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Groups Acting on Trees Without Involutive Inversions

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1 Abstract

We endowed the group of automorphisms of the d-regular tree denoted $\mathrm{Aut}(T_d)$ with a natrual topology such that for every $n \in \mathbb{N}$, the stabilizers $\mathrm{Stab}_{\mathrm{Aut}(T_d)}(B(v,n))$ are open sets in $\mathrm{Aut}(T_d)$. Furthermore, this topology dictates that the set of these stabilizers running over n form a neighbourhood basis of the identity $\mathrm{id} \in \mathrm{Aut}(T_d)$. The Weiss conjecture states that there are only finitely many conjugacy classes of subgroups of $\mathrm{Aut}(T_d)$ which are vertex transitive, topologically discrete, and locally primitive. For the case where $d = 3$ this conjecture further states that there are exactly seven such conjugacy classes of which five have been described in terms of how they act locally. Towards describing the remaining two of five as such, we modified the definition of the generalized Burger-Mozes universal group, to define an alternative universal group $EU_k(F)$ with prescribed local actions on edge, rather than vertex neighbourhoods. In doing so, upon restricting to the case where $d = 3$ we showed that our edge-universal group could be used to produce a subgroup of $\mathrm{Aut}(T_3)$ which contains no involutive inversions and which is topologically discrete. Thus we propose that this subgroup is a promising candidate for describing the classes which we seek in terms of how they act locally.

2 Introduction

Definitions and Basic Theory

The concept of symmetry, which is formalised using the algebraic notion of groups, is an ever-present field of interest within the mathematical community, with direct applications to many scientific disciplines. We can investigate such symmetry groups and their actions on mathematical objects to further our understanding of how different symmetries work, and to characterise new and interesting symmetry groups. Before we proceed, it is useful to formalize the general notion of a mathematical tree, and to introduce the notion of groups acting on such a tree.

We define a tree as the 5-tuple, $T := (V, E, o, t, r)$ where V is the set of vertices, E is the set of edges, $o : E \to V$ is the origin map which assigns an origin vertex $o(e)$ to the edge e, $t : V \to E$ is the terminus map which assigns a terminus vertex to the edge e, and $r : E \to E$ is the reversal map which swaps the origin and terminus vertices of the edge. We define an orientation of a tree as a choice of a subset $E^+ \subseteq E$ such that $E = E^+ \sqcup r(E)$.

We can define a notion of distance between vertices with the convention that the distance function between two vertices connected by an edge is equal to 1 i.e. $d(o(e), t(e)) = d(t(e), o(e)) = 1$.

For some vertex $v \in V$ we define $E(v)$ to be the set of edges extending from the vertex v.

Now, suppose $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are arbitrary graphs. Then, a graph homomorphism from Γ_1 to Γ_2 is defined to be the pair of mappings $\varphi = (\varphi_V, \varphi_E)$ where $\varphi_V : V \to V$ and $\varphi_E : E \to E$ are given such that:

- $o(\varphi_E(e)) = \varphi_V(o(e))$
- $t(\varphi_E(e)) = \varphi_V(t(e))$
- $r(\varphi_E(e)) = \varphi_E(r(e))$

In other words, we say that a graph homomorphism is a map between graphs which preserves the edge structures across the mapping. Building upon this definition, we say that graph isomorphism between Γ_1 and Γ_2 is a bijective graph homomorphism and furthermore, a graph automorphism of Γ_1 is a graph isomorphism from Γ_1 to itself. In fact, the set of these automorphisms $\mathrm{Aut}(\Gamma_1)$ becomes a group under the usual composition of maps such that we may study the actions of this group on different graphs.

We are interested particularly in the group $\mathrm{Aut}(T_d)$ where T_d is the d-regular tree defined such that for each vertex $v \in V(T_d)$ there are exactly a d-number of edges ($d \in \mathbb{N}_{\geq 3}$) attached to v and this edge structure is uniform throughout the entire tree. Let us now categorise the general properties of $\mathrm{Aut}(T_d)$ group actions via the Tits classification [CM18].

Proposition 1.2 The action of $\varphi \in \mathrm{Aut}(T_d)$ can be categorised into one of the following [Tit70]:
(i) $\varphi \in \mathrm{Aut}(T_d)$ is *elliptic* if it fixes some vertex v.
(ii) $\varphi \in \mathrm{Aut}(T_d)$ is an *inversion* if it inverts (or reflects) edges about some axis. Inversions of order n are effectively the identity transformation after applying the inversion n-times. *Involutive inversions* have order 2.
(iii) $\varphi \in \mathrm{Aut}(T_d)$ is a *translation* If it translates an array of edges and vertices in the usual way.

We may also introduce further tools to study these group actions. One particularly lucrative example of this is introduced via the notion of a topological group.

let $(G, \circ)$ be a group and $(G, \mathcal{T})$ a topological space. Then $(G, \circ, \mathcal{T})$ is a *topological group* if $\circ : (G, \mathcal{T}) \times (G, \mathcal{T}) \to (G, \mathcal{T})$ given by $\circ(x, y) = xy$ and $i : (G, \mathcal{T}) \to (G, \mathcal{T})$ given by $i(x) = x^{-1}$ are both continuous mappings. Note that in this case, we view $(G, \mathcal{T}) \times (G, \mathcal{T})$ as being equipped with the product topology.

Towards a general construction, let X be a set. There is a natural topology on $\mathrm{Sym}(X) := \{\varphi : X \to X \mid \varphi \text{ is bijective}\}$ such that the sets $\mathrm{Stab}_{\mathrm{Sym}(X)}(S)$ are open when $S \subseteq X$ is finite. Building upon this general construction, we can endow $\mathrm{Aut}(T_d)$ with the topology such that for each $n \in \mathbb{N}$, the stabilizers

$$\mathrm{Stab}_{\mathrm{Aut}(T_d)}(B(v, n)) =: U_n$$

over $\mathrm{Aut}(T_d)$ of a ball of radius n, centered around a fixed vertex v are open sets. Furthermore, this topology has the property that these stabilizers $\{U_n\}_{n \in \mathbb{N}}$ form a neighbourhood basis of the identity transformation $\mathrm{id} \in \mathrm{Aut}(T_d)$. This topology is a powerful tool for studying the properties of the automorphism group $\mathrm{Aut}(T_d)$.

To facilitate the study of group actions on T_d, we require a regular labelling map $l : E \to \{1, 2, ..., d\}$ where, upon restriction to the set of edges $E(v)$ we have $l_v := l \mid_{E(v)}: E(v) \to \{1, 2, ..., d\}$. This labelling is regular in the sense that no two edges attached to a given vertex are mapped to the same label. With this labelling we can then consider the map

$$\sigma : \mathrm{Aut}(T_d) \times V \to S_d$$

$$(g, v) \mapsto l_{gv} \circ g \circ l_v^{-1},$$

where the permutation $\sigma(g, v)$ is called the local action of g at v. The Burger-Mozes [BM00] universal group is then given for any $F \leq S_d$, by

$$U(F) = \{g \in \mathrm{Aut}(T_d) : \forall v \in V : \sigma(g, v) \in F\}.$$

Project Aim

The Weiss conjecture [Wei78] states that there are only finitely many conjugacy classes of discrete, vertex-transitive, locally primitive subgroups $H \leq \mathrm{Aut}(T_d)$. Furthermore, in the case where $d = 3$ it is known that there are precisely seven such conjugacy classes [Tut47][Tut59][DM80], five of which have been identified as universal groups similar to the above [Tor18, Example 11.28]. The aim of this project is to define universal groups with prescribed local actions on edge rather than vertex neighbourhoods such that we may identify the remaining two out of seven as such, see [Tor18, Example

11.28].

Statement of Authorship

Under the direction of Stephan Tornier, Jack Berry devised a definition for a universal group with prescribed edge rather than vertex local actions and investigated its properties with respect to the Weiss Conjecture.
Stephan Tornier devised the project aim and developed the approach which was taken towards achieving this aim. Stephan Tornier guided the project at each step and developed the theory implemented to obtain our results.

3 Universal Groups with Prescribed Vertex-Local Actions

In this section, the properties of the Burger, Mozes Universal group $U(F)$ will be outlined and expanded upon in lead up to presenting a proposed solution to the previously stated problem.

Lemma 2.1 Let $x \in V$ and $g, h \in \mathrm{Aut}(T_d)$. Then $\sigma(gh, x) = \sigma(g, hx)\sigma(h, x)$.

Proof. Consider $\sigma(gh, x) = l_{gh(x)} \circ gh \circ l_x^{-1} = l_{g(hx)} \circ g \circ h \circ l_x^{-1} = l_{g(hx)} \circ g \circ l_{hx}^{-1} \circ l_{hx} \circ h \circ l_x^{-1} = \sigma(g, hx)\sigma(h, x)$ as was required.

□

Lemma 2.2 Let $x \in V$ and $g \in \mathrm{Aut}(T_d)$. Then $\sigma(gg^{-1}, x) = e_F$ where e_F is the identity in F.

Proof. Consider $\sigma(gg^{-1}, x) = l_{gg^{-1}(x)} \circ gg^{-1} \circ l_x^{-1} = l_{\mathrm{id}(x)} \circ \mathrm{id} \circ l_x^{-1} = l_x \circ l_x^{-1} = e_F$.

□

Proposition 2.1 [BM00]. Let $U(F)$ be a universal group.
(i) $U(F)$ is closed in $\mathrm{Aut}(T_d)$.
(ii) $U(F)$ is locally permutation isomorphic to F.
(iii) The action of $U(F)$ on T_d is vertex transitive i.e. for all vertices $v_1, v_2 \in V$ there exists an element $g \in U(F)$ such that $gv_1 = v_2$.

(iv) The action of $U(F)$ on T_d is edge transitive if and only if $F \curvearrowright \{1, ..., d\}$ is transitive.

(v) $U(F)$ is discrete if and only if the action $F \curvearrowright \{1, ..., d\}$ is free.

(vi) $U(F)^+ = \{g \in U(F) \mid \exists e \in E : ge = e\}$ has all local permutations in $F^+ = \langle \mathrm{Stab}_F(i) \mid i \in \{1, ..., d\}\rangle$.

Proof. Suppose $g \in \mathrm{Aut}(T_d) \setminus U(F)$. Then $\sigma(g, x) \notin F$ for some $x \in V$. So the open neighbourhood given by $\{h \in \mathrm{Aut}(T_d) : h \mid_{(B(x,1)} = g \mid_{B(x,1)}\}$ of g is also contained in the compliment. So, the compliment is open and $U(F)$ is closed in $\mathrm{Aut}(T_d)$.

(ii) We wish to show that $\mathrm{Stab}_{U(F)}(v) \curvearrowright E(v)$ is isomorphic to $F \curvearrowright \{1, ..., d\}$ for all $v \in V$. Towards doing so, fix some $b \in V$ and let $a \in F \leq S_d$. Now, upon defining $\alpha \in \mathrm{Aut}(T_d)$ such that $\alpha(b) = b$ and $a \circ l = l \circ \alpha$, notice that $\sigma(\alpha, b) = l_{\alpha b} \circ \alpha \circ l_b^{-1} = a \circ l_b \circ l_b^{-1} = a \in F$. Thus, we have that each $\alpha \in \mathrm{Stab}_{U(F)}(b)$ acts on $E(b)$ like some $a \in F$ acting on $\{1, ..., d\}$ and conversely each $a \in F$ acts on $\{1, ..., d\}$ like some $\alpha \in \mathrm{Stab}_{U(F)}(b)$ acting on $E(b)$ i.e. the two actions are isomorphic as was required.

(iii) Fix vertices $b_1, b_2 \in V$. Since $\mathrm{Aut}(T_d)$ is the set of bijective homomorphisms from T_d to itself, there exists $g \in \mathrm{Aut}(T_d)$ such that $g(b_1) = b_2$. Considering such a map g, we add the further condition that $\sigma(g, x) = e_F \in F$ for any $x \in V$ and $F \leq S_d$ which completes the proof.

(iv) Suppose $F \curvearrowright \{1, ..., d\}$ is transitive. Given $e, e' \in E$, we can set $\alpha' \in U(F)$ such that $\alpha'(o(e)) = o(e')$ by vertex-transitivity of $U(F)$. Also, we can set $\alpha'' \in U(F)$ such that $\alpha''(\alpha'(e)) = e'$ since $U(F)$ is locally permutation isomorphic to F where the action of F is transitive. Thus, setting $\alpha = \alpha'' \circ \alpha'$ we have edge transitivity of $U(F)$.

Conversely, suppose that the action of $U(F)$ on T_d is edge transitive. Then $\mathrm{Stab}_{U(F)}(v)$ acts transitively on $E(v)$ for any $v \in V$. Thus, since $U(F)$ is locally permutation isomorphic to F we have that F acts transitively on $\{1, ..., d\}$.

(v) Suppose the action $F \curvearrowright \{1, ..., d\}$ is free. We wish to show that $\{\mathrm{id}\} \subseteq U(F)$ is open. To this end, we will show that one of the sets $U_n := \mathrm{Stab}_{U(F)}(B(v, n))$ (for some $v \in V(T_d)$) is equal to $\{\mathrm{id}\}$. Let us consider U_1. If $g \in U_1$ we know that since $F \curvearrowright \{1, ..., d\}$ is free, $\sigma(g, v) = e_F$ where e_F is the identity permutation in F. Now, consider the edge $e = (v, v') \in E$ with label $l(e) = i$. Since $g \in U(F)$ we have that $\sigma(g, v_1)$ fixes the edge i, which again means that $\sigma(g, v_1) = e_F$. Inductively extending across T_d, we find that $U_1 = \{\mathrm{id}\}$ so that $\{\mathrm{id}\}$ is indeed open, and $U(F)$ is discrete.

Conversely, Suppose that $F \curvearrowright \{1, ..., d\}$ is not free. Then, there exists some $a \in F \setminus \{e_F\}$ such that $a(i) = i$ for some $i \in \{1, ..., d\}$. To show $U(F)$ is in-discrete, we wish to show that there doesn't exist any U_n which is equal to $\{\text{id}\}$. To this end, we seek $g \in U(F)$ which fixes $B(v, n)$ but doesn't fix $B(v, n+1)$ for $v \in V(T_d)$ i.e. $g\mid_{B(v,n)} = \text{id}$ and $g|_{B(v,n+1)} \neq \text{id}$. Let $w, w' \in V$ be such that $d(v, w) = n$ and $d(v, w') = n - 1$ and furthermore, let $e = (w, w') \in E$ be the edge with label $l(e) = i$. We can achieve the desired result by setting $g\mid_{E(w)} = l_w^{-1} \circ a \circ l_w$ and $g\mid_{E(x)} = \text{id}$ for all $x \in B(v, n) \setminus \{w\}$. We see that the edge labelled i, is fixed by $a \in F \setminus \{e_F\}$ (this is permitted since the action of F is not free), whereas those edges extending from w which aren't inside $B(b, n)$ are permuted by a. Thus, we have $g\mid_{B(v,n)} = \text{id}$ and $g\mid_{B(v,n+1)} \neq \text{id}$ for any $n \in \mathbb{N}$.

(vi) Let $g \in U(F)^+$, then there exists $e \in E$ such that $ge = e$. Let such an edge e, be denoted $e = (x, y)$, for $x, y \in V$. Then we have $g(x) = x$, $g(y) = y$ and both $\sigma(g, x)$, $\sigma(g, y)$ in F^+. Now, assume inductively that all local permutations at vertices up to a distance n from the edge e are in F^+ and consider the vertex $v \in V$ at a distance $n+1$. Let e' be an edge with origin at v and terminus at $v' \in V$ where $d(v, e) = d(v', e) + 1$. Then, $\sigma(g, v) \in \sigma(g, v)\text{Stab}_F(l(e'))$ where $l(e')$ is the label of e'. Here, $\sigma(g, v)\text{Stab}_F(l(e'))$ is generated by the point stabilizer of $l(e')$ in F, which by definition gives us that $\sigma(g, v) \in F^+$ which completes the proof by induction.

□

We will now outline a generalisation of the universal group $U(F)$. Letting $B_{d,k}$ be the finite graph which is isomorphic to the ball $B(x, k) \subseteq T_d$ $(k \in \mathbb{N})$ for some vertex $x \in V(T_d)$, there exists a unique label preserving isomorphism

$$l_x^k : B(x, k) \to B_{d,k}$$

which maps the edges labelled $i \in \{1, ..., d\}$ in $B(x, k)$ to the edges labelled i in $B_{d,k}$. Then, we consider the map

$$\sigma_k : \text{Aut}(T_d) \times V \to \text{Aut}(B_{d,k})$$

$$(g, x) \mapsto l_{gx}^k \circ g \circ (l_x^k)^{-1}$$

which is called the k-local action of g at x. With this prescribed k-local action on vertex neighbourhoods we make the definition of a generalised universal group: for some $F \leq \text{Aut}(B_{d,k})$

$$U_k(F) := \{g \in \text{Aut}(T_d) \mid \forall x \in V : \sigma_k(g, x) \in F\}.$$

One might immediately note that the universal group outlined by Burger and Mozes satisfies exactly the case where $k = 1$. Furthermore, while $U_k(F)$ shares most of the properties outlined in proposition 2.1, it is by no means obvious when $U_k(F)$ is discrete or locally action isomorphic to F given some $F \leq \mathrm{Aut}(B_{d,k})$ when $k \geq 2$. To demonstrate such peculiarities, let us introduce some useful notation which will allow us to follow up with an example in which the universal group $U_k(F)$ is not locally action isomorphic to $F \leq \mathrm{Aut}(B_{d,k})$.

For some $x \in V$ and $\omega = (\omega_1, ..., \omega_n) \in \Omega^n$ with $n \in \mathbb{N}_0$, we denote the vertex $x_\omega := \gamma_{x,\omega}(n)$ where $\gamma_{x,\omega}$ is the unique label respecting morphism $\gamma_{x,\omega} : Path_n^{(\omega)} \to T_d$ which sends 0 to x i.e. x_ω is the vertex which is arrived at by taking the path along the edges $\omega \in \Omega^n$.

Example 2.1 Let $d = 3$ and $k = 2$ such that we have the 3-regular tree with labelling map l, (T_3, l) and the finite 3-regular tree with labelling map λ, $(B_{3,2}, \lambda)$. Fixing the central vertex $b \in B_{3,2}$ define $\Omega := \{1, 2, 3\}$ and let $a \in \mathrm{Aut}(B_{3,2})$ be the automorphism which swaps b_{13} with b_{12} in $B_{3,2}$. Supposing $F = \langle a \rangle = \{\mathrm{id}, a\}$ we ask the question: Is $U_2(F)$ locally action isomorphic to F? Or equivalently, if we fix $x \in V$, is there some $\alpha \in U_2(F)_x$ such that $\alpha\,|_{B(x,2)} = (l_x^2)^{-1} \circ a \circ l_x^2$?
It turns out that the answer to this question is no. If we Suppose such an $\alpha \in U_2(F)_x$ exists and we consider the local 2-action $\sigma_2(\alpha, v)$ where $v = x_{12}$, we see that this local 2-action is equal to neither $a \in F$ or $\mathrm{id} \in F$ such that $\alpha \notin U_2(F)$ contradicting our assumption.

The theoretical framework outlined thus far has been successful in characterising many of the conjugacy classes of subgroups which we are interested in. Specifically for the case where $d = 3$ we are able, using the generalised universal group with prescribed vertex-local actions, to describe five out of seven of these classes in terms of how they act locally as alluded to in the introduction. However, we are left with an incomplete picture. Towards addressing the central aim of this project, we formulate a definition for an analogous universal group with prescribed local action on edge, rather than vertex neighbourhoods with the goal of adding to this incomplete picture.

4 Universal Groups with Prescribed Edge-Local Actions

We define $E_{d,k}^{(i)}$ to be a finite graph which is isomorphic to the ball of radius k, centered around some edge $e \in E(T_d)$ which has label $l(e) = i \in \{1, ..., d\}$. Then there is a label preserving isomorphism between $B(e, k)$ and $E_{d,k}^{(i)}$. However it is not unique as it was for the case of vertex-central $B_{d,k}$. To see why this is the case, consider the central edge e of the ball $B(e,k)$. There are actually two different isomorphisms from $B(e,k)$ to $E_{d,k}^{(i)}$ which preserve labelling, one that maps the central edge as is, and the another that inverts the central edge. To obtain an isomorphism that is unique we choose some arbitrary orientation for T_d and furthermore, an orientation for the central edge in $E_{d,k}^{(i)}$. Then, we define the map

$$l_e^k : B(e,k) \to E_{d,k}^{(i)}.$$

to be the unique label preserving and orientation preserving isomorphism.

At this point, it would be tempting to define the local action of some $\alpha \in \mathrm{Aut}(T_d)$ at the edge e as the map $\mathrm{Aut}(T_d) \times E(T_d) \to \mathrm{Aut}(E_{d,k}^{(i)})$ where $(\alpha, e) \mapsto l_{\alpha e}^k \circ \alpha \circ (l_e^k)^{-1}$ but this turns out to be insufficient for reasons demonstrated in the following example.

Example 3.1 Suppose $d = 3$ and $k = 1$. Fix $e \in E(T_3)$ such that $l(e) = 1$ and let $\alpha \in \mathrm{Aut}(T_3)$ be such that $l(\alpha e) = 2$. Then, our proposed local action will send $E_{3,1}^{(1)}$ to $E_{3,1}^{(2)}$ which does not make sense for our purposes. To correct this error, we would need some map from $E_{3,1}^{(2)}$ to $E_{3,1}^{(1)}$ so that we start and finish in the same subtree.

Thus, in light of the issues outlined in Example 3.1, in order to complete our definition for an edge-local action we must introduce the isomorphism

$$\varphi_{i,j} : E_{d,k}^{(i)} \to E_{d,k}^{(j)}$$

which sends edges labelled $i \in \{1, ..., d\}$ to edges labelled $j \in \{1, ..., d\}$ and vice versa, whilst preserving the labelling otherwise and preserving the orientation (of the central edges). Then, we can provide a complete definition for this edge-local action of α at e:

$$\varepsilon_k : \mathrm{Aut}(T_d) \times E(T_d) \to \mathrm{Aut}(E_{d,k}^{(1)})$$

$$(\alpha, e) \mapsto \varphi_{(l(\alpha e),1)} \circ l_{\alpha e}^k \circ \alpha \circ (l_e^k)^{-1} \circ \varphi_{(l(e),1)}^{-1}.$$

Which allows the definition of our desired universal group with prescribed local actions on edge neighbourhoods:

$$\mathrm{EU}_k(F) := \{\alpha \in \mathrm{Aut}(T_d) \mid \forall e \in E(T_d) : \varepsilon_k(\alpha, e) \in F\}$$

where $F \leq \mathrm{Aut}(E_{d,k}^{(1)})$.

Proposition 3.1 $\mathrm{EU}_k(F)$ is a subgroup of $\mathrm{Aut}(T_d)$.

Proof. First, let $\text{id} \in \text{Aut}(T_d)$ be the identity and $e \in E(T_d)$. Then $\varepsilon_k(\text{id}, e) = \varphi_{(l(e),1)} \circ l_e^k \circ \text{id} \circ (l_e^k)^{-1} \circ \varphi^{-1}_{(l(e),1)}$ is the identity element in F such that $\text{id} \in \text{EU}_k(F)$.

Suppose $g, h \in \text{EU}_k(F)$. Consider $\varepsilon_k(gh, e) = \varphi_{(l(gh(e)),1)} \circ l^k_{gh(e)} \circ gh \circ (l_e^k)^{-1} \circ \varphi^{-1}_{(l(e),1)} = \varphi_{(l(gh(e)),1)} \circ l^k_{gh(e)} \circ g \circ (l^k_{he})^{-1} \circ \varphi^{-1}_{(l(he),1)} \circ \varphi_{(l(he),1)} \circ l^k_{he} \circ h \circ (l^k_e)^{-1} \circ \varphi^{-1}_{(l(e),1)} = \varepsilon_k(g, he)\varepsilon_k(h, e)$. Well, we know that $\varepsilon_k(h, e), \varepsilon_k(g, he) \in F$ by definition of $\text{EU}_k(F)$ and since $F \leq \text{Aut}(E^{(1)}_{d,k})$ we have that $\varepsilon_k(g, he)\varepsilon_k(h, e) \in F$ such that $gh \in \text{EU}_k(F)$.

Now, suppose that $g \in \text{EU}_k(F)$ and let $g^{-1} \in \text{Aut}(T_d)$ be the inverse element of g. We wish to show that $g^{-1} \in \text{EU}_k(F)$ or equivalently, $\varepsilon_k(g^{-1}, e) \in F, \forall e \in E(T_d)$. To this end, consider $\varepsilon_k(\text{id}, e) = \varepsilon_k(gg^{-1}, e) = \varepsilon_k(g, g^{-1}e)\varepsilon_k(g^{-1}, e)$ where id is the identity element in $\text{Aut}(T_d)$ and $\varepsilon_k(\text{id}, e)$ is the identity element in F. Well, we see that $\varepsilon_k(g^{-1}, e)$ is the inverse element of $\varepsilon_k(g, g^{-1}e)$ in $\text{Aut}(E^{(1)}_{d,k})$. Furthermore, we have $\varepsilon_k(g, g^{-1}e) \in F$ so since $F \leq \text{Aut}(E^{(1)}_{d,k})$ we also have $\varepsilon_k(g^{-1}, e) \in F$ which completes the proof.

□

Following from our definition for $\text{EU}_k(F)$ we look at its structure for different choices of F particularly for the case where $d = 3$ and $k = 1$.

5 Final Results, Conclusions and Further Work

Letting $d = 3, k = 1$ we choose to view an arbitrary $E_{3,1}$ sub-tree with respect to the labelling of its four leaves as follows:

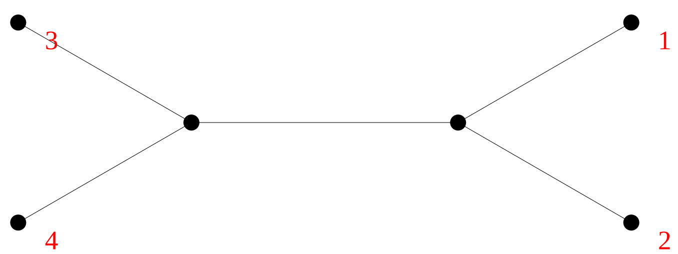

Figure 1: Arbitrary $E_{3,1}$ with labelled leaves.

Doing so allows us to look at the automorphisms $\text{Aut}(E_{3,1})$ as permutations on the set $\{1, 2, 3, 4\}$.

We found that

$$\mathrm{Aut}(E_{3,1}) \cong \langle (12), (34), (13)(24) \rangle$$

which generates a total of 8 elements:

- $g_1 = \mathrm{id}$
- $g_2 = (12)$
- $g_3 = (34)$
- $g_4 = (12)(34)$
- $g_5 = (13)(24)$
- $g_6 = (12)(34)(13)(24) = (14)(23)$
- $g_7 = (12)(13)(24) = (1423)$
- $g_8 = (34)(13)(24) = (1324)$

Of these elements, there are two g_7 and g_8, which are of particular interest. Looking back to the Weiss conjecture, a common feature among the conjugacy classes which we have successfully described so far is that they all contain involutive inversions. In addition, it is known that the remaining two do not contain involutive inversions, so when searching for descriptions of the remaining classes it makes sense to start by looking for groups of this type. Now we note that g_7 and g_8 are both non-involutive inversions and furthermore they both generate the same cyclic subgroup of order 4 $\langle g_7 \rangle = \langle g_8 \rangle = \{g_1, g_5, g_7, g_8\} = C_4 \leq \mathrm{Aut}(E_{3,1})$.

Upon setting $F = C_4$ and considering the corresponding edge-universal group, it is immediate by inspection that $\mathrm{EU}(C_4)$ contains no involutive inversions and we also claim that it is topologically discrete which we prove below.

Proposition 4.1 $\mathrm{EU}(C_4)$ is discrete.

Proof. We require that $\{\mathrm{id}\} \subseteq \mathrm{EU}(C_4)$ is open. Thus, we will show that $\{\mathrm{id}\} = \mathrm{stab}_{\mathrm{EU}(C_4)}(B(v, n))$ for some n. To this end, suppose that $g \in \mathrm{stab}_{\mathrm{EU}(C_4)}(B(v, n))$. Consider some vertex x on the boundary of $B(v, n)$ and the edge $e = (x, y)$ where y is the terminus vertex of e at distance $n - 1$ from v. If we consider the action of g on $B(e, 1)$, the only element in $\mathrm{EU}(C_4)$ which fixes $B(v, n)$ must also fix

$B(e, 1)$. Then, by induction on n we obtain $g = \mathrm{id}$. Thus $\{\mathrm{id}\} = \mathrm{stab}_{\mathrm{EU}(C_4)}(B(v, n))$ as was required.

□

So, via the definition we have proposed and a careful choice of F, we have been able to generate a subgroup of $\mathrm{Aut}(T_3)$ that contains no involutive inversions and which is topologically discrete.
While this result bodes well for our aims, it does not complete the picture, and so some interesting further work could be done on answering the following questions:

- Is $\mathrm{EU}(C_4)$ vertex transitive?
- Does $\mathrm{EU}(C_4)$ act locally like S_3 about vertices?

If the answer to these questions is yes, we will have outlined a promising candidate to describe one of the two of seven conjugacy classes that we seek, in terms of how they act locally.

We also propose that in order to look at describing the second of the two conjugacy classes that we seek, we would need to find more subgroup choices for F which have no involutive inversions. Since there exist no more of this type for the case where $k = 1$, one might consider setting $k = 2$ and investigating the edge-universal group with local actions over radius two rather than one to provide a complete picture for the case where $d = 3$.

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