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Calculating thermal diffusivity from laser-flash experiments

Christyn Wood

Supervised by Dr Elliot Carr

Queensland University of Technology

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1 Abstract

The thermal diffusivity of a material can be determined by using a laser-flash experiment where an energy heat-pulse is applied to the front-surface of a sample and the resulting temperature rise on the rear-surface is detected. In 1961, [Parker et al.](#) developed the half-rise time method to estimate the thermal diffusivity by analysing the time taken for the temperature to reach half of its maximum value. This report presents a new, more accurate method for calculating the thermal diffusivity involving the area between the rear-surface temperature rise and the steady-state temperature. Through numerical experiments, the effectiveness of this formula is demonstrated with the new approach producing more accurate and less variable estimates of the thermal diffusivity. To conclude, further extensions of this approach are discussed including investigating a system with layered samples and analysing an uninsulated material.

2 Introduction

From cars engines to coffee and air-conditioning to steam turbines, the world is continuously evolving to employ heat in more efficient ways. Each system in which heat flows from a hotter region to a cooler one is governed by the heat equation, a model that is essential for all thermodynamic systems. This equation describes how equilibrium can be met when heat is applied and distributed across a material, such that the change in temperature at a certain position in time is proportional to the concavity of the temperature profile.

The thermal diffusivity constant, α , is a key component of this equation that quantifies the rate at which heat diffuses across a sample, or equivalently signifies the rate needed to bring the temperature concavity to equilibrium. A high value of α indicates that the material has a greater ability to conduct heat relative to its ability to store heat¹. The thermal diffusivity constant can be expressed as $\alpha = k/(\rho c)$ ($\text{m}^2 \text{s}^{-1}$), where k is the thermal conductivity ($\text{W m}^{-1} \text{K}^{-1}$), ρ is the density of the material (kg m^{-3}) and c is the specific heat capacity (J K^{-1}).

Laser flash experiments are the most common method to determine the thermal diffusivity of a material ([Blumm and Opfermann, 2002](#); [Czél et al., 2013](#); [Vozár, 2003](#)), where an energy

¹<https://www.sciencedirect.com/topics/chemical-engineering/thermal-diffusivity>



heat pulse is applied to the front-surface of a material and the resulting rear-surface time dependent temperature rise is recorded. From this rear-surface temperature rise, [Parker et al. \(1961\)](#) formulated a method to estimate the thermal diffusivity of a material, by observing the time taken for the temperature to reach half of its maximum value. Parker et al.'s approach was developed under the assumption that the sample of material the laser-flash experiment is applied to is homogeneous, isotropic and thermally insulated with uniform thickness; and that the heat pulse applied is instantaneously and uniformly absorbed by a thin layer of the sample whose density and thermo-physical properties do not change with temperature ([Vozár, 2003](#)).

Although Parker et al.'s method remains standard practice under the above assumptions ([ASTM E1461-13, 2013](#)), many modifications to this approach have been investigated including heat loss in a system ([Cowan, 1961](#)), and layered samples ([Czél et al., 2013](#); [Larson and Koyama, 1968](#)). Recently, [Carr \(2019\)](#) presented an alternate method under the same assumptions as Parker et al., expressing the thermal diffusivity constant exactly in terms of the area enclosed between the rear-surface temperature rise curve and the steady-state temperature. This report extends Carr's method by deriving an exact equation for α using a more realistic set of initial and boundary conditions described by [Heckman \(1973\)](#), where the heat pulse applied to the sample is represented by a specified heat flux boundary condition that is continuously applied for a finite time period. For a typical laser-flash experiment, numerical experiments are used to show that this new method produces more reliable and less variable estimates of the thermal diffusivity constant in comparison to the half-rise time method.

Statement of Authorship

The workload was divided as follows:

- Christyn Wood developed the code in MATLAB, produced the mathematical results, performed the numerical experiments, reported and interpreted the results, and wrote this report.
- Elliot Carr developed the theory behind this report, supervised the work, assisted with the numerical experiments and proofread this report.



3 Thermal Diffusivity Calculation

3.1 Mathematical Model

The heat equation is given by

$$\frac{\partial T}{\partial t}(x, t) = \alpha \frac{\partial^2 T}{\partial x^2}(x, t), \quad 0 < x < L, \quad t > 0, \quad (1)$$

where T represents the temperature ($^{\circ}\text{K}$) at time t (s) at the spatial position x (m); α denotes the thermal diffusivity ($\text{m}^2 \text{s}^{-1}$) and L denotes the length of the one-dimensional sample (m).

In this work, we consider the following initial and boundary conditions:

$$T(x, 0) = 0, \quad -k \frac{\partial T}{\partial x}(0, t) = q(t), \quad \frac{\partial T}{\partial x}(L, t) = 0. \quad (2)$$

The finite heat-pulse time effect that is applied to the sample in the laser flash experiment is defined as $q(t)$, where $Q = \int_0^{\infty} q(t) dt$ is the total amount of heat absorbed in the system at $x = 0$. Any arbitrary $q(t)$ can be considered, provided $\lim_{t \rightarrow \infty} q(t) = 0$ and Q is finite, that is, after a period of time no further heat is applied to the sample. For this research, $q(t)$ was defined by the following linear piecewise function describing a triangular-shaped pulse:

$$q(t) = \begin{cases} \frac{F_0 t}{b\tau}, & 0 < t \leq b\tau \\ \frac{F_0(\tau - t)}{\tau - b\tau}, & b\tau \leq t \leq \tau \end{cases} \quad (3)$$

where τ is the length of the pulse (s), F_0 is the boundary flux ($\text{J m}^{-2} \text{s}^{-1}$) and b is an adjustable parameter, with $b\tau$ denoting the location of the peak of the pulse (Heckman, 1973).

3.2 Steady State Derivation

To develop a mathematical model for determining the thermal diffusivity, we rely on the knowledge of the steady-state temperature of the material. To determine the steady-state temperature theoretically, integrate the heat equation (1) over the length of the sample, recalling $\alpha = k/(\rho c)$, and use the FToc to give:

$$\rho c \int_0^L \frac{\partial T}{\partial t} dx = k \frac{\partial T}{\partial x}(L, t) - k \frac{\partial T}{\partial x}(0, t). \quad (4)$$

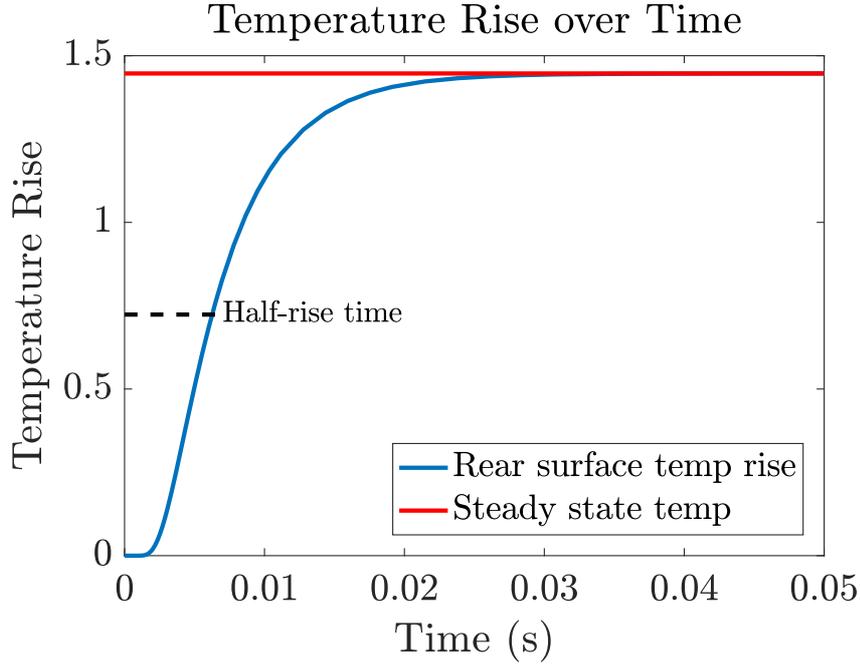


Figure 1: Plot of the rear-surface temperature rise curve (blue) and the steady-state temperature (red).

Noting the boundary conditions (2) and defining

$$\bar{T}(t) = \frac{1}{L} \int_0^L T(x, t) dx \quad (5)$$

to be the spatially-averaged temperature, Eq (4) reduces to the following differential equation:

$$\frac{d\bar{T}}{dt} = \frac{q(t)}{\rho c L}. \quad (6)$$

Integrating and using the initial condition (2) yields the solution:

$$\bar{T}(t) = \frac{1}{\rho c L} \int_0^t q(s) ds.$$

As $t \rightarrow \infty$, the spatially-averaged temperature becomes constant over time and tends to the steady-state temperature T_∞ . Hence,

$$T_\infty = \lim_{t \rightarrow \infty} \bar{T}(t) = \frac{Q}{\rho c L}, \quad (7)$$

where Q is the total amount of heat applied to the sample's front surface, defined previously. A plot of the steady-state temperature and the theoretical rear-surface temperature rise curve for a chosen set of parameters is shown in Figure 1.



3.3 Derivation of Thermal Diffusivity

To derive the thermal diffusivity as a closed-form expression, we consider the following function:

$$u(x) := \int_0^\infty T_\infty - T(x, t) dt. \quad (8)$$

The area between the steady-state temperature, T_∞ , and the rear-surface temperature rise curve, $T(L, t)$, between $t = 0$ and $t \rightarrow \infty$ is given by $u(L) = \int_0^\infty T_\infty - T_r(t) dt$, where $T_r(t) = T(L, t)$. From these definitions, a differential equation can be formed for $u(x)$ to derive an exact value of α .

Differentiating $u(x)$ twice and utilising the heat equation (1) yields the equation $u''(x) = \frac{1}{\alpha}[T(x, 0) - T_\infty]$. Recalling the form of T_∞ (7) and the initial condition (2), $u''(x) = -Q/(\alpha\rho cL)$. Integrating this equation twice with respect to x gives

$$u(x) = -\frac{Qx^2}{2\alpha\rho cL} + c_1x + c_2. \quad (9)$$

Boundary and auxiliary conditions must be formulated to identify the constants c_1 and c_2 . Considering the definition of $u(x)$ (8) and the boundary conditions (2) we can show that $u'(0) = Q/k$ and $u'(L) = 0$. The first two conditions are enough to identify c_1 , however a third condition is required to determine c_2 , namely $\int_0^L u(x) dx = a$, where a is a constant to be determined. Using the conditions above and rearranging produces the equation

$$u(x) = -\frac{Qx^2}{2\alpha\rho cL} + \frac{Qx}{\alpha\rho c} + \frac{a}{L} - \frac{QL}{3\alpha\rho c}. \quad (10)$$

By definition,

$$\int_0^L u(x) dx = \int_0^L \int_0^\infty T_\infty - T(x, t) dt dx,$$

however reversing the order of integration and separating yields

$$\int_0^L u(x) dx = \int_0^\infty \left[\int_0^L T_\infty dx - \int_0^L T(x, t) dx \right] dt,$$

where $\int_0^L T_\infty dx = \frac{QL}{\rho c}$ and $\int_0^L T(x, t) dx = \frac{1}{\rho c} \int_0^t q(s) ds$ from Eqs (5) and (7). Noting that $\int_0^t q(s) ds = \int_0^\tau q(s) ds$ for $t \geq \tau$, a formula for a can be derived by using substitution and separating the time integral at time τ , so that

$$a = \int_0^L u(x) dx = \int_0^\tau \left[\frac{QL}{\rho c} - \frac{1}{\rho c} \int_0^t q(s) ds \right] dt. \quad (11)$$



The area between the rear-surface temperature rise and the steady-state temperature can be evaluated from Eq (10), such that

$$u(L) = \int_0^\infty T_\infty - T_r(t) dt = \frac{QL}{6\alpha\rho c} + \frac{a}{L}. \quad (12)$$

Using the formula for the constant a (11) and rearranging Eq (12) gives an exact formula for the thermal diffusivity

$$\alpha = \frac{QL^2}{6(\rho cL \int_0^\infty T_\infty - T_r(t) dt - \int_0^\tau [Q - \int_0^t q(s) ds] dt)}. \quad (13)$$

Using the definition of $q(t)$ (3), the value of the integral $\int_0^t q(s) ds$ can be expressed as three separate cases:

$$\int_0^t q(s) ds = \begin{cases} \frac{F_0 t^2}{2b\tau}, & 0 < t \leq b\tau \\ \frac{F_0(2\tau t - t^2 - b\tau^2)}{2(\tau - b\tau)}, & b\tau \leq t \leq \tau \\ \frac{F_0\tau}{2}, & t \geq \tau. \end{cases}$$

4 Results and Discussion

4.1 Finite Volume Method

Our new formula for calculating the thermal diffusivity is verified by using synthetic data generated by the mathematical model (1)–(2). To solve this model we use the finite volume method over N nodes at equally spaced points over the length of the one-dimensional sample. Integrating the heat equation (1) between node i 's left and right finite volume boundary, given by $x_E = (x_i + x_{i+1})/2$ and $x_W = (x_{i-1} + x_i)/2$ respectively, and using the FToC, yields

$$\frac{d}{dt} \int_{x_W}^{x_E} T(x, t) dx = \alpha \frac{\partial T}{\partial x}(x_E, t) - \alpha \frac{\partial T}{\partial x}(x_W, t). \quad (14)$$

For each node i located at $x_i = (i - 1)h$ for $i = 1, \dots, N$, where $h = L/(N - 1)$, we apply the midpoint rule and central difference approximations to Eq (14) and rearrange to give a formula for the change in temperature at each node. Note that to determine the equations for



the boundary nodes we make use of the boundary conditions (2) in Eq (14). Thus,

$$\begin{aligned}\frac{dT_1}{dt} &= \frac{2\alpha}{h^2}[T_2 - T_1] + \frac{2}{h\rho c}q(t), \\ \frac{dT_i}{dt} &= \frac{\alpha}{h^2}[T_{i+1} - 2T_i + T_{i-1}], \quad i = 2, \dots, N-1, \\ \frac{dT_N}{dt} &= \frac{2\alpha}{h^2}[T_{N-1} - T_N],\end{aligned}$$

where T_i denotes the temperature of node i at time t . The above system is solved using MATLAB's inbuilt ODE solver ode15s, with absolute and relative tolerances of 10^{-12} .

4.2 Verification of Thermal Diffusivity Formula

To verify the new formula for the thermal diffusivity (13), we apply Gaussian noise at finite time points to the rear-surface temperature rise curve to create synthetic data mimicking a realistic laser flash experiment. Specifically, noise is generated in the form: $\tilde{T}_i = T_r(t_i) + z_i$, $i = 0, \dots, N$, where $t_i = it_N/N$, t_N is the final sample time and z_i is a random number drawn from a normal distribution with mean zero and standard deviation $\sigma(z)$.

To approximate the area between the synthetic data and the steady-state temperature appearing in the denominator of the thermal diffusivity formula (13), we use the trapezoidal rule to calculate an experimental value for the thermal diffusivity, denoted by $\tilde{\alpha}$. We then compare the resulting $\tilde{\alpha}$'s to the theoretical value of α by specifying the following parameter values: $L = 0.002$ m, $k = 222$ W m⁻¹ K⁻¹, $\rho = 2700$ kg m⁻³, $c = 896$ J kg⁻¹ K⁻¹, $b = 0.25$, $F_0 = 2.8 \times 10^7$ J m⁻² s⁻¹, $t_N = 0.05$ s. Thus, the target value for the experimental thermal diffusivity constant is $\alpha = k/(\rho c) = 9.1766 \times 10^{-5}$ m² s⁻¹, rounded to 5 significant figures.

Heckman (1973) applied Parker et al.'s half-rise time method to the mathematical model (1)–(2), deriving the formula $\alpha = 0.14325 L^2/t_{0.5}$, where $t_{0.5}$ is the half-rise time (see Figure 1). A comparative $\tilde{\alpha}$ value is produced by applying this formula to the same synthetic data that our method was applied to. Generating Gaussian noise and recording the resulting $\tilde{\alpha}$ values for both methods over 10,000 trials, allows us to analyse the resulting trends. Using the formula $\epsilon = (\tilde{\alpha} - \alpha)/\alpha \times 100\%$, we can calculate the relative error for both methods and display the error and $\tilde{\alpha}$ values in histograms for each level of noise, as seen in Figure 2. Comparing the methods, we can see that the new method produces significantly more accurate results with less



Formula	$\sigma(z)$	$\mu(\epsilon)$	$\sigma(\epsilon)$	$\min(\epsilon)$	$\max(\epsilon)$	$\mu(\tilde{\alpha})$	$\min(\tilde{\alpha})$	$\max(\tilde{\alpha})$
New Eq.	0.005	-0.0020	0.1062	-0.3962	0.3740	9.1768×10^{-5}	9.1423×10^{-5}	9.2129×10^{-5}
	0.02	-0.0057	0.4210	-1.6634	1.5946	9.1771×10^{-5}	9.0303×10^{-5}	9.3292×10^{-5}
	0.05	-0.0245	1.0693	-4.2775	4.2487	9.1788×10^{-5}	8.7867×10^{-5}	9.5691×10^{-5}
Heckman Eq.	0.005	0.2467	0.4415	-1.3038	1.7050	9.1540×10^{-5}	9.0201×10^{-5}	9.2962×10^{-5}
	0.02	-0.3140	1.5128	-7.9896	4.2490	9.2054×10^{-5}	8.7867×10^{-5}	9.9098×10^{-5}
	0.05	-3.1465	3.4107	-20.1635	6.5269	9.4653×10^{-5}	8.5776×10^{-5}	1.1027×10^{-4}

Table 1: Histogram Statistics: $\sigma(z)$ is the standard deviation for the Gaussian noise applied to the rear-surface temperature rise; $\mu(\epsilon)$, $\sigma(\epsilon)$ $\min(\epsilon)$ and $\max(\epsilon)$ refer to the mean, standard deviation, minimum and maximum values of the errors produced from calculating Eq (13) using numerical techniques and similarly $\mu(\tilde{\alpha})$, $\min(\tilde{\alpha})$ and $\max(\tilde{\alpha})$ refer to the mean, minimum and maximum values of the resulting $\tilde{\alpha}$ values.

error whereas the half-rise time method has a distinct bias for each noise level, each including a wide range of error.

The histogram statistics are displayed in Table 1, with each statistic evaluated to 5 significant figures. For low levels of noise, the new method performs better than the half-rise time method, producing nearly a quarter of the standard deviation in error. This percentage increases by 3% for the respectively increasing levels of noise. The minimum error for the half-rise time method is over 3 times greater than that from the new equation for each noise level, with a minimum error of approximately -4.3 for our method with $\sigma(z) = 0.05$ compared to approximately -20 for the half-rise time method. Similar results are seen with the maximum error, with the half-rise time method producing greater error results for each level of noise. The effects of the error values corresponding to the $\tilde{\alpha}$ values produced is evident in the final three columns of the table, where the mean value for each noise level for the new method is significantly closer to the true value of 9.1766×10^{-5} compared to the half-rise time method.

Thus, the research presented in this report has lead to a new method for calculating the thermal diffusivity that produces less variable results compared to the half-rise time method, and outputs more accurate estimates of the thermal diffusivity constant which provides more stable and reliable results when being applied practically.

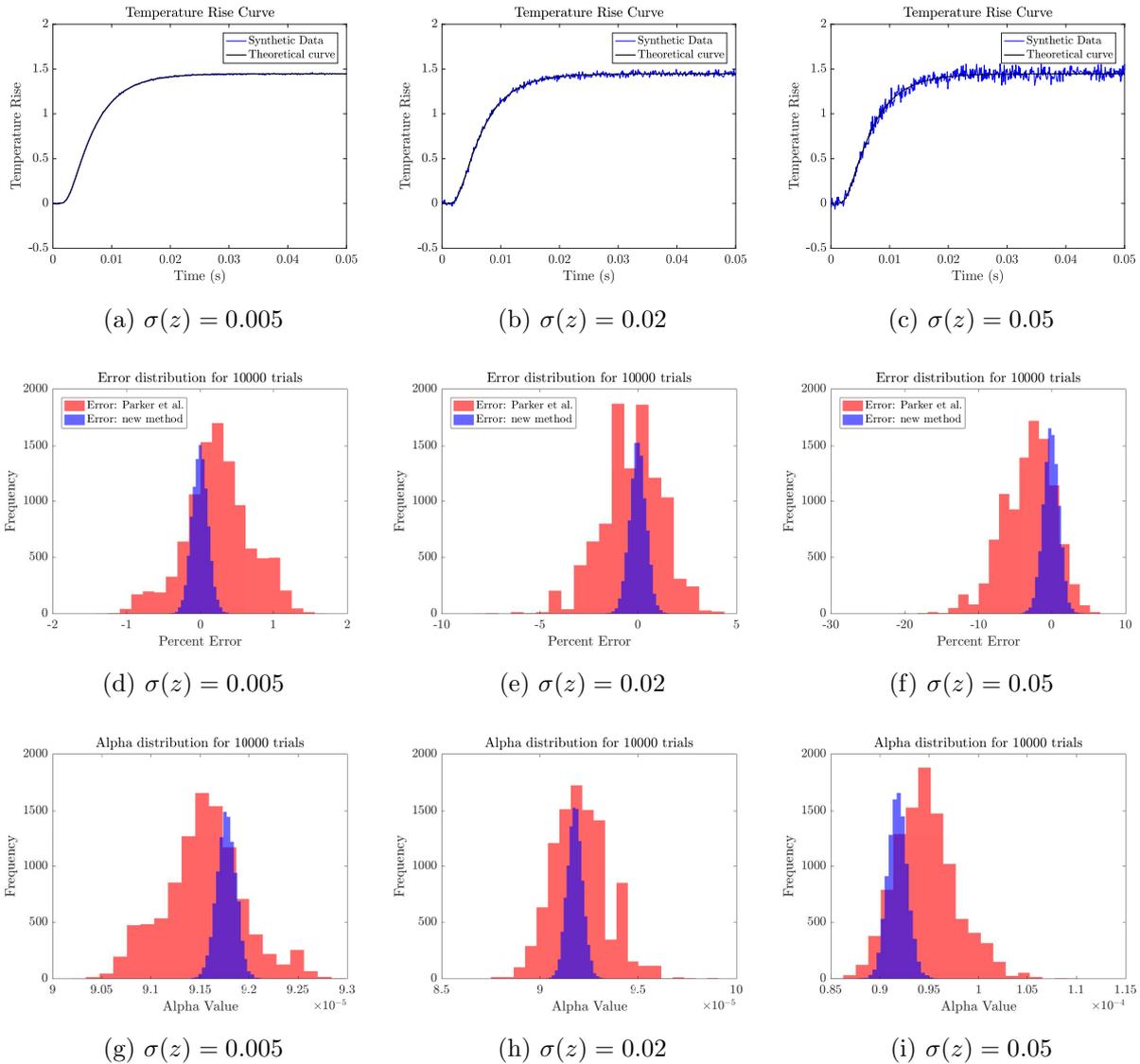


Figure 2: (a)-(c) Displays the synthetic data (blue) generated by applying Gaussian noise to the theoretical rear-surface temperature rise curve (black). (d)-(f) Displays the distribution of the relative error, ϵ , for 10,000 trials, comparing the new method (blue) with the half-rise time method (red). (g)-(i) Displays the resulting $\tilde{\alpha}$ values produced over 10,000 trials, comparing the new method (blue) with the half-rise time method (red).



5 Conclusion

A new method to determine the thermal diffusivity was explored by evaluating the area enclosed between the rear-surface temperature rise curve and the steady-state temperature to formulate an exact expression for the thermal diffusivity. Gaussian noise was applied to the rear-surface temperature rise curve to mimic a laser-flash experiment, and the area between this and the steady-state temperature was calculated using numerical techniques. This process was repeated for 10,000 trials and the resulting error and experimental thermal diffusivity values for both methods were displayed in histograms. The new method produces values for the thermal diffusivity constant that are less variable than the half-rise time method and confirms that the new method is a better approach for applications involving the thermal diffusivity where the laser-flash experiment is involved.

Some extensions to this approach to consider for future research would include analysing a system with heat loss ([Cowan, 1961](#)) and investigating materials consisting of layered samples ([Czél et al., 2013](#); [Larson and Koyama, 1968](#)), each composed of different properties.



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