$$

where

$$det(M) = AD - BC = 1. \tag{1.1.4}$$

The LCT X parameterised by M is given by

$$X_{(A,B,C,D)}[f](x) = \sqrt{\frac{1}{iB}} e^{i\pi \frac{D}{B} x^2} \int_{-\infty}^{\infty} e^{-i2\pi \frac{1}{B} x x' + i\pi \frac{A}{B} x'^2} f(x') dx' \tag{1.1.5}$$

for $B \neq 0$ *and*

$$X_{(A,0,C,D)}[f](x) = \sqrt{D} e^{i\pi C D x^2} f(Dx') \tag{1.1.6}$$

for $B = 0$.

The composition of two LCTs is an LCT corresponding to the multiplication of their two corresponding matrices. Reference: [1]

Definition 4 (The Fourier Transform) *The* Fourier Transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is an LCT relating the spatial or temporal representation of a function to its spatial or temporal frequency representation respectively. For* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*, it is defined by*

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(t) e^{-i2\pi\xi t} dt. \tag{1.1.7}$$

As an LCT, it is parameterised by the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{1.1.8}$$

The following properties hold:

$$\mathcal{F}^2 = \mathcal{P} \tag{1.1.9}$$

$$\mathcal{F}^3 = \mathcal{F}^{-1} \tag{1.1.10}$$

$$\mathcal{F}^4 = \mathcal{I} \tag{1.1.11}$$

where $\mathcal{P}$ is the parity operator ($\mathcal{P}f(x) = f(-x)$) and $\mathcal{I}$ is the identity operator. Energy is conserved under the Fourier transform:

$$||f||_2^2 = ||\mathcal{F}f||_2^2. \tag{1.1.12}$$

Reference: [1]

Definition 5 (The Fractional Fourier Transform) *The* Fractional Fourier Transform *is a generalised case of the Fourier Transform. The Fractional Fourier Transform of angle* θ $\mathcal{F}_\theta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is defined by:*

$$\mathcal{F}_\theta[f](u) = \sqrt{1 - icot(\theta)}e^{i\pi cot(\theta)u^2} \int_{\mathbb{R}} e^{-i2\pi\left[csc(\theta)ux - \frac{cot(\theta)}{2}x^2\right]} f(x)dx. \tag{1.1.13}$$

It is the LCT parameterised by the matrix

$$\begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}. \tag{1.1.14}$$

The following properties hold:

$$\mathcal{F}_{a\frac{\pi}{2}} = \mathcal{F}^a \tag{1.1.15}$$

where $a \in \mathbb{Z}$.

$$\mathcal{F}_\alpha \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta} \tag{1.1.16}$$

where $\alpha, \beta \in \mathbb{R}$.

$$\mathcal{F}_{-\alpha} = \mathcal{F}_\alpha^{-1} \tag{1.1.17}$$

where $\alpha \in \mathbb{R}$. Two notable consequences of these properties are:

$$\mathcal{F}_{\frac{\pi}{2}} = \mathcal{F} \tag{1.1.18}$$

$$\mathcal{F}_0 = \mathcal{F}_{2\pi} = \mathcal{I} \tag{1.1.19}$$

Reference: [1]

Definition 6 (Time and Band Limiting) *The* time-limiting *operator* Q_T *sets a function to zero outside of the interval* $[-\frac{T}{2}, \frac{T}{2}]$.

$$Q_T f(t) = \begin{cases} f(t) & |t| \le \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases} \tag{1.1.20}$$

The band-limiting *operator* B_Ω *sets the Fourier transform of a function to zero outside of the interval* $[-\Omega, \Omega]$.

$$B_\Omega f(t) = \mathcal{F}^{-1} Q_{2\Omega} \mathcal{F} f(t) \tag{1.1.21}$$

Reference: [2]

Theorem 1 (Heisenberg Uncertainty Principle) *For* $f \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} (t - t_0)^2 |f(t)|^2 dt \int_{\mathbb{R}} (\xi - \xi_0)^2 |\mathcal{F}f(\xi)|^2 d\xi \ge \frac{||f||_2^4}{16\pi^2} \tag{1.1.22}$$

$\forall t_0, \xi_0 \in \mathbb{R}$.

A consequence of this is that there can be no non-trivial analytic function which is both time and band-limited, and from (1.1.12) that if any time-limited analytic function is then band-limited, there will be a loss of energy. Reference: [1]

Definition 7 (Paley-Wiener Spaces) *The* Paley-Wiener Space PW_Ω *is the set of functions which are band-limited to the interval* $[-\Omega, \Omega]$*, ie whose Fourier Transforms are zero outside of this interval.*

Reference: [3]

Definition 8 (Legendre Polynomials) *A closed expression for the Legendre polynomial of degree* n*, normalised over* $[-1, 1]$*, is given by Rodrigues' formula:*

$$P_n(x) = \frac{\sqrt{n+\frac{1}{2}}}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n. \tag{1.1.23}$$

They form an orthonormal basis for $L^2([-1, 1])$. Reference: [4]

Definition 9 (Hermite Polynomials) *A closed expression for the* n*-th degree* Hermite polynomial *is given by:*

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}. \tag{1.1.24}$$

Reference: [1]

Definition 10 (Hermite-Gaussian Functions) *An expression for the* n*-th degree* Hermite-Gaussian function *(henceforth referred to as Hermite function) is given by*

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-\frac{x^2}{2}} H_n(x). \tag{1.1.25}$$

The Hermite functions are normalised over $\mathbb{R}$, and are eigenfunctions of the fractional Fourier transform:

$$\mathcal{F}_\alpha h_n = e^{-i\alpha n} h_n. \tag{1.1.26}$$

They form an orthonormal basis for $L^2(\mathbb{R})$. Reference: [1]

Definition 11 (Prolate Spheroidal Wave Functions) Prolate Spheroidal Wave Functions *(PSWFs)* ψ_n *are the eigenfunctions of the operator* $Q_T B_\Omega Q_T$*:*

$$Q_T B_\Omega Q_T \psi_n = \lambda_n \psi_n \tag{1.1.27}$$

Due to the conservation of energy under the Fourier transform (1.1.12), the eigenvalues have magnitude less than 1. Of all functions, the PSWFs minimise the energy loss of this operator. The PSWFs form an orthonormal basis for $L^2(\mathbb{R})$. Prolates of even or odd order are even or odd functions respectively. Reference: [2][5]

1.2 Optics

Definition 12 (Refractive Index) *The* refractive index *n of a material is a measure of how light is slowed within it. The velocity v of light in a medium with refractive index n is $v = \frac{c}{n}$ where c is the speed of light in a vacuum ($n = 1$). A medium whose refractive index is invariant through space is called* homogeneous. *A homogeneous medium with $n = 1$ is referred to as* free space.

Definition 13 (Quadratic Graded-Index Media) *A* quadratic graded-index medium *is an optical medium where the refractive index n varies with distance r from some optical axis z by:*

$$n^2(r) = n_0^2\left(1 - \frac{r^2}{\chi^2}\right) \tag{1.2.1}$$

where χ is an arbitrary real dimensional parameter.

The lower limit of a series of contacting thin lenses in free space as lens thickness approaches zero is a quadratic graded-index medium with n_0 as the refractive index of the lenses (Figure 1). Reference: [1]

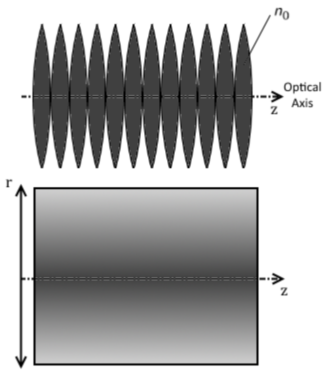

Figure 1: Series of thin lenses (top) and quadratic graded-index medium (bottom). The shade denotes the refractive index - white is free space, dark is higher n.

Definition 14 (The Wave Equation) *The wavefunction $f(x, y, z, t)$ of light evolves in time according to the* wave equation*:*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \tag{1.2.2}$$

where n is the refractive index of the medium as a function of space.

Reference: [1]

Solution 1 (The Wave Equation in a Homogeneous Medium from a Point Source)

Assuming we seek monochromatic solutions of frequency f_o, taking the temporal Fourier transform of the wave equation (1.2.2) gives the Helmholtz equation:

$$\frac{\partial^2 \hat{f}}{\partial x^2} + \frac{\partial^2 \hat{f}}{\partial y^2} + \frac{\partial^2 \hat{f}}{\partial z^2} + \frac{4\pi^2 n^2 f_o^2}{c^2}\hat{f} = 0 \tag{1.2.3}$$

where $\hat{f} = \mathcal{F}f$. A complete set of solutions is the set of monochromatic plane waves where r is the position vector and $\sigma = \frac{nf_o}{c}$ is the spatial frequency vector:

$$\hat{f}(r, \sigma) = e^{i2\pi\sigma\cdot r}. \tag{1.2.4}$$

The intensity of the wave is time-independent:

$$\hat{I}_{\hat{f}}(r) = \left|\hat{f}(r)\right|^2. \tag{1.2.5}$$

Reference: [1]

Theorem 2 (The Superposition Principle) *Since the wave equation is a homogeneous partial differential equation, any linear combination of solutions to the equation is also a solution, and the solution to a linear combination of initial conditions is the corresponding linear combination of their respective solutions.*

Reference: [1]

Definition 15 (Fresnel Approximation) *The* Fresnel approximation *assumes that, for* $r^2 = x^2 + y^2 + z^2$, *assuming* $z^2 \geq x^2 + y^2$, *a spherical wave* $f(x, y, z)$ *is approximately equal to a* parabolic wave*:*

$$f(x, y, z) = \frac{e^{i2\pi\sigma r}}{i\lambda r} \approx \frac{e^{i2\pi\sigma\left(z + \frac{x^2+y^2}{2z}\right)}}{i\lambda z}. \tag{1.2.6}$$

The approximation is valid for small deviations from the optical axis z.

Reference: [1]

Definition 16 (Multiplicative Filters) *A* multiplicative filter *is an operator which multiplies the function it operates on by another function of the same variables characteristic of the filter.*

An aperture is a multiplicative filter characterised by a step function over some interval. The action of a circular aperture of diameter T on a distribution is equivalent to Q_T. Reference: [1]

Definition 17 (Quadratic-Phase Systems) *A* quadratic-phase system *is an optical system equivalent to an LCT. An optical system consisting of arbitrary sequences of thin lenses, sections of quadratic graded-index media, and sections of free space (under the Fresnel approximation).*

Under the Fresnel approximation, propagation of light through any quadratic-phase system is described by an LCT. Reference: [1]

Definition 18 (Fourier Optical Systems) *A* Fourier optical system *is an arbitrary sequence of quadratic-phase systems separated by multiplicative filters. This is not necessarily describable by an LCT.*

Reference: [1]

Definition 19 (The Fresnel Transform) *The* Fresnel transform *is an LCT corresponding to the ABCD matrix*

$$\begin{bmatrix} 1 & \lambda d \\ 0 & 1 \end{bmatrix} \tag{1.2.7}$$

where λ and d are arbitrary real parameters.

Reference: [1]

1.3 Fourier Optics

Solution 2 (The Wave Equation in Free Space from a Planar Amplitude Distribution)

For an initial light intensity distribution on a plane perpendicular to the optical axis (a *planar amplitude distribution*), by decomposing the initial distribution into monochromatic plane waves and using the Fresnel approximation, it can be shown that the planar amplitude distribution a distance d along the optical axis is the Fresnel transform of the initial distribution, where λ is the wavelength of the light. Reference: [1]

Solution 3 (The Wave Equation in a Quadratic Graded-Index Medium)

If there is an initial planar amplitude distribution perpendicular to the optical axis z of a graded-index medium with parameter χ, by substituting equation (1.2.1) into the Helmholtz equation (1.2.3), it can be shown that the planar amplitude distribution at a distance d along the z axis is the fractional Fourier transform of angle $\theta = \frac{d}{\chi}$ of the initial planar amplitude distribution (Figure 2). That is to say, a planar light distribution is continuously fractionally Fourier transformed as it propagates through a quadratic graded-index medium. Reference: [1]

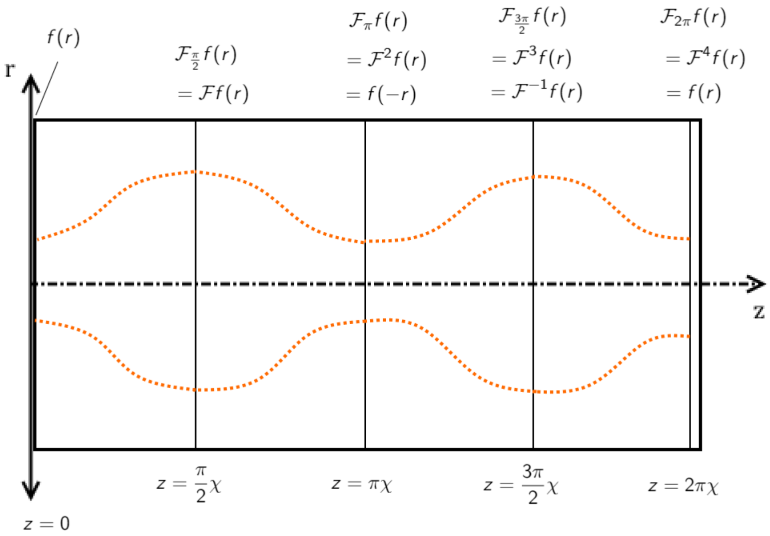

Figure 2: Propagation of a planar amplitude distribution of light through a quadratic graded-index medium.

2 Periodic Apertures in Quadratic Graded-Index Media

2.1 Description

Suppose we place circular apertures at regular intervals along an optical axis with sections of a graded-index medium between them (Figure 3). The apertures are mathematically equivalent to the time-limiting operator Q_2. We use the interval $[-1, 1]$ for simplicity. We restrict ourselves to the two-dimensional case, with one dimension along the optical axis and the other being the domain for the planar amplitude distribution. Between any two adjacent apertures, the optical signal is fractionally Fourier transformed by the same angle. The operator describing the action on a planar amplitude function incident on an aperture, taking the output after the subsequent aperture, is $J_t \equiv Q_2\mathcal{F}_tQ_2$ where $t = \frac{d}{\chi}$, d is the distance between adjacent apertures, and χ is the parameter describing the graded-index medium. Hence the operator describing propagation through n such intervals is $(J_t)^n$. Suppose the total length of these intervals is $z = nd$. If we fix z and χ, and take the limit of the system as $d \to 0$, then the operator describing the system with 'continuous' apertures becomes $\lim_{d\to 0}\left(J_{\frac{d}{\chi}}\right)^{\frac{z}{d}}$.

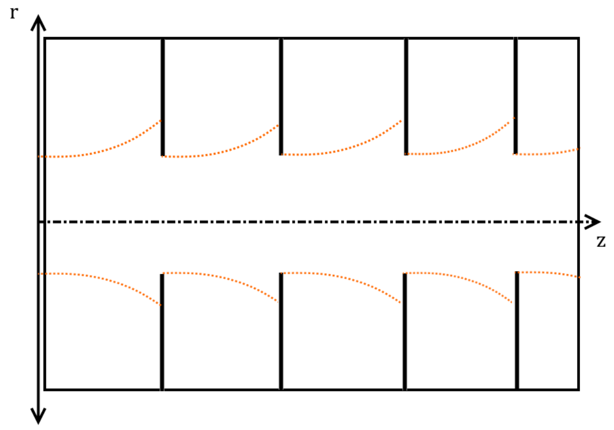

Figure 3: Periodic Apertures in a Quadratic Graded-Index Medium

2.2 Derivation

We wish to compute the action of the operator $Q_2\mathcal{F}_tQ_2$ on an arbitrary function f. We first expand the function in the basis of the normalised Legendre polynomials:

$$f = \sum_{n=0}^{\infty} a_n P_n. \tag{2.2.1}$$

We then apply the Q_2 operator:

$$Q_2 f = Q_2 \sum_{n=0}^{\infty} a_n P_n = \sum_{n=0}^{\infty} a_n Q_2 P_n. \tag{2.2.2}$$

We obtain an expression for $Q_2 f$ in terms of the time-limited Legendre polynomials. We then expand $Q_2 f$ in terms of the Hermite functions:

$$Q_2 f = \sum_{l=0}^{\infty} \langle Q_2 f, h_l \rangle h_l = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} a_n \langle Q_2 P_n, h_l \rangle h_l. \tag{2.2.3}$$

Note that the inner products integrate over the whole line, however are equivalent to inner products over $[-1, 1]$ since at least one of the functions is zero outside this interval. We then apply the fractional Fourier transform and use (1.1.26) to obtain:

$$\mathcal{F}_t Q_2 f = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} a_n \langle Q_2 P_n, h_l \rangle \mathcal{F}_t h_l = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} a_n \langle Q_2 P_n, h_l \rangle e^{-ilt} h_l. \tag{2.2.4}$$

We apply the Q_2 operator:

$$Q_2 \mathcal{F}_t Q_2 f = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} a_n \langle Q_2 P_n, h_l \rangle e^{-ilt} Q_2 h_l. \tag{2.2.5}$$

We expand $Q_2\mathcal{F}_tQ_2f$ in terms of the time-limited Legendre polynomials:

$$Q_2\mathcal{F}_tQ_2f = \sum_{m=0}^{\infty}\langle Q_2\mathcal{F}_tQ_2f, Q_2P_m\rangle Q_2P_m. \tag{2.2.6}$$

Substituting (2.2.5) into (2.2.6), we obtain:

$$Q_2\mathcal{F}_tQ_2f = \sum_{m=0}^{\infty}\langle\sum_{l=0}^{\infty}\sum_{n=0}^{\infty} a_n\langle Q_2P_n, h_l\rangle e^{-ilt}Q_2h_l, Q_2P_m\rangle Q_2P_m \tag{2.2.7}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_n\sum_{l=0}^{\infty}\langle Q_2P_n, h_l\rangle\langle h_l, Q_2P_m\rangle e^{-ilt}Q_2P_m. \tag{2.2.8}$$

This is equivalent to the infinite square matrix equation

$$A\bar{a} = \bar{b} \tag{2.2.9}$$

where

$$A_{mn}(t) = \sum_{l=0}^{\infty}\langle Q_2P_n, h_l\rangle\langle h_l, Q_2P_m\rangle e^{-ilt}, \tag{2.2.10}$$

$$\bar{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \end{bmatrix}, \tag{2.2.11}$$

$$\bar{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \end{bmatrix}, \tag{2.2.12}$$

and

$$Q_2\mathcal{F}_tQ_2f = \sum_{m=0}^{\infty} b_mQ_2P_m \tag{2.2.13}$$

If we impose that f is an eigenfunction of the operator:

$$Q_2\mathcal{F}_tQ_2f = \lambda Q_2f \tag{2.2.14}$$

then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_n\sum_{l=0}^{\infty}\langle Q_2P_n, h_l\rangle\langle h_l, Q_2P_m\rangle e^{-ilt}Q_2P_m = \lambda\sum_{m=0}^{\infty} a_mQ_2P_m \tag{2.2.15}$$

and hence

$$\sum_{n=0}^{\infty} a_n\left(\sum_{l=0}^{\infty}\langle P_n, h_l\rangle\langle h_l, P_m\rangle e^{-ilt}\right) = \lambda a_m. \tag{2.2.16}$$

This is equivalent to the infinite square matrix equation

$$A\bar{a} = \lambda\bar{a} \tag{2.2.17}$$

where

$$A_{mn}(t) = \sum_{l=0}^{\infty} \langle Q_2 P_n, h_l \rangle \langle h_l, Q_2 P_m \rangle e^{-ilt} \tag{2.2.18}$$

and

$$\bar{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \end{bmatrix} \tag{2.2.19}$$

is the vector of the coefficients of the original function in the basis of the Legendre polynomials.

For $t = \frac{\pi}{2}$, the eigenfunctions of the operator $Q_2 \mathcal{F}_t Q_2 = Q_2 \mathcal{F} Q_2$ are also eigenfunctions of the operator $(Q_2 \mathcal{F} Q_2)^2$. Since the prolates are either even or odd, they are eigenfunctions of the parity operator $\mathcal{P}$, and hence of $\mathcal{P} Q_T B_\Omega Q_T$ (1.1.27). We show that this operator is equivalent to $(Q_2 \mathcal{F} Q_2)^2$:

$$\mathcal{P} Q_T B_\Omega Q_T = \mathcal{P} Q_T \mathcal{F}^{-1} Q_{2\Omega} \mathcal{F} Q_T \tag{2.2.20}$$

$$= \mathcal{P} Q_T \mathcal{P} \mathcal{F} Q_{2\Omega} Q_{2\Omega} \mathcal{F} Q_T = \mathcal{P}\mathcal{P} \left(Q_T \mathcal{F} Q_{2\Omega} \right) \left(Q_{2\Omega} \mathcal{F} Q_T \right) = \left(Q_2 \mathcal{F} Q_2 \right)^2 \tag{2.2.21}$$

for $T = 2$ and $\Omega = 1$. Hence the eigenfunctions of $Q_2 \mathcal{F} Q_2$ are prolates corresponding to $Q_2 B_1 Q_2$ (but not with the same eigenvalues).

3 MATLAB Simulation

3.1 Computing the A matrix

We must truncate the matrix $A(t)$ at some dimension `dim` for computational practicality. From (2.2.10), $A = \mathcal{I}$ for $t = 0$. We can test the accuracy of the approximation to `dim` by comparing $A(0)$ to $\mathcal{I}$. To compute the A matrix corresponding to a certain `dim`, we begin by computing the `dim`×`dim` square matrix M whose entries are the inner products of the Hermite functions and time-limited Legendre polynomials:

$$M_{mn} = \langle Q_2 P_{n-1}, h_{m-1} \rangle = \int_{-1}^{1} P_{n-1}(x) h^*_{m-1}(x) dx = \int_{-1}^{1} P_{n-1}(x) h_{m-1}(x) dx \tag{3.1.1}$$

Since the Legendre polynomials and Hermite functions have the same parity as their order, the inner products such that $m + n$ is odd (*ie* an odd function is being integrated) were set to 0 rather than being calculated numerically to save computation time. For the inner products such that $m+n$ is even (*ie* an even function is being integrated), the integral was taken over the domain $[0, 1]$ and multiplied by 2 since a smaller interval takes less computation time.

Up to now, t has not been a factor in the computation. We now fix an arbitrary numerical value of t. We then compute the dim×dim diagonal matrix $E(t)$ such that:

$$E_{mn}(t) = \begin{cases} e^{-i(n-1)t} & m = n \\ 0 & m \neq n \end{cases}. \tag{3.1.2}$$

We find $A(t)$ as the matrix product of the transpose of M, $E(t)$, and M:

$$A(t) = M^T E(t) M. \tag{3.1.3}$$

3.2 Results

3.2.1 Variation of $A(0)$ with dimension

$A(0)$ should be the identity matrix. As dim was increased, the elements on the diagonal approached 1 and the elements off the diagonal approached 0. The entries further down the diagonal approached 0. The entries off the diagonal approached 0 as they got further from the diagonal (Figures 4-6).

$$\begin{bmatrix} 0.8982 & 0 & -0.2195 & 0 & 0.0199 & \dots \\ 0 & 0.6649 & 0 & -0.1598 & 0 & \dots \\ -0.2195 & 0 & 0.1623 & 0 & -0.0188 & \dots \\ 0 & -0.1598 & 0 & 0.0514 & 0 & \dots \\ 0.0199 & 0 & -0.0188 & 0 & 0.0022 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 4: The first 5×5 entries of $A(0)$ for dim$= 5$

$$\begin{bmatrix} 0.9495 & 0 & -0.1142 & 0 & -0.1119 & \dots \\ 0 & 0.8523 & 0 & -0.1931 & 0 & \dots \\ -0.1142 & 0 & 0.7206 & 0 & -0.3251 & \dots \\ 0 & -0.1931 & 0 & 0.6646 & 0 & \dots \\ -0.119 & 0 & -0.3251 & 0 & 0.4734 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 5: The first 5×5 entries of $A(0)$ for dim$= 20$

$$\begin{bmatrix} 0.9773 & 0 & -0.0502 & 0 & -0.0614 & \dots \\ 0 & 0.9321 & 0 & -0.1033 & 0 & \dots \\ -0.0502 & 0 & 0.8874 & 0 & -0.1431 & \dots \\ 0 & -0.1033 & 0 & 0.8365 & 0 & \dots \\ -0.0614 & 0 & -0.1431 & 0 & 0.8005 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 6: The first 5×5 entries of $A(0)$ for `dim`$= 100$

3.2.2 Eigenfunctions

We calculated the eigenvectors of the matrix for various values of t and reconstructed their relevant eigenfunctions (Figures 7, 8, 9). Using MATLAB code created by Roy R Lederman at `https://github.com/lederman/Prol_1D/` [6], we calculated the one-dimensional prolate functions that are eigenfunctions of the operator $Q_2 B_1 Q_2$ (Figure 10). The errors of our results were calculated by taking the difference between these prolates and the eigenfunctions we obtained for $t = \frac{\pi}{2}$ (Figure 11). The error for the first eigenfunction was 3 orders of magnitude less than that of the next eigenfunction, after which the error continues to increase in magnitude.

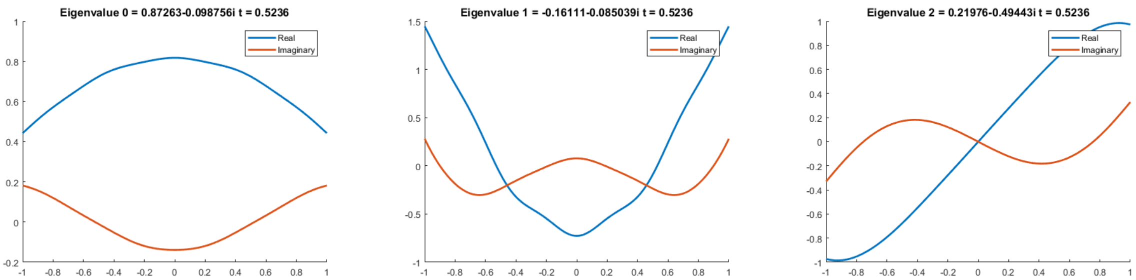

Figure 7: The first three eigenfunctions of $A\left(\frac{\pi}{6}\right)$

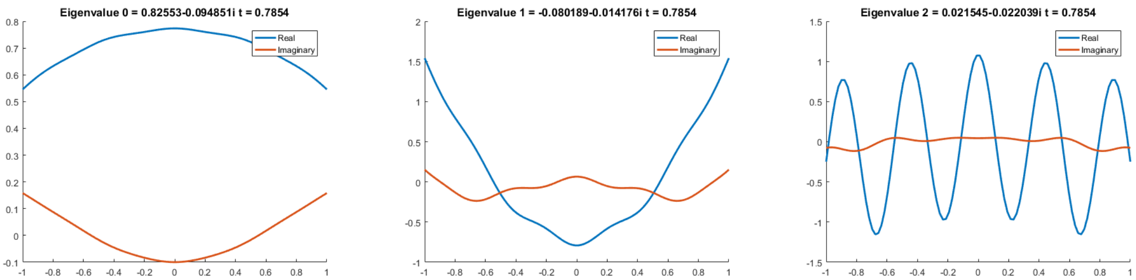

Figure 8: The first three eigenfunctions of $A\left(\frac{\pi}{4}\right)$

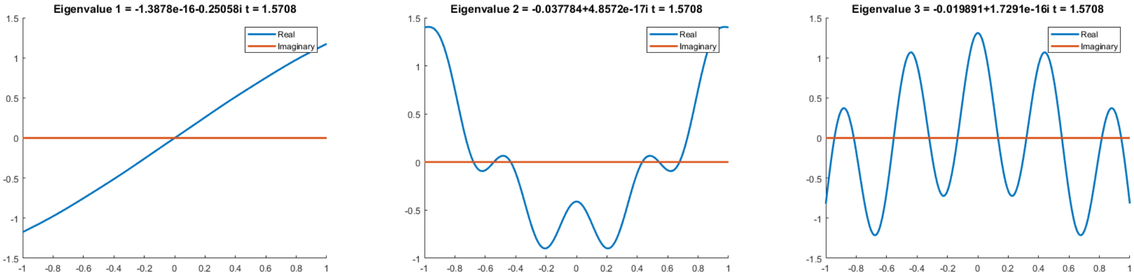

Figure 9: The first three eigenfunctions of $A\left(\frac{\pi}{2}\right)$ corresponding to prolates

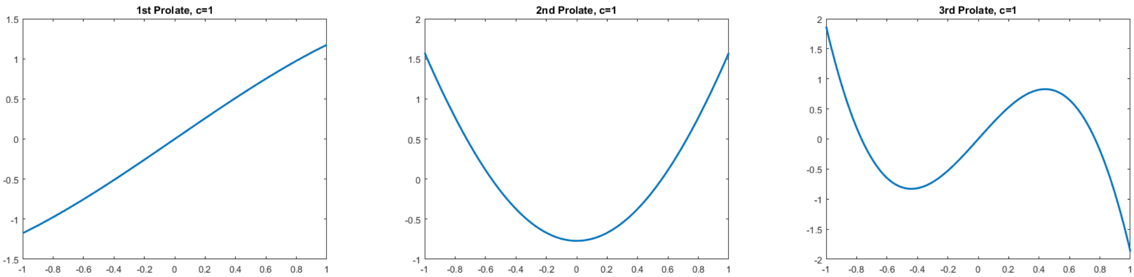

Figure 10: The first three prolates corresponding to $Q_2B_1Q_2$.

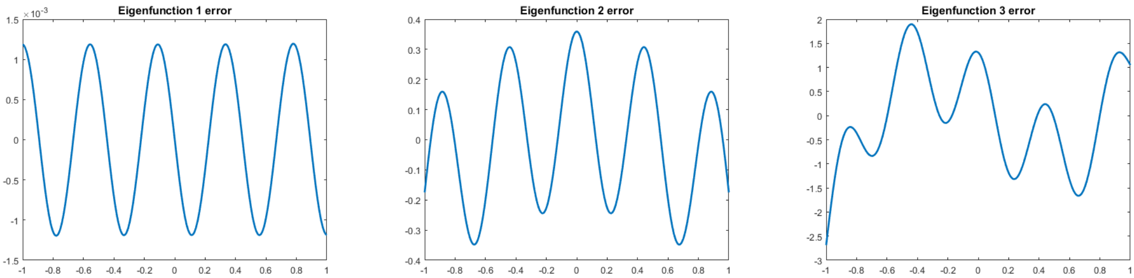

Figure 11: The error of the first three calculated eigenfunctions for $t = \frac{\pi}{2}$.

4 Conclusion

The error of the eigenfunctions of $A(\frac{\pi}{2})$ increased with the order of their corresponding prolates. This is potentially an effect of the truncation of the matrix. A higher `dim` may decrease the error for the earlier eigenfunctions, however this is computationally difficult. A potential improvement would be to approximate the Gaussian component of the Hermite functions by a truncated Taylor series and integrate the resulting polynomials analytically rather than numerically. This would decrease computation expense, allowing a higher `dim` matrix to be reasonably calculated, leading to more accurate results.

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