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Fusion categories from representations of quantum $\mathfrak{sl}_2$ at roots of unity

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Abstract

In this report we study category theory and we build towards the study of fusion category, as well as understanding quantum groups and the representation theory associated to them. We conclude by constructing fusion categories from finite dimensional representations of quantum $\mathfrak{sl}_2$ at roots of unity.

1 Introduction

Given a semi-simple Lie algebra $\mathfrak{g}$, the corresponding quantum group is deformation of the universal enveloping algebra $U(\mathfrak{g})$. Quantum groups depend on a parameter q which can either be taken to be an indeterminate or a non-zero complex number. We denote the quantum group as $U_q(\mathfrak{g})$. Quantum groups were motivated by the study of two dimensional solvable lattice models via the Yang-Baxter equation Baxter (1982). They were discovered by Drinfeld Drinfeld (1987) and Jimbo Jimbo (1985). Since their introduction quantum groups have seen much broader use in mathematics with applications to many different fields, one such application is to the theory theory of fusion categories which is motivated by its interaction with the field of quantum computing. The purpose of this report is to understand examples of fusion categories that come from the theory of quantum groups. We will try to understand different topics in category theory, such as what is a braided category or a ribbon category. We will look at the representation theory of $U_q(\mathfrak{sl}_2)$ for both the case when q is not a root of unity and when q is a root of unity. In the later case we will be looking at Lusztig's version of the quantum group, it will be shown that the category of finite dimensional representations is no longer semi-simple. By using the theory of tilting modules developed by Anderson and collaborators Andersen (1992) Andersen & Paradowski (1995) we will come to understand how one can build interesting structures from the representation theory at roots of unity. Which will be specifically applied to construct semi-simple fusion categories from the representation theory of $U_q(\mathfrak{sl}_2)$ where q is a odd root of unity. The process involves taking the quotient of known "negligible" tilting modules. The structure of a fusion category is richer than that of a tensor category. In particular in this paper we will look at the braided and ribbon structure of the category of finite dimensional $U_q(\mathfrak{sl}_2)$ modules.

2 Preliminary Representation Theory

Definition 2.1. *A representation of an algebra A (also called a left A-module) is a vector space V together with a homomorphism of algebras $\rho : A \to End(V)$. Notationally we denote this action by $a \cdot v := \rho(a)v$*

For the sake of a clear example of a representation rather working with a algebra we will instead work with a group, the definition of a group representation is equivalent to the one for an algebra. let G be the cyclic group of order 3 then

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \rho(r) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}, \ \rho(r^2) = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

is a representation on the vector space $V = span\{v_1, v_2\}$.

Directly leading on from this we have that a *sub-representation* (module) of a representation V of an algebra A is a subspace $W \subseteq V$ such that $\rho(A)(W) \subseteq W$. From the above example we can see that both $span\{v_1\}$ and $span\{v_2\}$ are one-dimensional sub-modules of V.
We say a nonzero representation V of A is *irreducible* if it's only sub-modules are $\{0\}$ and V. Once again sticking with the above example it is clear that V would not be irreducible but instead both of V's sub-modules themselves are irreducible as they are one-dimensional and hence trivially irreducible.
A nonzero representation V of A is said to be *indecomposable* if it is not isomorphic to a direct sum of two nonzero representations. It should be noted that a representation being irreducible implies that is also indecomposable but being indecomposable does not imply irreducibility as a indecomposable representation contains non-zero sub-representations. Examples of indecomposable representations will appear later when looking at the representation theory of $U_q(\mathfrak{sl}_2)$ with q a root of unity.

3 Category Theory

The following concepts come from Bakalov & Kirillov (2001), Rowell (2006).

Definition 3.1. *An Ab-category is a monoidal category in which all morphisms spaces are vector spaces over k and the composition and tensor product of morphisms are bilinear.*

Throughout this report all categories will be assumed to be Ab-categories.

Definition 3.2. *A monoidal category is a category C together with*

i) A bifunctor $\otimes : C \times C \to C$

ii) Natural isomorphisms $\alpha_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$

iii) A unit object $I \in ObC$

iv) And natural isomorphims $\rho_U : U \otimes I \to U$ and $\lambda_U : I \otimes U \to U$

such that it satisfies

$$\begin{CD}
U \otimes (V \otimes (W \otimes Z)) @>\alpha>> (U \otimes V) \otimes (W \otimes Z) @>\alpha>> ((U \otimes V) \otimes W) \otimes Z \\
@V{id \otimes \alpha}VV @. @AA{\alpha \otimes id}A \\
U \otimes ((V \otimes W) \otimes Z) @>>> @>\alpha>> (U \otimes (V \otimes W)) \otimes Z
\end{CD}$$

$$\begin{CD}
U \otimes (I \otimes V) @>\alpha>> (U \otimes I) \otimes V \\
@V{id \otimes \lambda}VV @VV{\rho \otimes id}V \\
U \otimes V @>id>> U \otimes V
\end{CD}$$

Definition 3.3. *A braided tensor category is a monodial category C, equipped with a natural isomorphisms $\sigma_{U,V} : U \otimes V \to V \otimes U$ for all $U, V \in ObC$ such that*

$$\begin{CD}
U \otimes (V \otimes W) @>\alpha>> (U \otimes V) \otimes W @>\sigma>> W \otimes (U \otimes V) \\
@V{id \otimes \sigma}VV @. @VV{\alpha}V \\
U \otimes (W \otimes V) @>\alpha>> (U \otimes W) \otimes V @>\sigma \otimes id>> (W \otimes U) \otimes V
\end{CD}$$

$$\begin{CD}
(U \otimes V) \otimes W @>\sigma \otimes id>> (V \otimes U) \otimes W @>\alpha^{-1}>> V \otimes (U \otimes W) \\
@V{\alpha^{-1}}VV @. @VV{id \otimes \sigma}V \\
U \otimes (V \otimes W) @>\sigma>> (V \otimes W) \otimes U @>\alpha^{-1}>> V \otimes (W \otimes U)
\end{CD}$$

$$\begin{CD}
I \otimes U @>\sigma>> U \otimes I \\
@V{\lambda}VV @VV{\rho}V \\
U @>id>> U
\end{CD}
\qquad
\begin{CD}
U \otimes I @>\sigma>> I \otimes U \\
@V{\rho}VV @VV{\lambda}V \\
U @>id>> U
\end{CD}$$

The reason for the name of this category is evident if one notices that the above relations are equivalent to the braid relations that is to say $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2 \otimes V_3$ where we view $\sigma_1 = \sigma_{V_1,V_2} \otimes id$ and $\sigma_2 = id \otimes \sigma_{V_2,V_3}$.

Definition 3.4. *Let C be a monoidal category and $V \in ObC$. A left dual to V is an object V^* along with two morphisms $ev_V : V^* \otimes V \to I$ and $\pi_V : I \to V \otimes V^*$ such that*

$$(id_V \otimes ev_V) \circ (\pi_V \otimes id) = id_V,$$

$$(ev_{V^*} \otimes id_{V^*}) \circ (id_{V^*} \otimes \pi_{V^*}) = id_{V^*}$$

A monoidal category is rigid if for every object in our category C it has corresponding left and right duals.
Similarly we can define a right dual of an object V to be an object ${}^*V$ along with morphisms $ev'_V : V \otimes^* V \to I$ and $\pi'_V : I \to {}^*V \otimes V$ satisfying similar conditions as above.

Definition 3.5. *A monoidal category is balanced if there exists a family of natural isomorphisms $\theta : V \to V$ for every object in our category such that*

$$\theta_{V \otimes W} = \sigma_{WV} \circ \sigma_{VW} \circ (\theta_V \otimes \theta_W),$$

$$\theta_I = id,$$

$$\theta_V^* = (\theta_V)^*$$

Definition 3.6. *A ribbon category is a balanced, rigid, braided tensor category.*

Definition 3.7. *An Ab-category is semisimple if it has the property that every object X is isomorphic to a finite direct sum of simple, that is objects X_i where $End(X) \cong k$ as well as $Hom(X_i, X_j) = 0$ for all $i \neq j$.*

Definition 3.8. *Let Λ be a set equipped with an involution $\lambda \mapsto \lambda^*$ and a distuished element ω such that $\omega = \omega^*$. A set of fusion rules indexed by Λ is a collection $\{N^\nu_{\lambda,\mu}\}_{\lambda,\mu,\nu \in \Lambda}$ of non-negative integers satisfying the following conditions:*
(i) for each $\lambda, \mu \in \Lambda, N^\nu_{\lambda,\mu} = 0$ for all except finitely many ν;
(ii) $N^\nu_{\lambda,\mu} = N^\nu_{\mu,\lambda}$;
(iii) $N^\nu_{\lambda,\omega} = N^\nu_{\omega,\mu} = \delta_{\lambda,\mu}$;
(iv) $N^\nu_{\lambda,\mu} = N^{\nu^*}_{\mu^*,\lambda^*}$;
(v) $N^\omega_{\lambda,\mu^*} = \delta_{\lambda,\mu}$
(vi) $\Sigma_{\alpha \in \Lambda} N^\beta_{\lambda,\alpha} N^\alpha_{\mu,\nu} = \Sigma_{\alpha \in \Lambda} N^\alpha_{\lambda,\mu} N^\beta_{\alpha,\nu}$.

Definition 3.9. *A fusion category is a semisimple Ribbon Ab-category generated by finitely many simple objects* $\{X_0, X_1, ..., X_n\}$ *such that* $X_i \otimes X_j \cong \bigoplus_k N_{i,j}^k X_k$ *satisfies the fusion rules given above.*

4 Quantum Groups

For $n \in \mathbb{Z}$, let $q \in \mathbb{C} \setminus \{\pm 1\}$, we define the *q-bracket* as such

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + ... + q^{-n+3} + q^{-n+1} \tag{1}$$

Notice if q is not a root of unity, then $[n]_q \neq 0$ for all non-zero integers. In the case of when q is a root of unity, let $d > 2$ denote its order, that is $q^d = 1$. Then define

$$e = \begin{cases} d & \text{if } d \text{ is odd} \\ d/2 & \text{if } d \text{ is even} \end{cases}$$

Then is is clear that

$$[n]_q = 0 \iff n \equiv 0 \text{ modulo } e$$

and we define the following relations as such

$$[m]_q! = [m]_q[m-1]_q \cdot ... \cdot [2]_q[1]_q$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$$

$$\begin{bmatrix} K; m \\ n \end{bmatrix}_q = \prod_{s=1}^{n} \frac{Kq^{m+1-s} - K^{-1}q^{s-1-m}}{q^n - q^{-n}}$$

Analogous to how a Lie algebra is a deformation of its corresponding Lie group The quantum group $U_q(\mathfrak{sl}_2)$ is a quantization of the universal enveloping algebra $U(\mathfrak{sl}_2)$

Definition 4.1. *Define c as the algebra generated by the variables* E, F, K, K^{-1} *subject to the* *relations*

$$KK^{-1} = K^{-1}K = 1$$
$$KEK^{-1} = q^2E,$$
$$KFK^{-1} = q^{-2}F$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

It has a Hopf algebra structure given by,

$$\Delta(E) = 1 \otimes E + E \otimes K,\ \Delta(F) = K^{-1} \otimes F + F \otimes 1, \tag{2}$$
$$\Delta(K) = K \otimes K,\ \Delta(K^{-1}) = K^{-1} \otimes K^{-1} \tag{3}$$
$$\epsilon(E) = \epsilon(F) = 0,\ \epsilon(K) = \epsilon(K^{-1}) = 1, \tag{4}$$
$$S(E) = -EK^{-1},\ S(F) = -KF,\ S(K) = K^{-1},\ S(K^{-1}) = K. \tag{5}$$

More so it is a ribbon hopf algebra, the structures are given in Chari & Pressley (1994)

When looking at the representations of $U_q(\mathfrak{sl}_2)$ one needs distinguish between the cases where q is a root of unity and the case when it is not. The need for this distinction will be clear in the succeeding section.

4.1 Representation theory of $U_q(\mathfrak{sl}_2)$ at q not a root of unity

In the case that q is not a root of unity the representation theory of $U_q(\mathfrak{sl}_2)$ will mimic the representation theory of $U(\mathfrak{sl}_2)$ this will be clear as one will be able to make a connection between e, f and E, F as the lowering and raising operators respectively h and K as the eigen operator. The theory presented in this section is derived from Kassel (1995).

Definition 4.2. *Let V be a $U_q(\mathfrak{sl}_2)$-module and λ a scalar. A vector $v \neq 0$ of V is a highest weight vector of weight λ if $Ev = 0$ and if $Kv = \lambda v$. A $U_q(\mathfrak{sl}_2)$-module is a highest weight module of highest weight λ if it is generated by a highest weight vector of weight λ.*

As will be shown later, all irreducible finite dimensional $U_q(\mathfrak{sl}_2)$ modules are highest weight modules.

Lemma 4.3. *Let v be a highest weight vector of weight lambda. Set $v_0 = v$ and $v_p = \frac{1}{[p]_q!}F^p v$ for $p > 0$. Then*

$$Kv_p = \lambda q^{-2p} v_p, \quad Ev_p = [\lambda; 1-p]v_{p-1}, \quad Fv_{p-1} = [p]_q v_p.$$

Proof.

$$\begin{aligned}
Kv_p &= \frac{1}{[p]_q!}KF^p v = \frac{1}{[p]_q!}q^{-2p}F^p K v = \frac{\lambda}{[p]_q!}q^{-2p}F^p v = \lambda q^{-2p} v_p, \\
Ev_p &= \frac{1}{[p]_q!}EF^p v = \frac{1}{[p]_q!}(F^p E + [p]_q[K; 1-p]F^{p-1})v \\
&= \frac{1}{[p-1]_q!}[\lambda; 1-p]F^{p-1}v = [\lambda; 1-p]v_{p-1} \\
Fv_{p-1} &= \frac{[p]_q}{[p]_q!}F^p v = [p]_q v_p
\end{aligned}$$

□

Theorem 4.4. *Let V be a finite-dimensional $U_q(\mathfrak{sl}_2)$-module generated by a highest weight vector v of weight λ. Then*

(i) $\lambda = \epsilon q^n$ where $\epsilon = \pm 1$ and n is the integer defined by Dim$(V) = n+1$.

(ii) Setting $v_p = \frac{F^p v}{[p]_q!}$, then have $v_p = 0$ for $p > n$. In addition, the set $\{v_0, v_1, ..., v_n\}$ is a basis for V.

(iii) The operator K acting on V is diagonalizable with the $(n+1)$ distinct eigenvalues $\{\epsilon q^n, \epsilon q^{n-2}, ..., \epsilon q^{-n+2},$

(iv) Any other highest weight vector in V is a scalar multiple of v and is of weight λ.

(v) The module V is irreducible.

Proof. (i) According to Lemma 4.3, the sequence $\{v_p\}_{p\geq 0}$ is a sequence of eigenvector for K with distinct eigenvalues. Since V is finite dimensional, there has to exist and integer n such that $v_n \neq 0$ but $v_{n+1} = 0$. From the formulas given in the lemma we have that $v_m = 0$ for all $m > n$ and $v_m \neq 0$ for all $m \leq n$. Lastly we have that

$$0 = Ev_{n+1} = \frac{q^{-1}\lambda - q^n\lambda^{-1}}{q - q^{-1}}v_n..$$

Hence, $q^{-n}\lambda = q^n\lambda^{-1}$, so $\lambda = \pm q^n$.

(iii) This is clear from the previous lemma.

(iv) Let v' be another highest weight vector. It is an eigenvector for the action of K hence a scalar multiple of some vector v_i. But from the previous vector v_i is killed by E if and only if $i = 0$.

(v) Suppose there exist a non-zero submodule V' of V and let v' be a highest weight vector of V'. Then v' is also a highest weight vector for V. From the above v' has to be a non–zero scalar of v. Therefore v is in V', hence as v is a generator of V one has $V \subset V'$. Proving V is irreducible. □

To rewrite the above information we have that, $V_{n,\epsilon} = span\{v_0^{(n)}, ..., v_n^{(n)}\}$ and $U_q(\mathfrak{sl}_2) \curvearrowright V_{n,\epsilon}$ as follows

$$K \cdot v_p^{(n)} = \epsilon q^{n-2p} v_p,\ E \cdot v_p = \epsilon [n-p+1]_q v_{p-1},\ F \cdot v_p = [p+1]_q v_{p+1}$$

Theorem 4.5. *Any simple finite-dimensional $U_q(\mathfrak{sl}_2)$-module is generated by a highest weight vector. Two finite-dimensional modules generated by highest weights vectors of the same weight are isomorphic.*

Proof. Let v be a highest weight vector of V. If V is simple then the submodule generated by v must be equal to V. If V and V' are generated by highest weight vectors v and v' with the same weight λ, then the map sending v_i to v'_i for all i is an isomorphism of $U_q(\mathfrak{sl}_2)$ - modules. □

At this point it one it would be ideal to look at the category of finite dimensional $U_q(\mathfrak{sl}_2)$-modules. Since $U_q(\mathfrak{sl}_2)$ is a Ribbon Hopf algebra it should be clear how this extends to make this category a ribbon category. Now it would be ideal that this category is also a fusion category but unfortunately this is not the case. Take the irreducible modules $\{V_0, ..., V_m\}$ to be the generating simple objects of our category. As with standard representations of Lie algebra the Clebsch-Gordan formula still holds, that is for $i, j \leq n$

$$V_i \otimes V_j = \bigoplus_{\substack{|i-j| \\ i+j=m \pmod 2}}^{i+j} V_m$$

Then it is clear that there is no choice of a finite number of irreducible modules such that a fusion category can be constructed from that collection, as say $V_{i+j} \notin Ob(C)$ and therefore we do not have a finite generating set of simple objects.

4.2 Representation Theory of $U_q(\mathfrak{sl}_2)$ at q a root of unity

Now without loss of generality suppose that if q is a odd l-th root of unity then it is clear that the representation theory in preceding section fails to hold as stated. Notice that since $[n]_q = 0 \pmod{l}$ one has that for any module if $l < n$ then it will not be irreducible and as a matter of fact will be indecomposable as it contains a sub-module generated by $\{v_0^{(n)}, ..., v_{l-1}^{(n)}\}$. Since the goal is to construct a fusion category from simple objects we could do one of two things. Firstly we could look at all the modules V_n where $n \leq l-1$ which we already know the theory of from the above. The other way to deal with this issue is instead to work with Lusztig's version of the quantum group.

The algebra $U^{res}_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{sl}_2)$ is a $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $U_q(\mathfrak{sl}_2)$ generated by the elements $E^{(r)}, F^{(r)}$ and $K^{\pm 1}$ for $r \geq 1$. The generating relations are given in page 297 Chari & Pressley (1994) Now for any $\varepsilon \in \mathbb{C}^\times$ the restricted specialisation is

$$U^{res}_{\varepsilon} = U^{res}_{\mathbb{Z}[q^{\pm 1}]} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Q}(\varepsilon)$$

This is Lusztig's version of the quantum group for any semi-simple Lie algebra that we'll be working with the main background for this is located in pages 299-302 in Chari & Pressley (1994). Let ε be a primitive l-th root of unity, where l is odd and greater than one. Let $r \in \mathbb{N}$, and write $r = r_0 + r_1 l$, where $0 \leq r_0, < l$ and $0 \leq r_1$. Then one can derive

$$X^{(l)} = X^{(r_0)} \frac{(X^{(l)})^{r_1}}{r_1!}, \text{ for } X = E \text{ or } F$$

It follows that $U^{res}_{\varepsilon}(\mathfrak{sl}_2)$ is generated as a $\mathbb{Q}(\varepsilon)$-algebra by the $E, F, E^{(l)}, F^{(l)}, K^{\pm 1}$ and $\begin{bmatrix} K;0 \\ r_0 \end{bmatrix}$.

Now that we've constructed a specialisation we wish to continue to find the irreducible modules and see if we can obtain a fusion category from them. The following definition will be for any quantum group associated to a semi simple Lie algebra $\mathfrak{g}$.

Definition 4.6. *Let $V_q(\lambda)$ be the irreducible $U_q(\mathfrak{g})$ -module with highest weight vector $\lambda \in P$ and highest weight vector v_λ. Let $V^{res}_{\mathbb{Z}[q^{\pm 1}]}(\lambda)$ be the $U^{res}_{\mathbb{Z}[q^{\pm 1}]}$ -submodule of $V_q(\lambda)$ generated by v_λ, that is $V^{res}_{\mathbb{Z}[q^{\pm 1}]}(\lambda) = U^{res}_{\mathbb{Z}[q^{\pm 1}]} \cdot v_\lambda$. Define the Weyl Module*

$$W^{res}_{\varepsilon}(\lambda) = V^{res}_{\mathbb{Z}[q^{\pm 1}]}(\lambda) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{C}$$

via the homomorphism $\mathbb{Z}[p^{\pm 1}] \to \mathbb{C}$ *given by* $q \mapsto \varepsilon$.

It should be noted that in the case of q being a non root of unity the Weyl modules for a given weight are actually the irreducible modules found in the previous section, but with ε being a root of unity the modules will be distinct and as will be shown the module is no longer irreducible but rather indecomposable.

As with all previous work we'll only be considering when $\mathfrak{g} = \mathfrak{sl}_2$. For any integer n, write $n = n_0 + n_1 l$, where $n_0, n_1 \in \mathbb{Z}$ and $0 \leq n_0 < l$. Then, if $m \in \mathbb{N}$,then the Weyl module $W_\varepsilon^{res}(m)$ with maximal weight m has a basis $\{v_0^{(m)}, ..., v_m^{(m)}\}$ on which the action of U_ε^{res} is given by

$$E \cdot v_r^{(m)} = [m-r+1]_\varepsilon v_{r-1}^{(m)},\ F \cdot v_r^{(m)} = [r+1]_\varepsilon v_{r+1}^{(m)},\ K \cdot v_r^{(m)} = \varepsilon^{m-2r} v_r^{(m)},$$
$$E^{(l)} \cdot v_r^{(m)} = ((m-r)_1 + 1) v_{r-l}^{(m)},\ F^{(l)} \cdot v_r^{(m)} = (r_1 + 1) v_{r+l}^{(m)}$$

Theorem 4.7. *Let V' be the subspace of $W_\varepsilon^{res}(m)$ spanned by the $v_r^{(m)}$ such that $m_0 < r_0 < l$ and $r_1 < l$ then one has*
(i) V' is the unique proper U_ε^{res}-submodule of W_ε^{res};
(ii) $V' \cong V_\varepsilon^{res}(m')$, where $m' = l - 2 - m_0 + l(m_1 - 1)$;
(iii) $W_\varepsilon^{res}(m)/V' \cong V_\varepsilon^{res}(m)$.

From this it is clear that $W_\varepsilon^{res}(m)$ is irreducible if either $m < l$ or $m_0 = l - 1$.
To illustrate an example of theorem above let $m = 7$ and $m = 9$. Below represent that action of the given generators on the bases for $W_\varepsilon^{res}(7)$ and $W_\varepsilon^{res}(9)$ respectively.

	$v_0^{(7)}$	$v_1^{(7)}$	$v_2^{(7)}$	$v_3^{(7)}$	$v_4^{(7)}$	$v_5^{(7)}$	$v_6^{(7)}$	$v_7^{(7)}$
E	0	$[1]_\varepsilon v_0^{(7)}$	0	$[2]_\varepsilon v_2^{(7)}$	$[1]_\varepsilon v_3^{(7)}$	0	$[2]_\varepsilon v_5^{(7)}$	$[1]_\varepsilon v_6^{(7)}$
F	$[1]_\varepsilon v_1^{(7)}$	$[2]_\varepsilon v_2^{(7)}$	0	$[1]_\varepsilon v_4^{(7)}$	$[2]_\varepsilon v_5^{(7)}$	0	$[1]_\varepsilon v_7^{(7)}$	0
K	$\varepsilon v_0^{(7)}$	$\varepsilon^2 v_1^{(7)}$	$v_2^{(7)}$	$\varepsilon v_3^{(7)}$	$\varepsilon^2 v_4^{(7)}$	$v_5^{(7)}$	$\varepsilon v_6^{(7)}$	$\varepsilon^2 v_7^{(7)}$
$E^{(l)}$	0	0	0	$2v_0^{(7)}$	$2v_1^{(7)}$	$v_2^{(7)}$	$v_3^{(7)}$	$v_4^{(7)}$
$F^{(l)}$	$v_3^{(7)}$	$v_4^{(7)}$	$v_5^{(7)}$	$2v_6^{(7)}$	$2v_7^{(7)}$	0	0	0

For $W_\varepsilon^{res}(7)$ we have $7 = 1 + 2l$ so $1 < r_0 < 3$ and $r_1 < 2$ so $V' = span\{v_2^{(7)}, v_5^{(7)}\}$

	$v_0^{(9)}$	$v_1^{(9)}$	$v_2^{(9)}$	$v_3^{(9)}$	$v_4^{(9)}$	$v_5^{(9)}$	$v_6^{(9)}$	$v_7^{(9)}$	$v_8^{(9)}$	$v_9^{(9)}$
E	0	0	$[8]_\varepsilon v_1^{(9)}$	$[7]_\varepsilon v_2^{(9)}$	0	$[5]_\varepsilon v_4^{(9)}$	$[4]_\varepsilon v_5^{(9)}$	0	$[2]_\varepsilon v_7^{(9)}$	$[1]_\varepsilon v_8^{(9)}$
F	$[1]_\varepsilon v_1^{(9)}$	$[2]_\varepsilon v_2^{(9)}$	0	$[4]_\varepsilon v_4^{(9)}$	$[5]_\varepsilon v_5^{(9)}$	0	$[7]_\varepsilon v_7^{(9)}$	$[8]_\varepsilon v_8^{(9)}$	0	0
K	$v_0^{(9)}$	$\varepsilon v_1^{(9)}$	$\varepsilon^2 v_2^{(9)}$	$v_3^{(9)}$	$\varepsilon v_4^{(9)}$	$\varepsilon^2 v_5^{(9)}$	$v_6^{(9)}$	$\varepsilon v_7^{(9)}$	$\varepsilon^2 v_8^{(9)}$	$v_9^{(9)}$
$E^{(l)}$	0	0	0	$3v_0^{(9)}$	$2v_1^{(9)}$	$2v_2^{(9)}$	$2v_3^{(9)}$	$v_4^{(9)}$	$v_5^{(9)}$	$v_6^{(9)}$
$F^{(l)}$	$v_3^{(9)}$	$v_4^{(9)}$	$v_5^{(9)}$	$v_6^{(9)}$	$v_7^{(9)}$	$v_8^{(9)}$	$v_9^{(9)}$	0	0	0

in the case of $W_\varepsilon^{res}(9)$ since $m = 3l$ then $0 < r_0 < 3$ and $r_1 < 3$ and $m' = 3 - 2 - 0 + 3(2) = 7$ therefore $V' = span\{v_1^{(9)}, v_2^{(9)}, v_4^{(9)}, v_5^{(9)}, v_7^{(9)}, v_8^{(9)}\} \cong V_\varepsilon^{res}(7)$

Definition 4.8. *A finite-dimensional U_ε^{res} - module V has Weyl filtration if there exists a sequence*

$$\{0\} = V_0 \subset V_1 \subset \cdots V_p = V$$

with $V_r/V_{r-1} \cong W_q^{res}(\lambda_r)$ for some $\lambda_r \in P^+, r = 1, ..., p$.
A finite-dimensional U_ε^{res}-module is a tilting module if both V and its dual V^* have Weyl filtrations.

Define the principal alcove as

$$\mathcal{C}_l = \{\lambda \in P^+ |\ (\lambda + \rho, \check{\alpha}) < l \text{ for all } \alpha \in \Delta^+\},$$

Here $\rho = \frac{1}{2}\Sigma_{\alpha\in\Delta^+}\alpha$. For $g = \mathfrak{sl}_2$ what that $\mathcal{C}_l = \{\lambda < l - 1, \lambda \in \mathbb{Z}^+\}$, suppose that $\lambda \in \mathcal{C}_l$ then from earlier we have that $W_\varepsilon^{res}(\lambda) = V_\varepsilon^{res}(\lambda)$ is irreducible. $V_\varepsilon^{res}(\lambda)^* \cong V_\varepsilon^{res}(\lambda)$. Hence $V_\varepsilon^{res}(\lambda)$ is a tilting module.

Proposition 4.9. *(i) The dual of a tilting module is tilting.*
(ii) Any (finite) direct sum of tilting modules is tilting.
(iii) Any direct summand of a tilting module is tilting.
(iv) Any (finite) tensor product of tilting modules is tilting.

The proof for this proposition is located in page 362 Chari & Pressley (1994), this proposition is useful because it tells us that with the appropriate choices of tilting modules one will be able to construct a category of representations out of them. Which as will be shown under certain constraints we can arrive to the fusion category we desire.

Example 4.10. *Let $\mathfrak{g} = \mathfrak{sl}_2$. From earlier we have that the tilting modules $T_\varepsilon(\lambda)$ with maximal weights in the range $0 \leq \lambda < l-1$ are irreducible. When $l-1 \leq \lambda \leq 2l-2$, $T_\varepsilon(\lambda)$ can be described explicitly as follows. First, by earlier $W_\varepsilon^{res}(l-1)$ is irreducible, so $T_\varepsilon(l-1) = V_\varepsilon^{res}(l-1)$. if $l \leq \lambda \leq 2l-2$, $T_\varepsilon(\lambda)$ is the $2l$-dimensional module with basis $\{t_r\}_{r=0,1,\ldots,\lambda} \cup \{t'_r\}_{r=0,1,\ldots,2l-2-\lambda}$ and the following action, with $r = r_0 + r_1 l$ with $0 \leq r_0 < l$, $r_1 = 0$ or 1.*

$T_\varepsilon(\lambda)$, for $l \leq \lambda \leq 2l-2$;

$$K \cdot t_r = \varepsilon^{\lambda - 2r}, \begin{bmatrix} K;0 \\ l \end{bmatrix}_\varepsilon \cdot t_r = r_1 t_r$$

$$E \cdot t_r = [\lambda - r + 1]_\varepsilon t_{r-1}, \ F \cdot t_r = [r+1]_\varepsilon t_{r+1},$$

$$E^{(l)} \cdot t_r = ((\lambda - r)_1 + 1) t_{r-l}, \ F^{(l)} \cdot t_r = (r_1 + 1) t_{r+l}$$

$$K \cdot t'_r = \varepsilon^{2l-2r-2\lambda} t'_r, \begin{bmatrix} K;0 \\ l \end{bmatrix}_\varepsilon \cdot t'_r = 0,$$

$$E \cdot t'_0 = [\lambda - l + 1]_\varepsilon t_{\lambda - l},$$

$$E \cdot t'_r = [2l - 1 - \lambda - r]_\varepsilon t'_{r-1} + \begin{bmatrix} \lambda + r - l \\ r \end{bmatrix}_\varepsilon t_{\lambda + r - l}, \text{if } 0 < r \leq 2l - 2 - \lambda,$$

$$F \cdot t'_r = [r+1]_\varepsilon t'_{r+1}, \text{if } 0 \leq r < 2l - 2\lambda,$$

$$F \cdot t'_{2l-2-\lambda} = \begin{bmatrix} l-1 \\ \lambda - l + 1 \end{bmatrix}_\varepsilon t_l,$$

$$E^{(l)} \cdot t'_r = 0, \ F^{(l)} \cdot t'_r = 0.$$

Here we can see that $T_\varepsilon(\lambda)$, $l < \lambda < 2l-2$ has a sub module specifically the Weyl module $W_\varepsilon^{res}(\lambda)$, but the tilting module is clearly not decomposable.

Definition 4.11. *For any $\lambda \in P^+$, there exists, up to isomorphism, a unique indecomposable tilting module T_ε with the following properties:*
(i) λ is the unique maximal weight of $T_\varepsilon(\lambda)$;
(ii) The weight space $T_\varepsilon(\lambda)_\lambda$ is one dimensional;
(iii) $T_\varepsilon(\lambda)^* \cong T_\varepsilon(-w_0(\lambda))$,

Let T be a tilting module for U_ε^{res}. Then,

$$T \cong \bigoplus_{\lambda \in P^+} T_\varepsilon(\lambda)^{\oplus n_\lambda(T)},$$

where the multiplicities $n_\lambda(T)$ are uniquely determined by T.

Given a finite-dimensional U_ε^{res}-module V and an endomorphism f of V, we define the *quantum trace* and *quantum dimension* as

$$\text{qtr}(f) = \text{trace}(K_{\rho^*} f),$$

$$\text{qdim}(V) = \text{trace}(K_{\rho^*})$$

and for the Weyl module we have that

$$\text{qdim}(W_\varepsilon^{res}(\lambda)) = \prod\nolimits_{\alpha \in \Delta^+} \frac{[(\lambda+\rho, \alpha)]_\varepsilon}{[(\rho, \alpha)]_\varepsilon}$$

In the case of $\mathfrak{sl}_2$ we have $\text{qdim}(T_\varepsilon(\lambda)) = [\lambda+1]_\varepsilon$ for $\lambda < l-1$, $\text{qdim}(T_\varepsilon(l-1)) = 0$

$$\begin{aligned} \text{qdim}(T_\varepsilon(\lambda)) &= \text{qdim}(W_\varepsilon^{res}(\lambda)) + \text{qdim}(W_\varepsilon^{res}(2l-2-\lambda)) \\ &= [\lambda+1]_\varepsilon + [2l-\lambda-1]_\varepsilon \\ &= 0. \end{aligned}$$

Let T_1 and T_2 be tiliting modules. Then,

$$T_1 \otimes T_2 \cong \bigoplus_{\lambda \in \mathcal{C}_l} V_\varepsilon^{res}(\lambda)^{\oplus m_\lambda} \oplus Z$$

where multiplicities $m_\lambda \in \mathbb{N}$ and Z is a U_ε^{res} - module with the property that qtr(f) for all homomorphisms of U_ε^{res} $f : Z \to Z$.

we have that

$$Z = \bigoplus_{\lambda \in P^+ \setminus \mathcal{C}_l} T_\varepsilon(\lambda)^{\oplus n_\lambda}$$

Theorem 4.12. *For $\mathfrak{g} = \mathfrak{sl}_2$ and $0 \le \lambda, \mu \le l-1$, then the decompostion of the tensor product $V_\varepsilon^{res}(\lambda) \otimes V_\varepsilon^{res}(\mu)$ into indecomposable tilting modules is as follows:*
if $\lambda + \mu \le l-2$

$$V_\varepsilon^{res}(\lambda) \otimes V_\varepsilon^{res}(\mu) \cong \bigoplus_{\substack{\nu=|\lambda-\mu| \\ \nu \equiv \lambda+\mu \pmod 2}}^{\lambda+\mu} V_\varepsilon^{res}(\nu), \tag{6}$$

if $l-1 \leq \lambda+\mu \leq 2l-2$

$$V_{\varepsilon}^{res}(\lambda) \otimes V_{\varepsilon}^{res}(\mu) \cong \bigoplus_{\substack{\nu=|\lambda-\mu| \\ \nu\equiv\lambda+\mu \pmod 2}}^{2l-4-\lambda-\mu} V_{\varepsilon}^{res}(\nu) \oplus \bigoplus_{\substack{\nu=l-1 \\ \nu\equiv\lambda+\mu \pmod 2}}^{\lambda+\mu} T_{\varepsilon}(\nu) \tag{7}$$

We call Z to be our negligible modules because their quantum dimension is zero. At this point we are very close to obtaining the fusion categories we desire, the issue at hand is getting rid of the negligible modules. Fortunately this can be easily done. For a given tilting module T let $\bar{T}$ be the sum of the irreducible summands whose maximal weights lie in $\mathcal{C}_l$. $\bar{T}$ is defined up to isomorphisms and if T_1 and T_2 are tilting modules, define a new tensor product as

$$T_1 \bar{\otimes} T_2 = \overline{T_1 \otimes T_2}$$

Hence let $\overline{Tilt}_l(\mathfrak{sl}_2)$ be the category whose objects are the isomorphism classes of tilting modules given by $\bar{T}$ or in other words the objects are $V_{\varepsilon}^{res}(\lambda)$ for $\lambda \in \mathcal{C}_l$ From proposition 4.9 this will be closed under direct sums, duals and the tensor product operation defined above. If we take the tilting modules $V_{\varepsilon}^{res}(\lambda),\ \lambda < l-1$ as the simple objects of our category then we will have a fusion category.

Theorem 4.13. *The category $\overline{Tilt_l}(\mathfrak{sl}_2)$ is a fusion category.*

Proof. We have to check the conditions in definition 3.9. The ribbon structure is a direct result of ribbon algebra structure of the quantum groups. It is semi simple as we know for each weight $\lambda < l-1$ our modules are irreducible and non isomorphic as well as that for any finite dimensional module it is just the sum of the irreducible modules. The fusion coefficients are given by

$$N_{\lambda,\mu}^{\nu} = \begin{cases} 1 \text{ for } |\lambda-\mu| \leq \nu \leq \lambda+\mu,\ \nu \leq 2l-(\lambda+\nu),\ \nu+\lambda+\mu \in 2\mathbb{Z}, \\ 0 \text{ otherwise.} \end{cases}$$

Then one can check that the fusion rules are satisfied. Hence we have a fusion category. □

Example 4.14. *Here we give an example of the fusion coefficients of $\overline{Tilt_l}(\mathfrak{sl}_2)$ in the case that*

$l = 5$.

$$V_\varepsilon^{res}(0)\bar{\otimes}V_\varepsilon^{res}(\lambda) \cong V_\varepsilon^{res}(\lambda),\ \lambda = 0,1,2,3$$

$$V_\varepsilon^{res}(1)\bar{\otimes}V_\varepsilon^{res}(\lambda) \cong \begin{cases} V_\varepsilon^{res}(0) \oplus V_\varepsilon^{res}(2),\ \lambda = 1 \\ V_\varepsilon^{res}(1) \oplus V_\varepsilon^{res}(3),\ \lambda = 2 \\ V_\varepsilon^{res}(2),\ \lambda = 3 \end{cases}$$

$$V_\varepsilon^{res}(2)\bar{\otimes}V_\varepsilon^{res}(\lambda) \cong \begin{cases} V_\varepsilon^{res}(0) \oplus V_\varepsilon^{res}(2),\ \lambda = 2 \\ V_\varepsilon^{res}(1),\ \lambda = 1 \end{cases}$$

$$V_\varepsilon^{res}(3)\bar{\otimes}V_\varepsilon^{res}(\lambda) \cong V_\varepsilon^{res}(0),\ \lambda = 3$$

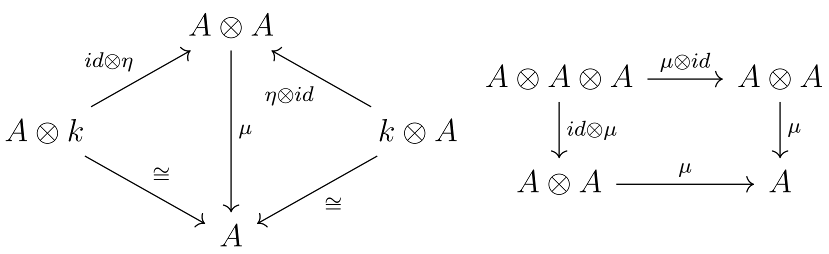
5 Conclusion

In this report we introduced the concepts of quantum groups and fusion categories. We considered Lusztig's version of the quantum group $U_q(\mathfrak{sl}_2)$ at a root of unity. We showed that taking category of tilting modules and quotienting out by the negligible modules produces a fusion category. This text will hopefully be a good introductory resource for anyone wishing to learn about fusion categories. And it is meant as a basis future work directed at obtaining different fusion categories from different quantum groups such as $U_q(\mathfrak{osp}(1|2))$. These new fusion categories may have applications in fields such as quantum computing or topological invariants.

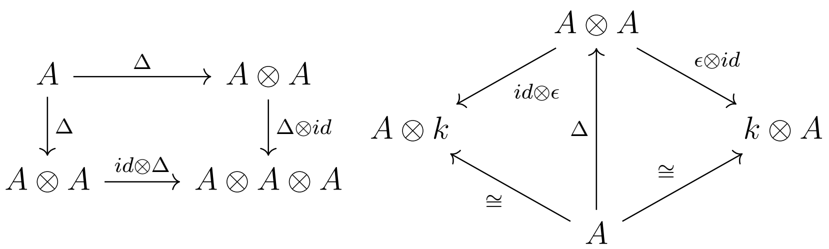
6 Appendix

In this appendix we gather some basic diffenitions that will be useful for the work presented above.

Definition 6.1. *An algebra is a triple (A, μ, η) where A is a vector space and μ, η are the multiplication and unit respectively, such that they satisfy the commutative diagrams given below.*



Definition 6.2. *An Co-algebra is a triple (A, Δ, ϵ) where A is a vector space and μ, η are the co-multiplication and co-unit respectively, such that they satisfy the commutative diagrams given below.*



Definition 6.3. *A morphism of algebras $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a linear map such that*

$$\mu' \circ (f \otimes f) = f \circ \mu \ \ \text{and} \ \ f \circ \eta = \eta'$$

Definition 6.4. *A morphism of coalgebras $f : (A, \Delta, \epsilon) \to (A', \Delta', \epsilon')$ is a linear map such that*

$$\Delta' \circ f = (f \otimes f) \circ \Delta \ \ \text{and} \ \ \epsilon' \circ f = \epsilon$$

Definition 6.5. *A bialgebra is a quintuple $(A, \mu, \eta, \Delta, \epsilon)$ where (A, μ, η) is an algebra and (A, Δ, ϵ) is a coalgebra such that*

(i) The maps μ and η are morphisms of coalgebras.

(ii) The maps Δ and ϵ are morphisms of algebras.

We define the convolution product as such $f * g = \mu \circ (f \otimes g) \circ \Delta$.

Definition 6.6. *A Hopf Algebra is a biagebra along with the map S called the antipode such that $S * id = id * S = \eta \circ \epsilon$.*

Definition 6.7. *A category C is a the following collection of data:*

(i) A class of objects $Ob(C)$.

(ii) For all objects $X, Y \in Ob(C)$, the class $Hom(X,Y)$ of morphisms from X, Y.

(iii) For any objects $X, Y, Z \in Ob(C)$ a composition map

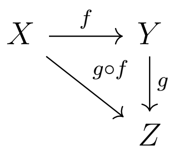
$$Hom(Y,Z) \times Hom(X,Y) \to Hom(X,Z),\ (f,g) \mapsto f \circ g.$$

such that the data satisfies the following conditions:

(i) Composition is associative, i.e. $(f \circ g) \circ h = f \circ (g \circ h)$.

(ii) For each $X \in Ob(C)$, there is a morphism $1_X \in Hom(X,X)$, called the unit morphism, such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any f, g for which the compositions make sense.

The diagram below illustrates one of the simplest examples of a category, where the unit morphisms are implicitly implied to be there.



Definition 6.8. *A Hopf algebra is braided if there exists an invertible element $R \in A \otimes A$ such that*

(i) $R\Delta(x) = \Delta(x)^{op}R$ for all $x \in A$

(ii) $(\Delta \otimes 1)(R) = R_{13}R_{23}$

(iii) $(1 \otimes \Delta)(R) = R_{13}R_{12}$ here R is the universal R-matrix.

Definition 6.9. *A ribbon Hopf algebra is a braided which posses an invertible central element θ such that*

$$\theta^2 = uS(u),\ \ S(\theta) = \theta,\ \ \epsilon(\theta) = 1, \Delta(\theta) = (R_{21}R_{12})^{-1}(\theta \otimes \theta)$$

where $u = \mu(S \otimes id)(R_{21})$.

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