AMSI
VACATION
RESEARCH
SCHOLARSHIPS
2018-2019

Symmetrisations and other rearrangement inequalities

Siksha Sivaramakrishnan

Supervised by Julie Clutterbuck

Monash University

28th February 2019

Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Australian Government
Department of Education and Training

1 Introduction

A rearrangement is the process of manipulating the shape of a geometric object while preserving its size. Rearrangements are extremely useful analytic tools that mix geometry and integration theory in an essential way [3]. They have many applications in PDE theory, optimisation problems and proving classical theorems like the isoperimetric inequality [2].
In this report we discuss two different types of rearrangements, the symmetric decreasing rearrangement and polarizations. We further discuss the connections between these two. Next we prove the Pólya-Szegő inequality, and state some other classical inequalities. Finally, we discuss some further research paths we looked at during this project. The majority of the content presented in this paper was based off the lecture notes by Almut Burchard [1].

2 Symmetric Decreasing Rearrangements

2.1 Definitons

Definition 2.1. The symmetric decreasing rearrangement of a set $A \subset \mathbb{R}^n$ is the open ball A^* of the same volume, i.e.,

$$A^* = \{x \in \mathbb{R}^n \ | \ \omega_n |x|^n < Vol(A)\}$$

where ω_n is the volume of the unit ball.

Figure 1: Symmetric decreasing rearrangement of a subset of $\mathbb{R}^n$

This rearrangement gives us a radially symmetric set as seen in figure 1. We can also define this rearrangement for functions.

Definition 2.2. The symmetric decreasing rearrangement of a non-negative function that vanishes at infinity is given by taking the symmetric decreasing rearrangements of the upper level sets, i.e.,

$$f^*(x) = \int_0^\infty \chi_{\{f(y)>t\}^*}(x) dt$$

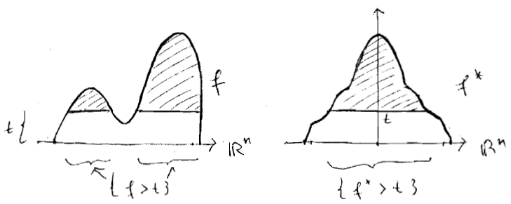

Figure 2: Symmetric decreasing rearrangement of a non-negative function that vanishes at infinity

We can define the distribution function as follows to characterize the size of these upper level sets as a way to uniquely determine f^*.

Definition 2.3. The distribution function of f is given by,

$$\mu_f(t) = Vol(\{x \mid f(x) > t\})$$

2.2 Properties and main results

Some properties of the symmetric decreasing rearrangement are discussed below.

Proposition 2.4. f^* is equimeasurable with f.

Proof. To get f^* equimeasurable with f, we want $\mu_f(t) = \mu_{f^*}(t)$. Notice,

$$\begin{aligned} \mu_f(t) &= Vol(\{x \mid f(x) > t\}) \\ &= Vol(\{x \mid f(x) > t\}^*) \\ &= Vol(\{x \mid f^*(x) > t\}) \\ &= \mu_{f^*}(t) \end{aligned}$$

Hence, we get the desired result.

□

This tells us that the shaded regions seen in figure 2 have equal volume.

Proposition 2.5. The symmetric decreasing rearrangement preserves L^p-norms.

Proof. We use the layer-cake decomposition and Fubini's theorem to write

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^p dx &= \int_{\mathbb{R}^n} \int_0^\infty \chi_{\{f(y)^p > t\}}(x) dt dx \\ &= \int_0^\infty Vol(\{f(x)^p > t\}) dt \\ &= \int_0^\infty Vol(\{f(x) > s\}) ps^{p-1} ds \\ &= \int_0^\infty \mu_f(s) ps^{p-1} ds\end{aligned}$$

But, we know that f^* is equimeasurable with f. So,

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^p dx &= \int_0^\infty \mu_f(s) ps^{p-1} ds \\ &= \int_0^\infty \mu_{f^*}(s) ps^{p-1} ds \\ &= \int_{\mathbb{R}^n} |f^*(x)|^p dx\end{aligned}$$

Hence, we get the desired result of $\|f\|_p = \|f^*\|_p$. □

Hence, we see that by performing the symmetric decreasing rearrangement, we do not change the p-norm of the function.
Further, we can see that the symmetric decreasing rearrangement is also order preserving.

Proposition 2.6. The symmetric decreasing rearrangement is order preserving, i.e., $f(x) \leq g(x) \ \ \forall x \in \mathbb{R}^n \implies f^*(x) \leq g^*(x) \ \ \forall x \in \mathbb{R}^n$.

Proof. Consider,

$$\begin{aligned}f^*(x) &= \int_0^\infty \chi_{\{f(y)>t\}^*}(x) dt \\ &\leq \int_0^\infty \chi_{\{g(y)>t\}^*}(x) dt \\ &= g^*(x)\end{aligned}$$

Since $f(x) \leq g(x) \implies \{f(y) > t\}^* \subseteq \{g(y) > t\}^*$. Hence, we have that the symmetric decreasing rearrangement is order-preserving. □

Further, there are some classical results that arise from looking at symmetric decreasing rearrangements, such as the isoperimetric inequality which states that balls minimise perimeter. Another big result is the Hardy-Littlewood inequaity. Both of these results are stated below.

Theorem 2.7 (Isoperimetric inequality). *$Per(A) \geq Per(A^*)$ for any subset A of $\mathbb{R}^n$.*

Theorem 2.8 (Hardy-Littlewood Inequality). *Let f, g be non-negative, measurable functions that vanish at infinity. Then*

$$\int fg \leq \int f^* g^*$$

Proof. Firstly, consider the case where $f = \chi_A$ and $g = \chi_B$ are characteristic functions of measurable sets A and B of finite volume. The rearrangements A^* and B^* are centred balls, and their intersection $A^* \cap B^*$ is the smaller of the two balls. Thus,

$$Vol(A^* \cap B^*) = \min\{Vol(A), Vol(B)\} \geq Vol(A \cap B),$$

which gives us the inequality in this case. In general, we use the layer-cake decomposition and Fubini's theorem to get

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)g(x)dx &= \int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty \chi_{\{f(y)>s\}}(x)\chi_{\{g(y)>t\}}(x)dsdtdx \\
&= \int_0^\infty\int_0^\infty Vol(\{f(x) > s\} \cap \{g(x) > s\})dsdt \\
&\leq \int_0^\infty\int_0^\infty Vol(\{f(x) > s\}^* \cap \{g(x) > s\}^*)dsdt \\
&= \int_{\mathbb{R}^n} f^*(x)g^*(x)dx
\end{aligned}$$

The above uses the proof above of Hardy-Littlewood for characteristic functions. This gives us the result of the Hardy-Littlewood inequality. □

A consequence of the Hardy-Littlewood inequality is that the L^p distance between two functions is larger than the L^p distance between their symmetric decreasing rearrangements, as seen in the following proposition.

Proposition 2.9. The symmetric decreasing rearrangement decreases L^p distances:

$$\|f - g\|_p \geq \|f^* - g^*\|_p \quad 1 \leq p \leq \infty$$

.

Proof. We consider,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f(x) - g(x)|^p dx \\
&= \int_{\mathbb{R}^n} p \int_0^\infty [f(x) - t]_+^{p-1} \chi_{\{g(x) \le t\}} + [g(x) - t]_+^{p-1} \chi_{\{f(x) \le t\}} dt dx \\
&= \int_{\mathbb{R}^n} p \int_0^\infty [f(x) - t]_+^{p-1} (1 - \chi_{\{g(x) > t\}}) + [g(x) - t]_+^{p-1} (1 - \chi_{\{f(x) > t\}}) dt dx \\
&= \|f\|_p + \|g\|_p + p \int_0^\infty \int_{\mathbb{R}^n} (-[f(x) - t]_+^{p-1} - [g(x) - t]_+^{p-1}) \chi_{\{g(x) > t\}} \chi_{\{f(x) > t\}}
\end{aligned}$$

By Hardy-Littlewood inequality, we see that

$$\begin{aligned}
&p \int_0^\infty \int_{\mathbb{R}^n} (-[f(x) - t]_+^{p-1} - [g(x) - t]_+^{p-1}) \chi_{\{g(x) > t\}} \chi_{\{f(x) > t\}} \\
&\ge p \int_0^\infty \int_{\mathbb{R}^n} (-[f^*(x) - t]_+^{p-1} - [g^*(x) - t]_+^{p-1}) \chi_{\{g(x) > t\}^*} \chi_{\{f(x) > t\}^*}
\end{aligned}$$

Further, we know that symmetric decreasing rearrangement preserves L^p-norms. So, we get

$$\int_{\mathbb{R}^n} |f(x) - g(x)|^p dx \ge \int_{\mathbb{R}^n} |f^*(x) - g^*(x)|^p dx$$

Hence, $\|f - g\|_p \ge \|f^* - g^*\|_p \quad 1 \le p \le \infty$. □

3 Polarizations

3.1 Definitions

Another type of rearrangement is polarization, or two-point rearrangement. This rearrangement is not symmetric like the previous one, and in fact even breaks convexity as seen in figure 3. The definition is given as follows.

Definition 3.1. Polarization of a set about a hyperplane X_0 (X_+ containing the origin and X_- the other side), with σ being reflection in this hyperplane, is given by:

$$\begin{cases}
A^\sigma \cap X_+ = (A \cup \sigma A) \cap X_+) \\
A^\sigma \cap X_- = (A \cap \sigma A) \cap X_-) \\
A^\sigma \cap X_0 = A \cap X_0
\end{cases}$$

Further, like for symmetric decreasing rearrangements, we can define polarizations for functions as well as follows.

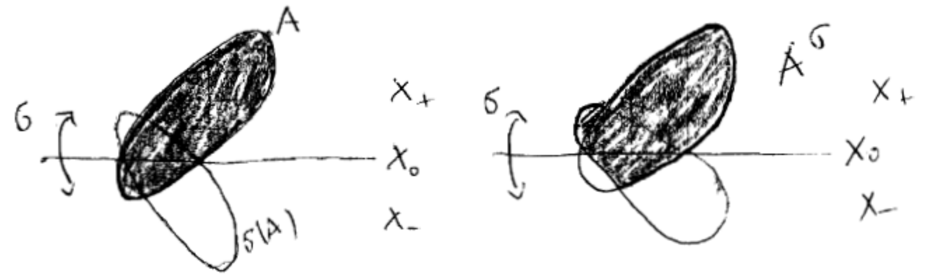

Figure 3: Polarization of a set in $\mathbb{R}^n$

Definition 3.2. Polarization of a a function f about a hyperplane X_0 is given by:

$$f^\sigma(x) = \begin{cases} \max\{f(x), f(\sigma x)\}, \text{ if } x \in X_+ \\ \min\{f(x), f(\sigma x)\}, \text{ if } x \in X_- \\ f(x), \text{ if } x \in X_0 \end{cases}$$

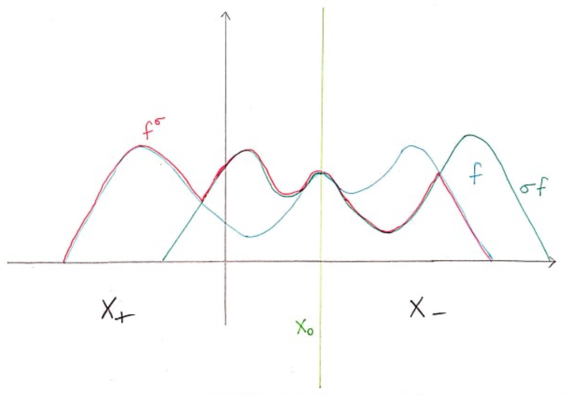

Figure 4: Polarization of a non-negative function that vanishes at infinity

3.2 Properties and main results

We notice that there are similarities in the inequalities that fall out of symmetric decreasing rearrangements and polarizations. The two results provided below are exact counterparts of the results shown in the symmetric decreasing rearrangement section.

Theorem 3.3 (Hardy-Littlewood inequality for polarizations)**.** $\int fg \leq \int f^\sigma g^\sigma$.

Proof. Consider

$$\begin{aligned}\int (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx &= \int_{X_+} (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx \\ &+ \int_{X_-} (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx + \int_{X_0} (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx \\ &= \int_{X_+} (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx + \int_{X_-} (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx\end{aligned}$$

Now we consider cases.

Case 1: When $f(x) \geq f(\sigma x)$ and $g(x) \geq g(\sigma x)$ for $x \in X_+$, then the integral over X_+ disappears. Further, for $y = \sigma x \in X_-$, $f(y) < f(\sigma y)$ and $g(y) < g(\sigma y)$. And so the integral over X_- disappears as well. Hence, we get $\int (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx = 0$.

Case 2: When $f(x) < f(\sigma x)$ and $g(x) < g(\sigma x)$ for $x \in X_+$, then the integral over X_- disappears. So, we get $\int (f^\sigma g^\sigma - fg)dx = \int_{X_+} (f^\sigma g^\sigma - fg)dx \geq 0$.

Case 3: When $f(x) \geq f(\sigma x)$ and $g(x) < g(\sigma x)$ for $x \in X_+$, we see that $g(x) < g(\sigma x)$ on X_+ and $f(y) \leq f(\sigma y)$ on X_-. Further we know that f, g are non-negative. So, we get

$$\begin{aligned}&\int (f^\sigma(x)g^\sigma(x) - f(x)g(x))dx \\ &= \int_{X_+} (f(x)g(\sigma x) - f(x)g(x))dx + \int_{X_-} (f(\sigma x)g(x) - f(x)g(x))dx \\ &= \int_{X_+} f(x)(g(\sigma x) - g(x))dx + \int_{X_-} g(x)(f(\sigma x) - f(x))dx \geq 0\end{aligned}$$

By symmetry of case 3, WLOG this gives us the same result for $g(x) \geq g(\sigma x)$ and $f(x) < f(\sigma x)$ for $x \in X_+$.

This gives us all possible cases. Hence, we get the desired inequality. □

Proposition 3.4. Polarization decreases L^p distances:

$$\|f - g\|_p \geq \|f^\sigma - g^\sigma\|_p$$

Idea of proof: The proof here uses the exact same trick as for symmetric decreasing rearrangements as we have the Hardy-Littlewood inequality for polarizations. □

The connection between polarizations and symmetric decreasing rearrangements do not stop there. The next subsection discusses a deeper relationship between the two rearrangements.

3.3 From Polarizations to Symmetric Decreasing Rearrangements

Lemma 3.5. Let f be a non-negative function that vanishes at infinity. Then:

$$f = f^* \iff f = f^\sigma \;\; \forall \sigma$$

and

$$f = f^* \circ \tau \text{ for some translation } \tau$$

$$\iff \quad \forall \sigma, \text{ either } f = f^\sigma \text{ or } f = f^\sigma \circ \sigma.$$

As seen above, there is an inherent relationship between polarizations and symmetric decreasing rearrangements. In fact, this relationship is even deeper.

Theorem 3.6. *Assume that f is a non-negative continuous function with compact support in $\mathbb{R}^n$, and let*

$$Pol_f = \{f^{\sigma_1,\ldots,\sigma_k} \;\mid\; k \geq 0, \sigma_1, \ldots, \sigma_k \text{ reflections}\}$$

There exists a sequence $\{g_k\}_{k\geq 1} \subset Pol_f$ such that

$$g_k \longrightarrow f^* \text{ uniformly}$$

In this surprising way, polarizations, a rearrangement that doesn't even seem to preserve convexity, can be used to approximate any symmetric decreasing rearrangement. A proof of both of these can be found in Almut's notes [1].

4 Some classical inequalities

This section deals with introducing some of the classical inequalities found in the literature in his area of mathematics.

Definition 4.1. $W^{1,p}(\mathbb{R}^n)$ is the set of all L^p functions on $\mathbb{R}^n$ whose first derivatives are also L^p functions.

Lemma 4.2 (Co-area formula)**.**

$$\int_{\Omega} g(x)|\nabla f(x)|dx = \int_0^\infty \int_{f^{-1}(t)} g(x) d\sigma dt.$$

for $\Omega \subset \mathbb{R}^n$ open set, every measurable function g and Lipschitz function f, where dσ is integration with respect to (n-1)-dimensional Hausdorff measure.

A direct consequence of the co-area formula gives us the following corollary.

Corollary 4.3. For every interval $(t_1, t_2]$,

$$\int_{t_1}^{t_2} \int_{f^{-1}(t)} |\nabla f|^{-1} d\sigma dt = Vol(\{x \mid t_1 < f(x) \leq t_2, |\nabla f(x)| \neq 0\})$$

Theorem 4.4 (Pólya-Szegő inequality)**.** *If $f \in W^{1,p}(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$, then*

$$\|\nabla f\|_p \geq \|\nabla f^*\|_p$$

Proof. By the co-area formula

$$\begin{aligned} \|\nabla f\|_p^p &= \int |\nabla f(x)|^p dx \\ &= \int_0^\infty \int_{f^{-1}(t)} |\nabla f|^{p-1} d\sigma dt. \end{aligned}$$

Jensen's inequality states for a convex function ϕ

$$\phi\left(\frac{1}{\mu(\Omega)} \int_\Omega g d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_\Omega (\phi \circ g) d\mu$$

Since $s \mapsto s^{-(p-1)}$ is a convex function, take $\phi(s) = s^{-(p-1)}$, $g = |\nabla f|^{-1}$ and use Fleming-Rischell's result to get $\mu(\Omega) = Per(\{f > t\})$ [1]. Now, by Jensen's inequality, we get

$$\left(\int_{f^{-1}(t)} |\nabla f|^{-1} \frac{d\sigma}{Per(\{f > t\})}\right)^{-(p-1)} \leq \int_{f^{-1}(t)} |\nabla f|^{p-1} \frac{d\sigma}{Per(\{f > t\})}.$$

By a direct consequence of the co-area formula, and since the volume of critical points decrease with the symmetric decreasing rearrangement, we know that

$$\begin{aligned} \int_{t_1}^{t_2} \int_{f^{-1}(t)} |\nabla f|^{-1} d\sigma dt &= Vol(\{x \mid t_1 < f(x) \leq t_2, |\nabla f(x)| \neq 0\}) \\ &\leq Vol(\{x \mid t_1 < f^*(x) \leq t_2, |\nabla f^*(x)| \neq 0\}) \\ &= \int_{t_1}^{t_2} \int_{(f^*)^{-1}(t)} |\nabla f^*|^{-1} d\sigma dt \end{aligned}$$

So, we get that,

$$\int_{f^{-1}(t)} |\nabla f|^{-1} d\sigma \leq \int_{(f^*)^{-1}(t)} |\nabla f^*|^{-1} d\sigma \tag{1}$$

Further, by replacing f with f^*, the perimeter of the upper level sets decreases by the isoperimetric inequality. So, this along with Jensen's inequality and (1) gives us,

$$\begin{aligned}\int_{f^{-1}(t)} |\nabla f|^{p-1} d\sigma &\geq Per(\{f > t\})^p . \left(\int_{f^{-1}(t)} |\nabla f|^{-1} d\sigma\right)^{-(p-1)} \\ &\geq Per(\{f^* > t\})^p . \left(\int_{(f^*)^{-1}(t)} |\nabla f^*|^{-1} d\sigma\right)^{-(p-1)} \\ &= \int_{(f^*)^{-1}(t)} |\nabla f^*|^{p-1} d\sigma\end{aligned}$$

where Jensen's inequality become an equality when $f = f^*$. Hence, we have that

$$\begin{aligned}\int |\nabla f(x)|^p dx &= \int_0^\infty \int_{f^{-1}(t)} |\nabla f|^{p-1} d\sigma dt \\ &\geq \int_0^\infty \int_{(f^*)^{-1}(t)} |\nabla f^*|^{p-1} d\sigma dt \\ &= \int |\nabla f^*(x)|^p dx\end{aligned}$$

So, we get the desired result of $\|\nabla f\|_p \geq \|\nabla f^*\|_p$.

□

A proof of the Pólya-Szegő inequality can by shown using theorem 3.6 as well. A sketch of this proof is given below.

Alternative sketch of proof: When $1 \leq p < \infty$ we use the following argument. One can show that if $f \in L^p$, then we have $\nabla g_k \longrightarrow \nabla f^*$ weakly in L^p.

Further, we need to use the Pólya-Szegő identity for polarization which gives us $\|\nabla f\|_p = \|\nabla f^\sigma\|_p$ for any reflection σ. So, $\|\nabla f\|_p = \|\nabla g_k\|_p$. Now, since the p-norm is convex, it is weakly lower semicontinuous, and we get that

$$\|\nabla f\|_p = \lim \|\nabla g_k\|_p \geq \|\nabla f^*\|_p$$

which is the desired result.

An argument using Lipschitz continuity gives us the result for $p = \infty$.

□

Some other classical inequalities are given below without proof.

Theorem 4.5 (Riesz' rearrangement inequality). *Let f, g, h be non-negative measurable functions on $\mathbb{R}^n$ that vanish at infinity. Then*

$$\int\int f(x)g(y)h(x-y)dxdy \leq \int\int f^*(x)g^*(y)h^*(x-y)dxdy,$$

in the sense that the left-hand side is finite whenever the right-hand side is finite.

Theorem 4.6 (Talenti's inequality). *Let f be a smooth non-negative function with compact support on $\mathbb{R}^n$ for some $n > 2$, and let f^* be it's symmetric decreasing rearrangement. If u and v vanish at infinity and solve*

$$-\Delta u = f, \quad -\Delta v = f^*$$

then $u^*(x) \leq v^*(x)$ for all $x \in \mathbb{R}^n$.

5 Further Research

During the research project, some time was spent looking at interesting questions regarding polarizations of sets. Questions regarding the perimeter and diameter of polarized sets arose quite naturally from drawing pictures like figure 3. A couple of these conjectures are listed below.

Theorem 5.1. *The diameter of a set after polarization is non-increasing.*

The main idea for the proof of this is the triangle inequality.

Conjecture 5.2. The polarization of a convex set with a line of symmetry parallel to X_0 is the set itself if the line of symmetry lies in X_+, and the reflection of the set if the line of symmetry lies in X_-. i.e., $A^\sigma = A$ or $A^\sigma = \sigma A$

The idea for this is seen in figure 5.

Conjecture 5.3. The perimeter of a set after polarization is non-increasing.

Conjecture 5.3 intuitively seems to make sense as we have seen from theorem 3.6, the set of finitely many polarizations can be used to approximate symmetric decreasing rearrangements. Further, symmetric decreasing rearrangements follow the isoperimetric inequality which states that the perimeter will decrease due to this rearrangement. An example of perimeter decreasing due to polarization is seen in figure 6.

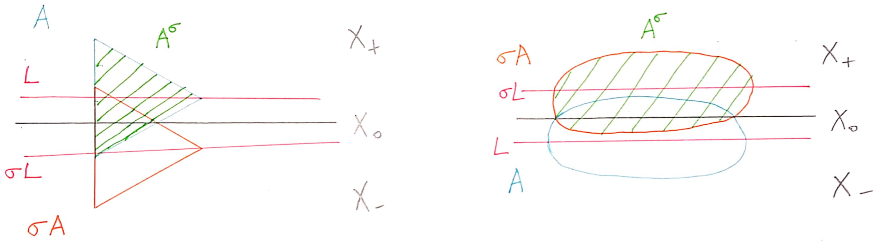

Figure 5: Polarization of a set with line of symmetry parallel to X_0

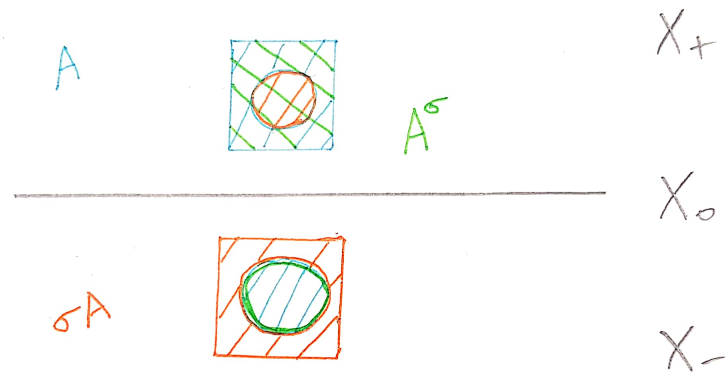

Figure 6: An example of perimeter decreasing after polarization of a set

References

[1] Almut Burchard. *A Short Course on Rearrangement Inequalities.* URL: `http://www.math.utoronto.ca/almut/rearrange.pdf`. (accessed: 09.02.2019).

[2] Bernd Kawohl. *Symmetrization - or how to prove symmetry of solutions to a PDE.* URL: `https://www.researchgate.net/publication/266506870_Symmetrization_-_or_how_to_prove_symmetry_of_solutions_to_a_PDE`. (accessed: 09.02.2019).

[3] Elliot H. Lieb and Michael Loss. "Analysis". In: American Mathematical Society, 2001. Chap. 3.