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Zappa-Szép Product Construction on Compact Quantum Groups

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Abstract

In this paper, we look at compact quantum groups, following their definition and treatment by S.L. Woronowicz. We review the literature on the topic, starting with the quantum group framework formulated for locally compact groups, as discussed by Ann Maes and Alfons Van Daele. We then look at the Zappa-Szép product of groups and introduce a similar construction for compact quantum groups, as done by Dr. Nathan Brownlowe, Dr. David Robertson and Penelope Drastik. Finally, we look at finding the Zappa-Szép product of quantum permutation groups $A_s(n!-1)$ and $A_s(n)$, discuss our approach (albeit unfinished business) and present problems we encountered during this research project as well as a simpler proof for coassociativity of the Zappa-Szép comultiplication.

1 Introduction

Compact quantum groups can be seen as "quantum versions" of compact groups. In section 2, we give an introduction to compact quantum groups, following their definition and approach by S.L. Woronowicz. We follow closely the approach discussed by Ann Maes and Alfons Van Daele in their *Notes on Compact Quantum Groups* (1998), starting with some motivation for quantum groups, presenting a precise quantum group framework for locally compact groups, and moving into the precise definition of a compact quantum group.

How might we put two of these groups together to get a new one? In 2015-2016 during a previous AMSI Vacation Research Project, VRS scholar Penelope Drastik supervised by Dr. Nathan Brownlowe and Dr. David Robertson produced a theorem explaining precisely how to take the product of two compact quantum groups, and produce a new one. Their approach was inspired by the Zappa-Szép product of groups, which generalises the direct and semi-direct products of groups. In section 3, we present the Zappa-Szép product of groups and follow their approach closely, seeing how they applied a similar construction to compact quantum groups. Finally, we present the theorem. We omit the proof concerning linear density of $\Delta(A \otimes B)(1_{A\otimes B} \otimes (A \otimes B))$ as required by the comultiplication Δ of a compact quantum group (which may or may not rely on some extra assumptions and is a work in progress). We provide a simpler proof for coassociativity of Δ.

The aim of this research project was to apply the theorem produced by Penelope Drastik supervised by Dr. Nathan Brownlowe and Dr. David Robertson to instances of compact quantum groups. In particular, we wanted to see what the product of quantum permutation groups, $A_s(n!-1)$ and $A_s(n)$, which are the quantum analogues of $C(S_{n!-1})$ and $C(S_n)$ respectively, looks like. In what ways might it be dual to the group identity $S_{n!} \cong S_n \bowtie S_{n!-1}$? In section 4, we look at finding $A_s(n!-1) \bowtie A_s(n)$, discuss our approach (albeit unfinished business) and present problems we encountered during this research project.

1.1 Acknowledgements & Statement of Authorship

I would like to acknowledge and immensely thank my supervisor, Dr. Nathan Brownlowe, for his assistance and time in numerous meetings throughout the duration of this extended project. I am grateful for the experience and exposure to mathematical research I gained from the project; the ways in which I developed my thinking and research skills; as well as the valuable public speaking skills I was taught in order to deliver a carefully-thought-out presentation.

I would like to acknowledge the work[1] of Penelope Drastik, Dr. Nathan Brownlowe and Dr. David Robertson, which provided a rich and super interesting basis for my first research project.

Finally, I would like to acknowledge and express my gratitude to Dr. David Robertson and Dr. Nathan Brownlowe for helping me understand and re-write proofs (in particular, Dr. Robertson produced the simplified coassociativity proof[2] using string diagrams). They directed the progress made on the quantum permutation groups problem (section 4) and will continue to think about the problem together.

The content presented in this report is either review of existing literature (with perhaps modified or new[3] proofs), otherwise, the statement of the quantum permutation groups problem which was directed by Dr. Nathan Brownlowe and Dr. David Robertson.

2 Compact Quantum Groups

2.1 Quantum Groups

Since their development, quantum groups have evolved in different directions motivated from both physics and mathematics. Johan Kustermans and Lars Tuset[4] loosely describe quantum groups as *"essentially groups or group-like objects that are quantizations of groups"* - otherwise, *quantum versions of groups.*

As the field of quantum physics developed, it was found that non-commutative algebras could be used to describe atomic and molecular geometry in the *quantum* context. In a similar spirit to that in which J. von Neumann established a rigorous mathematical foundation for quantum mechanics via von Neumann algebras, effectively generalising the classical theory of Borel integration, quantum groups aim to generalise the study of 'classical' geometric spaces to the quantum setting. This is done by expanding the space of commutative algebras (which describe[5] classical spaces) to include the non-commutative.

The quantum group first appeared in quantum integrable systems, and was formalized by Vladimir Drinfeld and Michio Jimbo as a particular class of the Hopf algebra. There seems to be less work

[1] [4]

[2] section 3.3 Theorem - see appendix A for proof

[3] see A.2, thanks to Dr. Nathan Brownlowe

[4] [1] *A Survey of C^*-algebraic Quantum Groups*

[5] see section 1.2 on Gelfand's Theorem

focused on a broader treatment of quantum groups in the C^*-algebra framework. Gelfand's Theorem allows a natural generalisation of classical locally compact Hausdorff spaces by considering non-commutative C^*-algebras, which we put together with extra structure to impose a group-like structure on the underlying spaces. We formalise this in the next section.

2.2 Gelfand's Theorem

Gelfand's Theorem allows us to have a sensible definition of quantum groups, which we will see are quantum analogues of $C(G)$, where the underlying space G is locally compact. For this reason, we will present the following relevant results.

Lemma 1: Let X be a locally compact, Hausdorff space. Then, the continuous functions vanishing at infinity, $C_0(X) = \{f \in C(X) : \forall \epsilon > 0, \{x : |f(x)| \geq \epsilon\}$ is compact$\}$, together with the supremum-norm $\|f\|_\infty = sup\{|f(x)| : x \in X\}$, is a commutative C^*-algebra.

Definition 2: The **maximal ideal space**, Δ_A, of a commutative, complex Banach algebra, A, is the set of all characters (non-zero homomorphisms from $A \to \mathbb{C}$) of A. The maximal ideal space endowed with the weak*-topology is a locally compact, Hausdorff space.

Theorem 3 (Gelfand's Theorem): Let A be a commutative C^*-algebra. Then, A is isomorphic to $C_0(\Delta_A)$. $A \cong C_0(\Delta_A)$ is unital[6] if and only if Δ_A is compact. In this case, $A \cong C_0(\Delta_A) = C(\Delta_A)$.

2.3 Locally Compact Quantum Semi-groups

It is now clear how 'classical' spaces (locally compact Hausdorff spaces) can be studied using commutative C^*-algebras. We view these C^*-algebras as the continuous functions on our 'classical spaces'. We are particularly interested in the case where our underlying space (on which we look at continuous functions) is a group. Remembering that we aim to generalise our study to the quantum setting, we introduce the object of a *comultiplication* - an instrument through which we will capture the group structure of the underlying space.

Definition 4: Let A be a C^*-algebra. A non-degenerate *-homomorphism $\Phi : A \to M(A \otimes A)$ (where $M(\cdot)$ denotes the multiplier algebra of a space) is called coassociative if $(\Phi \otimes id)\Phi = (id \otimes \Phi)\Phi$. We then call Φ a **comultiplication** on A.
Throughout this paper, $A \otimes A$ will denote the spatial tensor product of C^*-algebra A with itself.
Since Φ is non-degenerate and Φ and id are continuous, upon considering the natural unique extensions of $\Phi \otimes id$ and $id \otimes \Phi$ to maps on $M(A \otimes A)$, we have that the maps $(\Phi \otimes id)\Phi$ and $(id \otimes \Phi)\Phi$ make sense as compositions of maps, and they go from $A \to M(A \otimes A \otimes A)$.

[6]The theorem reduced to this case is also known as the Gelfand-Naimark Theorem.

Suppose now that G is a locally compact, Hausdorff space endowed with a continuous, associative multiplication (so that G is a semi-group).
Let $A = C_0(G)$. $M(A \otimes A)$ can be identified[7] with $C_b(G \times G)$.

Let $\Phi : A \to M(A \otimes A)$ **be defined by** $\Phi(f)(p, q) = f(pq)$**.**

It follows that Φ is non-degenerate and coassociative[8]. We have then associated with G a pair (A, Φ) of a commutative C^*-algebra and a comultiplication on it.
Conversely, suppose we have a pair (A, Φ) where A is a commutative C^*-algebra and Φ a comultiplication on A. Gelfand's theorem says that $A \cong C_0(G)$ where $G = \Delta_A$. Define a multiplication on G by $\Phi(f)(p, q) = f(pq)$ for all $f \in C_0(G)$[9]. It follows from the coassociativity of Φ (and this is not obvious and requires some effort to prove) that this multiplication is associative. We now have that G is a locally compact semi-group.
There is therefore a one-to-one correspondence between pairs (A, Φ) with A a commutative C^*-algebra and Φ a comultiplication on A, and locally compact semi-groups G.

Finally, our goal is to have the underlying space G of (A, Φ) be a group.
In the case where A is finite, the extra structure on (A, Φ) that turns G necessarily into a locally compact *group* translates to (A, Φ) being a Hopf *-algebra. (Conversely, if G is a finite, locally compact group, then (A, Φ) is a Hopf *-algebra.) To impose a group structure on the underlying space G, then, the following definition makes sense:

Definition 5: A **finite quantum group** is a pair (A, Φ) where A is a finite-dimensional C^*-algebra and Φ is a comultiplication on A such that (A, Φ) is a Hopf *-algebra.
Note that we do not require A to be commutative - we are, in this way, generalising our study of functions on 'classical spaces' (finite compact Hausdorff groups, G) using commutative algebras, to what we can informally consider to be functions on 'quantum versions of G' (though we cannot say what this means).

2.4 Compact Quantum Groups

We finally look at the case where G is compact, to yield the definition of a compact quantum group.
Suppose G is a compact Hausdorff semi-group. Then, $A = C_0(G) = C(G)$ is unital, so our multiplier algebras become just our algebras. (In fact, recall that conversely, if A is unital, then Δ_A is compact).

[7] See Appendix B, B17 of [2]: Iain Raeburn and Dana P. Williams, *Morita Equivalence and Continuous-Trace C^*-algebras* (1998) for identification of $C_0(S) \otimes C_0(T)$ with $C_0(S \times T)$ where S, T are locally compact Hausdorff spaces.
[8] See Appendix A, A.1
[9] This is well defined because Φ is non-degenerate.

We can identify $M(A \otimes A) = A \otimes A$ with $C(G \times G)$ and therefore $M(A \otimes A \otimes A) = A \otimes A \otimes A$ with $C(G \times G \times G)$. We still have that (A, Φ), with Φ defined the same way as it was above, is a pair of a commutative, unital C^*-algebra and a coassociative comultiplication (see Appendix A for a visual representation of how we translate associativity in G into 'quantum language': coassociativity).

We need G to have an identity and inverses in order for G to be a group. What structure do we now require in order to impose a group structure on our compact semi-group, G?

Lemma 6[10]: A compact semi-group in which the cancellation law holds is a group.
We use this result to find precisely what extra structure to put on (A, Φ).

Proposition 7[11]: Let G be a compact semi-group, let $A = C(G)$ and $\Phi : A \to A \otimes A$ be defined by $\Phi(f)(p, q) = f(pq)$. Then, the cancellation law holds in G if and only if the sets $\Phi(A)(1 \otimes A) = \{\Phi(c)(1 \otimes b) : b, c \in A\}$ and $\Phi(A)(A \otimes 1) = \{\Phi(c)(b \otimes 1) : b, c \in A\}$ are linearly dense subsets of $A \otimes A$.

Combining the above result with Lemma 6, we see that the extra structure on (A, Φ) which imposes a group-structure on the underlying space, G, is precisely that the sets $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are linearly dense in $A \otimes A$.

Now that we have abstracted the properties of G being a group in the case that G is compact to properties of the pair (A, Φ), the following definition makes sense:

Definition 8: A **compact quantum group** is a pair (A, Φ) where A is a unital C^*-algebra, Φ is a coassociative comultiplication and the sets $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are linearly dense in $A \otimes A$.

3 The Zappa-Szép Product

3.1 Zappa-Szép Product of Groups

The Zappa-Szép product of groups generalises the direct and semi-direct products. We first look at the internal product.

Suppose that $G, H \leq K$ are groups with $K = GH$ and $G \cap H = \{e\}$, the identity element. Then, $|K| = |G \times H|$. Furthermore, for every $k \in K$, there are **unique** $g \in G$ and $h \in H$ such that $k = gh$. In particular, $\forall\ h \in H$ and $g \in G$, $\exists\ h \cdot g \in G$ (which we call the *action* map on $H \times G$) and $h|_g \in H$

[10]See [3]: Ann Maes and Alfons Van Daele, *Notes on Compact Quantum Groups* (Maart, 1998), for proof.
[11]See Appendix A, , for proof.

(which we call the *restriction* map on $H \times G$), so that $hg = (h \cdot g)h|_g \in GH$.

Our action and restriction maps give rise to the group multiplication $m_{G\times H} : (G \times H) \times (G \times H) \to G \times H$ given by $(g, h)(g', h') \mapsto (g(h \cdot g'), h|_{g'}h')$. We then have that $(G \times H, m_{G\times H}) =: G \bowtie H \cong K$.

Our action and restriction maps turn out to have the following properties:

1 $\forall h \in H$, $g \mapsto h \cdot g$ is a bijection

2 $\forall g \in G$, $h \mapsto h|_g$ is a bijection

3 $\forall g \in G$, $e \cdot g = g$

4 $\forall h \in H$, $h|_e = h$

5 $\forall h_1, h_2 \in H$ and $g \in G$, $(h_1h_2) \cdot g = h_1 \cdot (h_2 \cdot g)$

6 $\forall h_1, h_2 \in H$ and $g \in G$, $(h_1h_2)|_g = (h_1|_{h_2\cdot g})(h_2|_g)$

7 $\forall h \in H$ and $g_1, g_2 \in G$, $h \cdot (g_1g_2) = (h \cdot g_1)(h|_{g_1} \cdot g_2)$

8 $\forall h \in H$ and $g_1, g_2 \in G$, $h|_{g_1g_2} = (h|_{g1})|_{g_2}$.

Furthermore, we have the following properties (which are not independent of the above):

- $\forall h \in H$, $h \cdot e = e$
- $\forall g \in G$, $e|_g = e$.

Taking two arbitrary groups, G and H, and finding action and restriction maps on $H \times G$ with properties 1-8, we can define a group multiplication $m_{G\times H}$ given by the same formula. We then have that $(G \times H, m_{G\times H}) =: G \bowtie H$ is a group (the external Zappa-Szép product). In fact, $G \bowtie H$ is an internal Zappa-Szép product of $G \times \{e\} \cong G$ and $\{e\} \times H \cong H$.

It is also a fact that the Zappa-Szép product is symmetrical, that is, $G \bowtie H \cong H \bowtie G$. If (considering all products as internal, using the identification $G \times \{e\} \cong G$ and $\{e\} \times H \cong H$ for external products of G and H mentioned above) $h'g' = gh$, we can express the action and restriction maps, $*$ and $\|$, from $H \bowtie G$ in terms of the action and restriction maps, $\cdot$ and $|$, from $G \bowtie H$. The expressions are

$$g * h = h|_{h^{-1}\cdot g^{-1}}$$

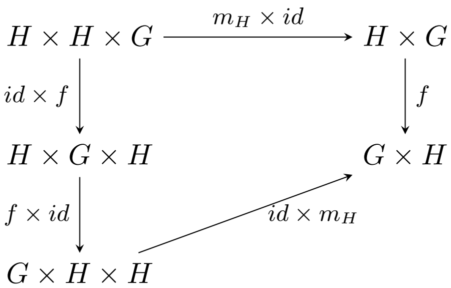
$$g\|_h = h^{-1}|_{g^{-1}} \cdot g.$$

3.2 Zappa-Szép Product Construction for Compact Quantum Groups

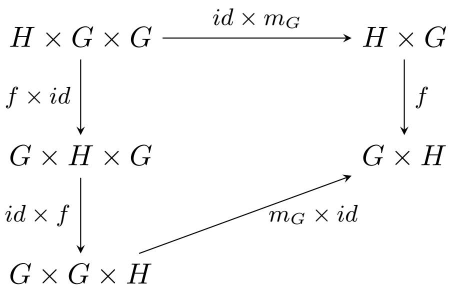
We aim to achieve the analogue of a Zappa-Szép product for compact quantum groups. To this aim, we first express the properties 1-8 in the language of commutative diagrams.

First, let $f : H \times G \to G \times H$ record the action and restriction maps, that is, $(h, g) \mapsto (h \cdot g, h|_g)$. This is a bijection by properties 1-2.

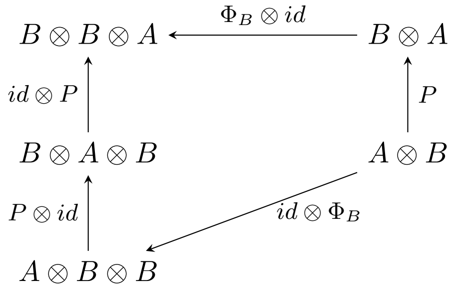
We can express properties 5-6 as commutativity of the following diagram

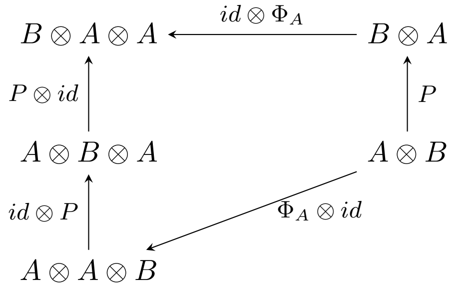


and properties 7-8 as commutativity of



Now, suppose that G, H are compact and our action and restriction are continuous. We translate the above two diagrams, respectively, into commutative diagrams about $C(G)$ and $C(H)$ in a similar way to the translation of associativity in G to coassociativity in $(C(G), \Phi)$ (as seen in appendix A). We take this a step further and suppose we have this commutativity for arbitrary compact quantum groups, (A, Φ_A) and (B, Φ_B). Further, suppose that there exists an isomorphism, $P : A \otimes B \to B \otimes A$ which shall act as the "quantum version" of f. This yields the commutativity of the following two diagrams





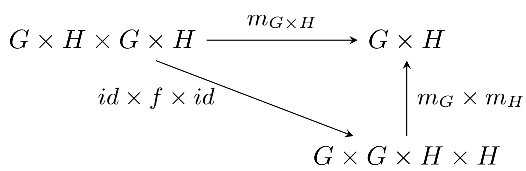
Motivated by these properties of f, suppose that

$$(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$$

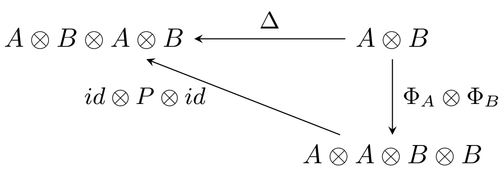
$$(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi_A \otimes id_B).$$

The question of taking two compact quantum groups A and B and taking their product, $A \otimes B$ begs the question: *what comultiplication, Δ, together with $A \otimes B$ yields a compact quantum group?*

Similarly to $m_{G\times H}$ in the group setting, we expect Δ to rely on the comultiplications of A and B as well as P, which (informally) plays the role of recording the "action" and "restriction" information. In the group setting, we had



Translating this once again into "quantum language", suppose that



That is, $\Delta = (id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$.

The question is, has all that we supposed about P and Δ been enough to create a compact quantum group? The following tells us: yes, with some extra information.

3.3 Theorem

Let (A, Φ_A), (B, Φ_B) be Compact Quantum Groups. Suppose $P : A \otimes B \to B \otimes A$ is an isomorphism satisfying:

$$(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$$

$$(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi_A \otimes id_B).$$

Let $\Delta : A \otimes B \to (A \otimes B) \otimes (A \otimes B)$ be given by $\Delta = (id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$.

Then, $(A \otimes B, \Delta)$ is a **Compact Quantum Group.**

See appendix A for proof.

4 Zappa-Szép Product of Quantum Permutation Groups

4.1 Quantum Permutation Groups

Quantum permutation groups are the "quantum versions" of continuous functions on symmetric groups. They are motivated by the following realisation:

$$C(S_n) = C^*_{comm.}\{u_{ij} | u \text{ is an } n \times n \text{ magic unitary matrix}\}_{ij}$$

where a magic unitary matrix is one in which all its entries are projections ($u^2 = u$ and $u^* = u$), entries are mutually orthogonal with all other entries in the same column or row, and entries sum to 1 in each column or row. $C^*_{comm.}$ denotes the C^*-algebra generated by elements of the indicated set, which we require additionally to be commutative.

It is a fact that the following universal algebra is a finitely generated Hopf algebra:

$$A_s(n) := C^*\{u_{ij} | u \text{ is an } n \times n \text{ magic unitary matrix}\}_{ij}.$$

Together with comultiplication given by $\Delta_n(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, this is a type of compact quantum group. Initially, the motivation was to see whether the identity $S_{n!} \cong S_n \bowtie S_{n!-1}$ translated to the quantum version - namely, whether $(A_s(n!), \Delta_{n!}) \cong (A_s(n) \otimes A_s(n!-1), \Delta_{n \bowtie (n!-1)})$, where $\Delta_{n \bowtie (n!-1)}$ is the comultiplication induced by the Zappa-Szép construction from section 3. Of course, this cannot be true due to discrepancies in commutativity of each side, but it is still interesting to see how the result for compact quantum groups would be dual to that for symmetric groups.

4.2 Product of Quantum Permutation Groups

Let us look at the product, $A_s(n!-1) \bowtie A_s(n)$. To motivate what our comultiplication might be (what P might be), we first examine the "classical" commmutative case. We write out the magic unitary entries generating $C(S_{n!-1})$ and $C(S_n)$. Suppose R is the matrix corresponding to $C(S_{n!-1})$ indexed by elements of $S_n \backslash \{e\}$, and Q the matrix corresponding to $C(S_n)$ indexed by elements in $\{1, 2, ..., n\}$:

$$R_{\pi',\pi} = 1_{\{\rho \in bijections(S_n):\rho(\pi)=\pi',\rho(e)=e\}}, \text{ and}$$
$$Q_{i,j} = 1_{\{\sigma \in S_n:\sigma(j)=i\}}.$$

Now, suppose $\zeta \in C(G)$ and $\eta \in C(H)$ for arbitrary groups G, H. We use that in general, the following logic was used to derive properties of P (i.e. go from a commutative diagram about f to one about P): $P(\zeta \otimes \eta)(h, g) = (\zeta \otimes \eta) \circ f(h, g) = \zeta(h \cdot g)\eta(h|_g)$, again using the identification of $C(S_{n!-1}) \otimes C(S_n)$ with $C(S_{n!-1} \times S_n)$. Then,

$$\begin{aligned} P(1_{\{g\}} \otimes 1_{\{h\}})(h', g') &= 1_{\{g=h' \cdot g'\}} 1_{\{h=h'|g'\}} \\ &= 1 \iff (h' \cdot g')h'|_{g'} = gh. \end{aligned}$$

Now, by definition,

$$h'g' = (h' \cdot g')h'|_{g'} = gh.$$

Using that the Zappa-Szép product is symmetric in G and H, this is equivalent to

$$h' = g * h$$

$$g' = g\|_h.$$

We have then that

$$P(1_{\{g\}} \otimes 1_{\{h\}}) = 1_{\{g*h\}} \otimes 1_{\{g\|_h\}}.$$

Finally, where it is understood that $\tau \in bijections(S_n)$ and $\sigma \in S_n$,

$$\begin{aligned} P(R_{\pi',\pi}, Q_{i,j}) &= \sum_{\tau,\sigma|\tau(e)=e,\tau(\pi)=\pi',\sigma(j)=i} P(1_{\{\tau\}} \otimes 1_{\{\sigma\}}) \\ &= \sum_{\tau,\sigma|\tau(e)=e,\tau(\pi)=\pi',\sigma(j)=i} 1_{\{\tau*\sigma\}} \otimes 1_{\{\tau\|_\sigma\}} \\ &= \sum_{\tau,\sigma|\tau(e)=e,\tau(\pi)=\pi',\sigma(j)=i} 1_{\{\tau(\sigma)\}} \otimes 1_{\{[\tau(\sigma)]^{-1}\circ\tau(\sigma\circ\cdot)\}} \end{aligned}$$

where $[\tau(\sigma)]^{-1} \circ \tau(\sigma \circ \cdot) : S_n \backslash \{e\} \to S_n \backslash \{e\}$ is defined by $\sigma_0 \mapsto [\tau(\sigma)]^{-1} \circ \tau(\sigma \circ \sigma_0)$.

The difficulty in this problem was that in order to answer the questions of what expression our comultiplication $\Delta = (id_{A_s(n!-1)} \otimes P \otimes id_{A_s(n)})(\Delta_{A_s(n!-1)} \otimes \Delta_{A_s(n)})$ should take, as well as whether P satisfied the necessary assumptions, we require P to ideally be expressed in terms of entries of R and Q; otherwise, in terms of the elements of our quantum permutation groups. It has not been easy to abstract maps specific to actual permutations S_n and $S_{n!-1}$ = bijections of $S_n \to S_n$ fixing e to arbitrary elements of quantum permutation groups. This is a work in progress and I aim to continue trying.

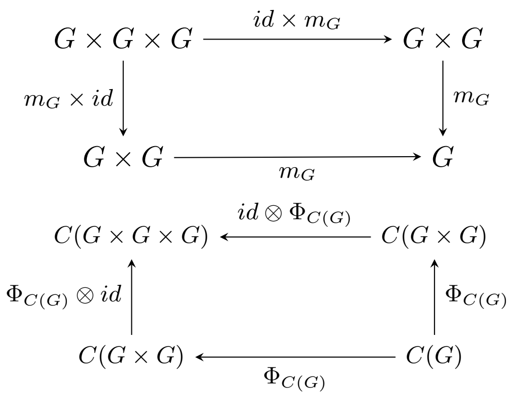
5 Next Steps

As discussed in above sections, our next steps are to work on the 'density' part of the proof of section 3.3's Theorem, as well as continue working on the problem presented in section 4.

A Appendix A

A.1 Commutative diagrams expressing associativity and coassociativity

We can say that the following diagrams are 'dual'.



A.2 Associativity gives rise to coassociativity

Let G be a compact, Hausdorff semi-group.

Claim: $(\Phi_{C(G)} \otimes id)\Phi_{C(G)} = (id \otimes \Phi_{C(G)})\Phi_{C(G)}$.

Proof:
We have $(\Phi_{C(G)} \otimes id)\Phi_{C(G)} : C(G) \to C(G) \otimes C(G) \otimes C(G)$.
Identify $C(G) \otimes C(G) \otimes C(G)$ with $C(G \times G \times G)$, and let $f \in C(G)$.

$$[(\Phi_{C(G)} \otimes id)\Phi_{C(G)}](f)(g,h,k) = (\Phi_{C(G)} \otimes id)[\Phi_{C(G)}(f)](g,h,k)$$

Write $\Phi_{C(G)}(f) = lim_{k\to\infty}\sum_{i=1}^{k}(\xi_i \otimes \eta_i)$[12].

$$= (\Phi_{C(G)} \otimes id)[lim_{k\to\infty}\sum_{i=1}^{k}(\xi_i \otimes \eta_i)](g,h,k)$$

$$= lim_{k\to\infty}\sum_{i=1}^{k}[(\Phi_{C(G)} \otimes id)(\xi_i \otimes \eta_i)(g,h,k)]^{13}$$

$$= lim_{k\to\infty}\sum_{i=1}^{k}[(\Phi_{C(G)}(\xi_i) \otimes \eta_i)(g,h,k)]$$

$$= lim_{k\to\infty}\sum_{i=1}^{k}[(\Phi_{C(G)}(\xi_i) \otimes \eta_i)((g,h),k)]$$

By definition of $\Phi_{C(G)}$,

$$= lim_{k\to\infty}\sum_{i=1}^{k}[\xi_i \otimes \eta_i(m(g,h),k)]$$

$$= [lim_{k\to\infty}\sum_{i=1}^{k}(\xi_i \otimes \eta_i)](m(g,h),k)$$

$$= \Phi_{C(G)}(f)(m(g,h),k)$$

$$= f(m(m(g,h),k))$$

Using associativity in G,

$$= f(m(g,m(h,k)))$$

$$= \Phi_{C(G)}(f)(g,m(h,k))$$

$$= lim_{k\to\infty}\sum_{i=1}^{k}[\xi_i \otimes \eta_i(g,m(h,k))]$$

$$= lim_{k\to\infty}\sum_{i=1}^{k}[\xi_i \otimes \Phi_{C(G)}(\eta_i)(g,h,k)]$$

$$= lim_{k\to\infty}\sum_{i=1}^{k}[(id \otimes \Phi_{C(G)})(\xi_i \otimes \eta_i)(g,h,k)]$$

$$= (id \otimes \Phi_{C(G)})[lim_{k\to\infty}\sum_{i=1}^{k}(\xi_i \otimes \eta_i)](g,h,k)$$

$$= (id \otimes \Phi_{C(G)})[\Phi_{C(G)}(f)](g,h,k)$$

$$= [(id \otimes \Phi_{C(G)})\Phi_{C(G)}](f)(g,h,k)$$

[12] The elementary tensors are linearly dense in the spatial tensor product, C(G)$\otimes C(G)$

[13] using that homomorphisms between C^*-algebras are continuous

Hence, $(\Phi_{C(G)} \otimes id)\Phi_{C(G)} = (id \otimes \Phi_{C(G)})\Phi_{C(G)}$ as claimed.

The same proof gives the result for the case in which G is a locally compact Hausdorff semi-group and we are looking at $\Phi_{C_0(G)}$.

A.3 Proof of Proposition 7

Proposition 7: Let G be a compact semi-group, let $A = C(G)$ and $\Phi : A \to A \otimes A$ be defined by $\Phi(f)(p,q) = f(pq)$. Then, the cancellation law holds in G if and only if the sets $\Phi(A)(1 \otimes A) = \{\Phi(c)(1 \otimes b) : b, c \in A\}$ and $\Phi(A)(A \otimes 1) = \{\Phi(c)(b \otimes 1) : b, c \in A\}$ are linearly dense subsets of $A \otimes A$.

Proof:
Assume $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are linearly dense in $A \otimes A$. We prove this direction of the proposition assuming only that G is locally compact (compactness is a special case).
Let $\Psi : A \otimes A \to C_0(G \times G)$ be the isomorphism[14] given by $\Psi(f \otimes g)(p,q) = f(p)g(q)$.
Extend the isomorphism in the natural way to an isomorphism $\Psi : M(A \otimes A) \to C_b(G \times G)$, so that $\Phi : C_0(G) \to C_b(G \times G)$.
Let $p, q, r \in G$ and suppose $pq = qr$. $\forall f, g \in A$,

$$\Psi[\Phi(f)(1 \otimes g)](p,r) = f(pr)g(r)$$

$$\Psi[\Phi(f)(1 \otimes g)](q,r) = f(qr)g(r).$$

By supposition, these numbers are equal. By our density assumption and using that the isomorphism Ψ between C^*-algebras $A \otimes A$ and $C_0(G \times G)$ is isometric,
$\forall h \in C_0(G \times G) = \Psi(A \otimes A)$,

$$h(p,r) = h(q,r).$$

Using that $C_0(G \times G)$ has the separation property (whenever $p \neq q$, $\exists f \in C_0(G \times G)$ such that $f(p) \neq f(q)$), we have that therefore, $p = q$ and hence, the right cancellation law holds in G.
Similarly, since $\Phi(A)(A \otimes 1)$ is linearly dense in $A \otimes A$, the left cancellation law holds in G. Together, we have that the cancellation law holds in G.

Conversely, suppose the cancellation law holds in G, so that using Lemma 6, G is a group. It is clear then that the maps

$$(p,q) \mapsto (pq, q)$$

$$(p,q) \mapsto (p, pq)$$

[14] See Appendix B, B17 of [2]: Iain Raeburn and Dana P. Williams, *Morita Equivalence and Continuous-Trace C^*-algebras* (1998) for proof.

are homeomorphisms $G \times G \to G \times G$, which we denote by h. By the Gelfand–Kolmogorov Theorem, these maps give rise to isomorphisms Γ of $A \otimes A \cong C(G \times G)$ given by $f \mapsto f \circ h \; \forall f \in C(G \times G)$. That is,

$$\Gamma(f \otimes g)(p, q) = f \otimes g(pq, q) = \Phi(f)(1 \otimes g)$$

$$\Gamma(f \otimes g)(p, q) = f \otimes g(p, pq) = \Phi(g)(f \otimes 1).$$

Hence, $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are linearly dense in $A \otimes A$.

A.4 Section 3.3 Theorem (Proof)

Let (A, Φ_A), (B, Φ_B) be Compact Quantum Groups. Suppose $P : A \otimes B \to B \otimes A$ is an isomorphism satisfying:

$$(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$$

$$(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi_A \otimes id_B).$$

Let $\Delta : A \otimes B \to (A \otimes B) \otimes (A \otimes B)$ be given by $\Delta = (id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$.

Then, $(A \otimes B, \Delta)$ is a **Compact Quantum Group.**

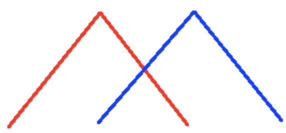
Proof:

Δ is coassociative.

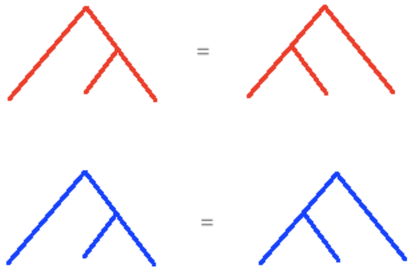
We use string diagrams to demonstrate that Δ is coassociative.

Identify Φ_A and Φ_B with the following two diagrams, respectively 

and P with  .

Then, Δ is  .

Coassociativity of Φ_A and Φ_B are expressed respectively as

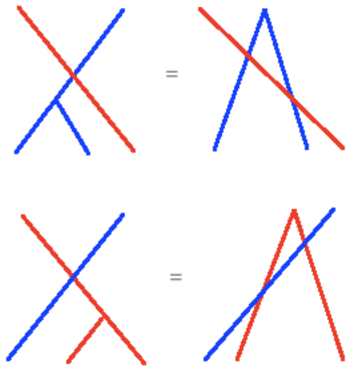


and the following properties of P

$$(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$$

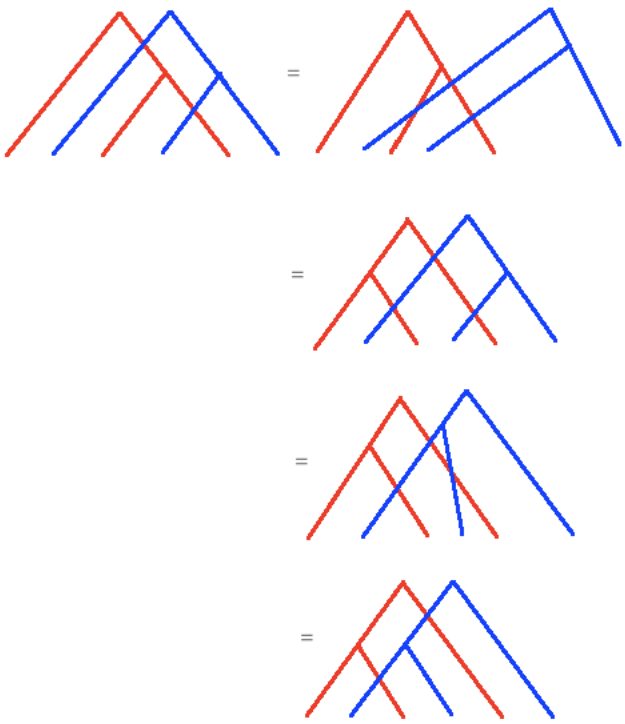
$$(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi_A \otimes id_B).$$

are expressed as



.

Finally, we see that $(id \otimes \Delta)\Delta =$



$= (\Delta \otimes id)\Delta.$

Hence, Δ is coassociative.

$\Delta(A \otimes B)(1_{A\otimes B} \otimes (A \otimes B))$ and $\Delta(A \otimes B)((A \otimes B) \otimes 1_{A\otimes B})$ are linearly dense in $((A \otimes B) \otimes (A \otimes B))$.

The proof of the fact that $\Delta(A \otimes B)(1_{A\otimes B} \otimes (A \otimes B))$ and $\Delta(A \otimes B)((A \otimes B) \otimes 1_{A\otimes B})$ are linearly dense in $((A \otimes B) \otimes (A \otimes B))$ is still in the works. It is possible we need some extra assumption about P relating to density, or otherwise encoding more information available in the Zappa-Szép group case, but hopefully not.

In Penelope Drastik's project[15], there is a working proof relying on an extra assumption about P: namely, that $(1_A \otimes B)P^{-1}(1_B \otimes A)$ is dense in $A \otimes B$. We feel that this assumption seems independent of our existing assumptions about P, and it may be too strong to require it as an extra assumption. We are currently trying to prove this without any extra assumptions, otherwise, possible weaker assumptions that we must add to the formulation of the theorem.

B References

1 Johan Kustermans and Lars Tuset, *A Survey of C^*-algebraic Quantum Groups, Part I* (2014)

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[15] [4]

3 Ann Maes and Alfons Van Daele, *Notes on Compact Quantum Groups* (Maart, 1998).

4 Penelope Drastik supervised by Dr. N. Brownlowe *Zappa Szép Products of Compact Quantum Groups* (2016).

5 Nathan Brownlowe, David Pask, Jacqui Ramagge, David Robertson and Michael F. Whittaker, *Zappa Szép Product Groupoids and C^* C-blends* (2016) Springer.

6 Teodor Banica, Julien Bichon and Benoˆit Collins, *Quantum Permutation Groups: A survey* (2007).