AMSI VACATION RESEARCH SCHOLARSHIPS 2019–20

EXPLORE THE MATHEMATICAL SCIENCES THIS SUMMER

Defining a universal discrete-variable Wigner function of discrete physical systems for all dimensions

Lucky K. Antonopoulos

Supervised by A/Prof Nicolas C. Menicucci

RMIT University - QuRMIT - CQC2T

Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Australian Government Department of Education

Contents

Abstract

Similar to the wave function representation of states of quantum mechanics, the *continuous variable* Wigner function is a similar but more intuitive representation and is very useful in the fields of quantum information and quantum computing. Over the last few decades, scientists such as William K. Wootters (1987) have created a *discrete-variable* Wigner function (representing states of discrete systems) using ad hoc approaches, leading to many different definitions of such a function. A subset of such definitions however have a commonality being that they are only initially defined over prime dimensions (systems that have a prime number of total possible states). To extend this definition to non-prime dimensions, extra work implementing complex mathematical methods are required. In this work, we aim to define a universal discrete-variable Wigner function which would be defined for all possible dimensions. We attempt to accomplish this task by implementing our coarse grain mapping method to the continuous-variable Wigner function of Gottesman, Kitaev, and Preskill (GKP) encoded states; these are continuous systems which represent information of a discrete system. This methodology has been shown to be consistent with the results obtained by Wooters for the first seven prime dimensions. I conjecture that this method works for all prime d and checking this claim is left to future work.

1 Introduction

The continuous-variable (CV) Wigner function is a quasiprobability distribution – i.e., it satisfies all conditions for a probability distribution except positivity, and it represented in phase-space[9] – i.e., the multidimensional space of position and momentum required to specify the state of the system for a CV system. A discrete-variable (DV) Wigner function has been defined for qudits – i.e., d-dimensional quantum systems, by many scientists[3][5][6][7][8] over the decades such as William K. Wooters[3] in 1987. However, many functions such as these are only initially defined for a subset of all possible dimensions – i.e., they are only initially defined for prime and infinite dimensional systems.

Gottesman, Kitaev, and Preskill (GKP) developed a method[2] for representing information of a discrete system within a continuous system. The wave functions of these GKP-encoded states are a Dirac comb – i.e., they are a regularly spaced comb of Dirac deltas and consequently are doubly periodic – i.e., periodic in both the position and momentum representations. This leads to a conclusion that the logical subspace of the state has a natural toroidal symmetry in these variables.

To define a quasiprobability-description of DV systems, we will apply the CV Wigner function to GKP-encoded states. This is a way of filling the gap in the currently available state representations of DV systems. It is possible to do this since the CV Wigner function works for any CV state – including a GKP-encoded qudit of any dimension. This is in contrast to a subset of previously defined DV Wigner functions, which are only initially defined for a subset of all possible dimensions (primes).

In this project, we take the tools that we know work – i.e., GKP encoding and the CV Wigner function – and apply them to a DV system. We then look for patterns relating our description to the known DV Wigner function. In doing so, we have found that what we call a 'coarse grain mapping' method to be a possible link to defining a DV Wigner function from the CV Wigner function of GKP-encoded states.

1.1 Statement of Authorship

Lucky K. Antonopoulos developed the Mathematica codes to generate the CV Wigner functions of GKP-encoded d-dimensional states as well as Wootters' DV Wigner function, worked through the algebra in section 2.5.1, began work on developing a generalised analytic expression for a DV Wigner function using the coarse graining method, discussed and interpreted his findings and wrote this report.

A/Prof Nicolas Menicucci is Lucky's supervisor and was also the one who suggested the coarse graining method which he noted for qubits as a possible mapping technique for higher dimensional systems.

2 Theory

2.1 The wave function representation

To talk about the Wigner representation of a system in physics, we must first introduce one of the most common ways of describing the state of a physical in system and that is the wave function representation. The wave function is a complex valued probability amplitude which has the property that when you take the square of the magnitude, you get real valued, measure able probabilities. As an example, the wave function, $\psi(x)$, for a particle in one dimension is calculated from Schrödinger's equation[4] which is a 2nd order differential equation of the form

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t} \tag{1}$$

where $\hbar$ is the reduced Planck's constant, m is the mass of the particle, x and t are the position and time respectively, and i is the imaginary unit. (This is the time dependant version of Schrödinger's equation.)

2.2 The continuous-variable Wigner representation

With the wave function in hand, the CV Wigner function can be defined. This function was first defined in a paper by Eugene Wigner in 1932[1]. Unlike the wave function representation – which is complex valued, the CV Wigner function is real valued, and it is defined in phase-space. Phase-space is a useful tool in that it can provide physical intuition about the system as opposed to wave function representation which is complex valued and difficult to distill intuition from. Another property that differs between the two representations is that where as the wave function can only define pure states (and hence superposition states), a Wigner function can define both pure and mixed states of a quantum system, which is a great feature to have at hand. The CV Wigner function for a pure state is defined as

$$W(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\ \psi^*(q+y)\psi(q-y)e^{\frac{2ipy}{\hbar}} \tag{2}$$

where the asterisk denotes complex conjugation, q and p are the quadratures of position and momentum, and y is just the integration variable.

2.3 Distilling intuition for phase-space and a qudit and its dimensionality

I would like to briefly provide an example here to help with building an intuition of what phase-space exactly is and for what a qudit and its dimensionality might be. For intuition, the easiest way to visualise phase-space is by considering a swinging pendulum as shown in figure

1, where both the position and momentum of the pendulum can be described by a single point in phase-space.

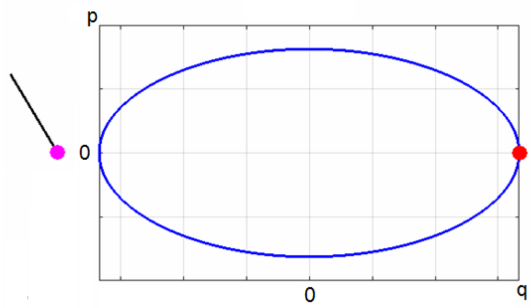

Figure 1: The state of a continuous-variable system – a pendulum (pink), described by a point (red) on a phase-space diagram.[10]

In quantum physics, there are quantum physical systems which we call qudits; these are quantum systems with a particular dimensionality. The dimensionality, d, of a system is the number of possible states it can have. As an example, a classical coin flip has 2 possible states, heads and tails, and therefore is a 2-dimensional classical system (a bit). For the quantum analog, we could, for example have a single electron with its spin property either being aligned along the positive or negative z-axes – i.e., up or down along z; this is one such quantum equivalent of the classical bit is referred to as a qubit (a quantum bit).

The main difference between a classical a quantum system is that whereas classical systems are defined by there states, quantum systems can be in superpositions of their defining states.

2.4 A discrete-variable Wigner representation

As already mentioned, a DV Wigner function describes the state of a discrete physical system. However, the interesting aspect to note about these functions is that there are many ad hoc mathematical definitions by many scientists over the decades. A common (and interesting) feature among some these definitions (such as Wootters' DV Wigner function[3]) is that they are only initially defined for systems with prime dimensions, or systems with infinite dimensions (which would then define a continuous system). There are then extra mathematical methods implemented to alter the function's properties to define them for non-prime dimensional systems. Therefore, we see that there there is a gap in the definition of DV Wigner functions which doesn't allow for a singular, unique DV Wigner function to be defined, let alone for any d-dimensional system.

2.4.1 Wootter's discrete-variable Wigner function

One of the first DV Wigner functions was defined by William K. Wooters in 1986[3]; this function is defined in equation (3)

$$W_{\alpha}^{d} = \frac{1}{d} tr(\hat{\rho}\hat{A}_{\alpha}) \tag{3}$$

where we have the value of the DV Wigner function for a particular coordinate $\alpha := (a_1, a_2)$ for a d-dimensional system, where $\hat{\rho}$ is the density operator which defines the state of the system and $\hat{A}$ is the phase point operator which is a $d \times d$ matrix defined for each coordinate and is defined in equation (4)

$$A_{\alpha,\, jk} = \delta_{2a_1, j+k}\, e^{\frac{2i\pi}{d} a_2 (j-k)} \tag{4}$$

where j and k are the j^{th} and k^{th} row and column respectively of the $d \times d$ matrix, and δ is a Dirac delta. (Note that the product $2\alpha_1$ and the sum $j+k$ are both modulo d.)

As an example, the DV Wigner function for a 2-dimensional qudit (a qubit) as defined by Wootters is shown in figure 2.

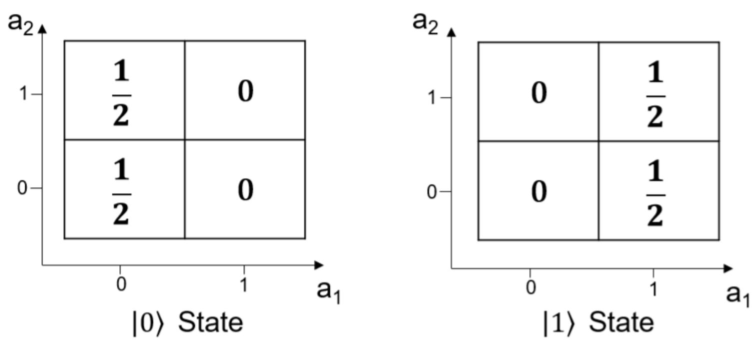

Figure 2: The $|0\rangle$ state and $|1\rangle$ state of definite position for a 2-dimensional qudit. The values shown represent the value of the point coordinates, indicated by the tick marks. (Note that the horizontal axis is in position and the vertical axis is in momentum.

Interestingly, similar to the CV Wigner function, the DV Wigner function produces measurable probabilities when one integrates over a line i.e., sums the values over a line. As an example, figure 3 shows that to get a measurable probability for position of the $|0\rangle$ state or $|1\rangle$ state of a qubit, you sum over the values of the opposite quadrature and indeed each state has a unique, definite position with 100% certainty.

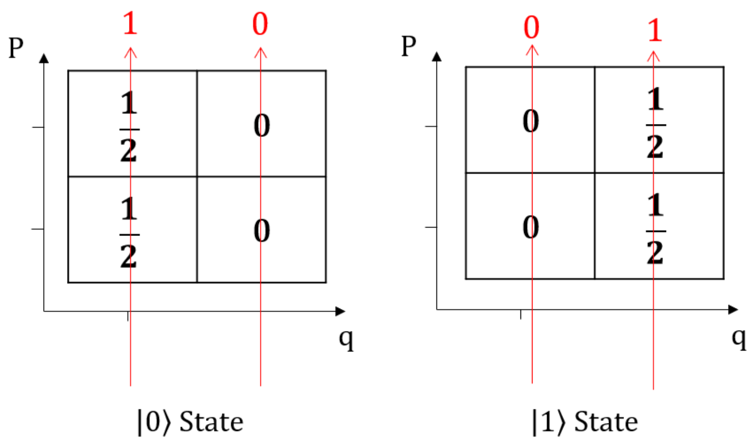

Figure 3: The $|0\rangle$ state and $|1\rangle$ state of definite position for a 2-dimensional qudit. The values shown represent the value of the point coordinates, indicated by the tick marks. The red lines show which values are summed over to get measurable probabilities (numbers indicated in red) for position.

2.5 GKP-encoded states

Gottesman, Kitaev, and Preskill (GKP) defined a method of encoding a qubit in an oscillator – i.e., representing information, such as the state of a discrete system within a continuous system, specifically the harmonic oscillator; we call these types of states GKP-states[2]. The wave function of the j^{th} state of a d-dimensional GKP-state is a Dirac comb which is a sum of equally spaced Dirac deltas in position and momentum and they are defined using Dirac notation as shown in equations (5) and (6) respectively.

$$\psi^d_{|j\rangle}(q,p) := |\alpha(j+ds)\rangle_q = \sum_{s=-\infty}^{\infty} \delta(q-\alpha j-\alpha ds) \tag{5}$$

$$\psi^d_{|j\rangle}(q,p) := \left|\frac{2\pi}{d\alpha}(j+dt)\right\rangle_p = \sum_{t=-\infty}^{\infty} \delta(p-\frac{2\pi}{d\alpha}j-\frac{2\pi}{\alpha}t) \tag{6}$$

where $\alpha=\sqrt{\frac{2\pi}{d}}$ and is defined as the spacing between spikes (each Dirac delta) and j is the j^{th} state of the system where $j\in\{0,1,..,d-1\}$; in the wave function j acts as some horizontal shift of $j \times \alpha$. (Note that the subscript q on the ket indicates that it is an eigenstate in the position basis.)

2.5.1 Derivation of the continuous-variable Wigner function of GKP-encoded qudits

To obtain the CV Wigner function of a GKP-encoded qudit, such as those defined in equations (5) and (6), we substitute their wave function into the general CV Wigner function defined in equation (2). Using the position state as in equation (5) and substituting we get equation (7), a form of the CV Wigner function of a GKP-encoded qudit.

$$W^d_{|j\rangle}(q,p) = \frac{1}{\pi}\int_{-\infty}^{\infty} dy \sum_{s=-\infty}^{\infty} \delta(q+y-\alpha j - d\alpha s) \sum_{t=-\infty}^{\infty} \delta(q-y-\alpha j-d\alpha t)e^{2iyp} \tag{7}$$

where we have set $\hbar$ equal to 1, and changed one of the summation indices so we can keep track of them explicitly. Next, using the property that Dirac deltas are defined at a single horizontal point, and that we can commute an integral and summation for linear functions, which we have, we can integrate by simply setting one of the Dirac deltas' arguments to zero (here we take the delta summed over t) to get equation (8).

$$y = q - \alpha j - d\alpha t \tag{8}$$

Doing so sets this delta equal to 1 (the total area under the delta) where we now make use of equation (8) by substituting it back into equation (7) to get equation (9)

$$W^d_{|j\rangle}(q,p) = \frac{1}{\pi}\sum_{s,t=-\infty}^{\infty} \delta(2q-2\alpha j - d\alpha t - d\alpha s)e^{(2q-2\alpha j-2d\alpha t)ip} \tag{9}$$

where we now choose to work on the Dirac delta and exponential terms separately so we can simplify our expression further. First, we arm ourselves with the Poisson summation relation (equation (10)) and use it to change the Dirac delta into a sum of exponentials. This allows us to split the exponential into multiple exponentials where we get nice simplifications

$$\sum_{s=-\infty}^{\infty} \delta(x-sT) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{2\pi i x \frac{n}{T}} \tag{10}$$

and so from equation (9) we can set $x = 2q - 2\alpha j - d\alpha t$ and $T = d\alpha$ and begin simplifying as follows:

$$\sum_{s=-\infty}^{\infty} \delta(2q-2\alpha j-d\alpha t-d\alpha s) = \sum_{s=-\infty}^{\infty} \frac{1}{d\alpha} e^{2\pi i \frac{s}{d\alpha}(2q-2\alpha j-d\alpha t)}$$

$$= \sum_{s=-\infty}^{\infty} \frac{1}{d\alpha} e^{2\pi i(2q-2\alpha j)\frac{s}{d\alpha}} \, e^{-2\pi i t s}$$

where

$$e^{-2\pi i t s} = (e^{-2\pi i})^{ts} = (1)^{ts} = 1$$

and so

$$\sum_{s=-\infty}^{\infty} \delta(2q-2\alpha j-d\alpha t-d\alpha s) = \sum_{s=-\infty}^{\infty} \frac{1}{d\alpha} e^{2\pi i(2q-2\alpha j)\frac{s}{d\alpha}}$$

Now, setting $x = 2q - 2\alpha j$ and $T = d\alpha$ we get the simplification of our delta term which is

$$\sum_{s=-\infty}^{\infty} \delta(2q-2\alpha j-d\alpha t-d\alpha s) = \sum_{s=-\infty}^{\infty} \delta(q-\alpha j-\frac{d\alpha}{2}s) \tag{11}$$

As for the exponential term from equation (9), we again use the Poisson summation except this time to turn it into a delta functions as follows:

$$\sum_{t=-\infty}^{\infty} e^{(2q-2\alpha j-2d\alpha t)ip}$$

$$= \sum_{t=-\infty}^{\infty} e^{2ip(q-\alpha j)} \, e^{-2di\alpha tp}$$

$$= \sum_{t=-\infty}^{\infty} e^{2ip(q-\alpha j)} \, \pi\frac{1}{\pi} e^{-2i\pi(d\alpha p)\frac{t}{\pi}}$$

and then using the Poisson summation one last time with $x = d\alpha p$ and $T = \pi$, we arrive at the simplification that

$$\sum_{t=-\infty}^{\infty} e^{(2q-2\alpha j-2d\alpha t)ip} = \pi \sum_{t=-\infty}^{\infty} e^{2ip(q-\alpha j)} \delta(p-\frac{\pi}{d\alpha}t) \tag{12}$$

Substituting equations (11) and (12) into equation (9) we get

$$W^{d}_{|j\rangle}(q,p) \approx \sum_{s,t=-\infty}^{\infty} \delta(q-\alpha j-\frac{d\alpha}{2}s)\delta(p-\frac{\pi}{d\alpha}t)e^{2ip(q-\alpha j)} \tag{13}$$

The final simplification we now make is by using the property of the Dirac delta, which is that it is non-zero when its argument is equal to zero, we can get equations for q and p, to substitute into the last remaining exponential.

$$q = \alpha j + \frac{d\alpha}{2}s$$

$$p = \frac{\pi}{d\alpha}t$$

and so by substitution into

$$e^{2ip(q-\alpha j)}$$

we get

$$= e^{2i(\frac{\pi}{d\alpha}t)((\alpha j+\frac{d\alpha}{2}s)-\alpha j)}$$

$$= e^{\pi ist}$$

$$= (-1)^{st} \tag{14}$$

Finally, substituting equation (14) into equation (13) we get our final form of the CV Wigner function of a d-dimensional GKP-state, which is a sum of deltas and is

$$W^{d}_{|j\rangle}(q,p) \approx \sum_{s,t=-\infty}^{\infty} (-1)^{st}\delta(q-\alpha j-\frac{d\alpha}{2}s)\delta(p-\frac{\pi}{d\alpha}t) \tag{15}$$

which is the expression defined in the paper by Gotteseman, Kitaev and Preskill[2]

3 Methodology

3.1 The continuous-variable Wigner function of GKP-encoded qudits

With the general form of the CV Wigner function of a general GKP-encoded qudit (equation (15)), we can now begin to plot them on a phase-space diagram, where we may distil some important properties. Figures 4 and 5 show plots for a qubit ($d = 2$) and qutrit ($d = 3$).

3.1.1 Important properties

According to equation 15, the point values of the CV Wigner function are equal to either ± 1 and given the fact that these values represent probabilities, it is useful to normalise them. To do this, we sum each value along the first column of the $|0\rangle$ state and require that it sums to 1; this is because the system has been chosen to have a definite position and so the probability

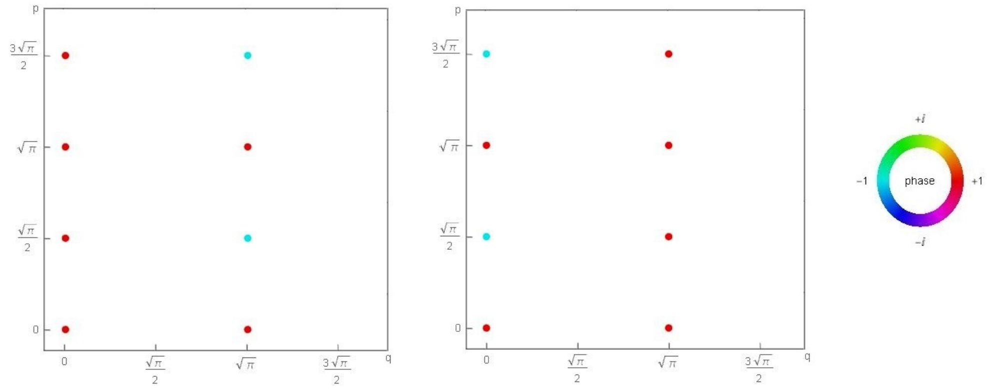

Figure 4: The CV Wigner function of definite position for a GKP-encoded 2-dimensional qudit. We have the $|0\rangle$ state (left) where $j = 0$ and $|1\rangle$ state (right) where $j = 1$. Each point is normalised to a value of $\pm\frac{1}{4}$ (sign dependant on phase).

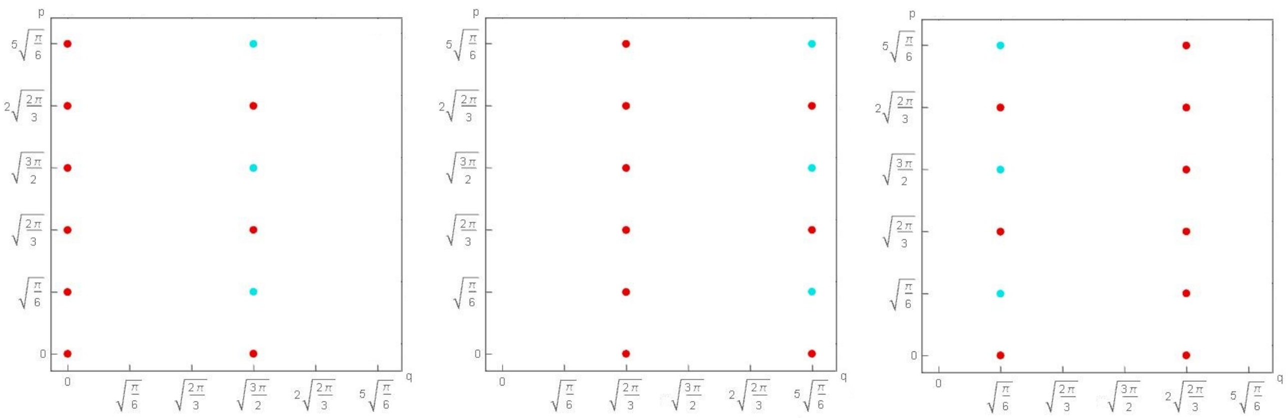

Figure 5: The CV Wigner function of definite position for a GKP-encoded 3-dimensional qudit. We have $|0\rangle$ state (left) where $j = 0$, $|1\rangle$ state (middle) where $j = 1$ and $|2\rangle$ state (right) where $j = 2$. Each point is normalised to a value of $\pm\frac{1}{6}$ (sign dependant on phase).

of a measurement of position must be 1. For example, each red point in figure 4 is worth $+\frac{1}{4}$, and each blue point is worth $-\frac{1}{4}$. Likewise, each red point in figure 5 is worth $+\frac{1}{6}$, and each blue point is worth $-\frac{1}{6}$. This pattern leads to the rule that the points are normalised to values as $\frac{1}{2d}$. (Note as well that each tick mark signifies a valid coordinate of the Wigner function and that no dot means the value is zero).

The other property to note is the j parameter. As can be seen in equation (15), the j parameter always translates each point to the right by two tick marks, which corresponds to a translation of α After d shifts, the Wigner function is back in it's $|0\rangle$ state as the defined unit cell is periodic; that is, the the unit cell has a toroidal topology.

Finally, it is interesting to note that the number of points horizontally is always two whereas the number of point vertically is always equal to $2d$, giving a total number of non-zero points equal to $4d$.

3.2 Coarse grain mapping

The notion of coarse graining the CV Wigner function of GKP-encoded states is easy to understand and I have been able to show that it produces the correct DV Wigner function when compared to Wootters' version for the first seven prime dimensions.

To coarse grain, we first section off our unit cell into a grid of d^2 boxes where each box always contains four points. Then it is as simple as adding the value of each of the four points, per box. As an example, lets look at a qutrit system. Taking the plots from figure 5, we coarse grain by drawing a grid on top to produce figure 6.

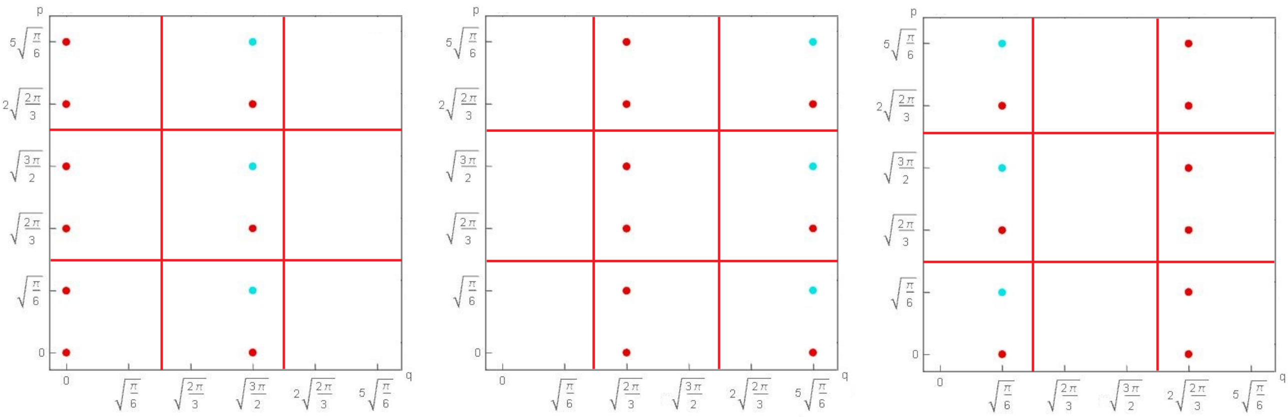

Figure 6: The CV Wigner function of definite position for a GKP-encoded 3-dimensional qudit with initial a coarse grained grid drawn on top. Each point is normalised to a value of $\pm\frac{1}{6}$ (sign dependant on phase). We have the $|0\rangle$ state (left) where $j = 0$, $|1\rangle$ state (middle) where $j = 1$ and $|2\rangle$ state (right) where $j = 2$.

Then by summing each box's points, not forgetting the zero points at particular tick-mark coordinates, we arrive at figure 7 which looks identical to that of Wootters' DV Wigner function for the 3 states of a qutrit.

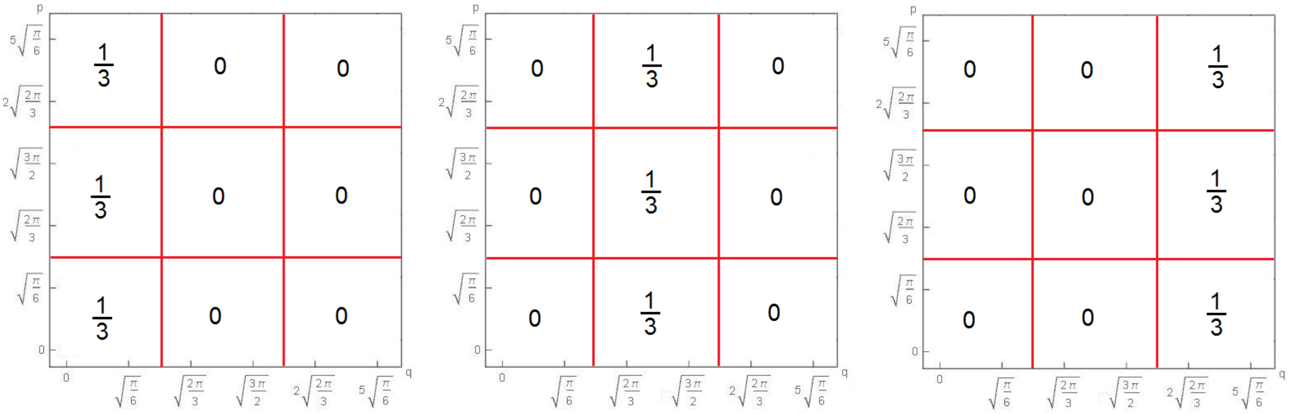

Figure 7: A DV Wigner function from the coarse grained CV Wigner function of definite position for a GKP-encoded 3-dimensional qudit. This is also identical to Wootter's DV Wigner function. We have the $|0\rangle$ state (left) where $j = 0$, $|1\rangle$ state (middle) where $j = 1$ and $|2\rangle$ state (right) where $j = 2$.

The coarse grain mapping method has been shown to work for the first seven prime dimensions by comparison to Wootters'[3] work and I conjecture that this method works for all prime d; checking this claim is left to future work.

4 Conclusion

By comparison to William K. Wooters' work on defining a discrete-variable Wigner function for discrete systems, we have provided a method called coarse grain mapping which I have shown successfully maps the continuous-variable Wigner function of GKP-encoded qudits to an equivalent discrete-variable Wigner function as defined by Wooters for the first seven prime dimensions. As Wooters' discrete-variable Wigner function, as well as the many other definitions of such representations, are only initially defined for prime dimensional systems, our method aims to provide a universal discrete-variable Wigner function for a d-dimensional system.

It is my conjecture that the coarse grain mapping method works for all prime d and checking this claim is left to future work. This includes defining a mathematical definition of the coarse grain mapping method and showing that we can obtain the appropriate discrete-variable Wigner function for any d-dimensional system in comparison to the many other defined discrete-variable Wigner functions.

5 References

[1] E.P. Wigner (1932), 'On the Quantum Correction for Thermodynamic Equilibrium', *Physical Review* **40**, 749

[2] D. Gottesman, A. Kitaev, and J. Preskill (2001), 'Encoding a Qubit in an oscillator', *Physical Review A* **64**, 012310

[3] W.K. Wooters (1987), 'A Wigner-function formulation of finite-state quantum mechanics', *Annals of Physics* 176, 1-21.

[4] Griffiths, David J. (2004), 'Introduction to Quantum Mechanics' (2nd ed.), Prentice Hall, ISBN 978-0-13-111892-8

[5] N.M. Atakishiyev, S.M. Chumakov, K. B. Wolf (1998), 'Wigner distribution function for finite systems', *Journal of Mathematical Physics* 39, 6247

[6] M. Hillery, R.F. O'Connel, M.O. Scully, E.P. Wigner (1984), 'Distribution functions in physics: Fundamentals', *Physics Reports* 106, Pages 121-167

[7] D. Gross (2006), 'Hudson's theorem for finite-dimensional quantum systems', *Journal of Mathematical Physics* 47, 122107

[8] ULF Lionhardt (1996), 'Discrete Wigner function and quantum-state tomography', *Physical Review A* 53, 2998

[9] C. Miquel, J.P. Paz, M. Saraceno (2002), 'Quantum computers in phase space', *Physical Review A* 65, 062309

[10] I. Cooper, Image accessed February (2020), 'Dynamics of oscillating and chaotic systems - rigid Pendulum', *School of Physics, University of Sydney*