Lie Symmetries and Compactly Supported Solutions of Reaction-Diffusion Equations

Dylan Green

Supervised by Dr. Maureen Edwards

University of Wollongong

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1 Abstract

Theory of Lie symmetries is developed and applied to partial differential equations (PDEs). This theory is expanded upon to develop the "non-classical" symmetry method for PDEs. Both classical and non-classical methods are applied to one-dimensional reaction-diffusion equations, and further similar results are sourced from literature. In order to explain physical observations, compactly supported solutions to reaction-diffusion equations are sought from known solutions. These attempts are successful to varying degrees.

2 Introduction

Reaction-diffusion equations are used to describe physical situations in which a diffusive substance (eg. concentration of a compound or virus, population density of particular organisms) diffuses through a medium which has some active response to the presence of the diffusive substance. The one-dimensional form of these equations are partial differential equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x}\right) + Q(u),$$

where u is the diffusive substance (which will henceforth be called concentration), x is the one spatial dimension considered and t is time. The functions $K(u)$ and $Q(u)$ are the diffusion and reaction terms respectively, and describe the rate of diffusion and the way that the medium affects the concentration.

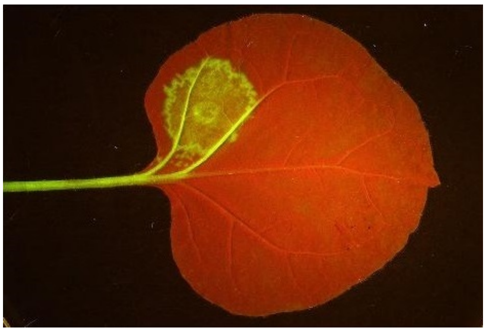


Figure 1: Diffusion of a virus through a potato leaf. (Edwards et al. 2016)

Such equations are well studied objects, however most efforts to study these equations have resulted in solutions of unbounded support. In many physical and biological contexts, it is clear that a diffusive quantity is bounded spatially at any given time. To see this, we need only consider the time evolution of a diffusive process acting upon an initial condition in which the diffusive quantity has bounded support: due to the finite speed at which objects can travel, in any finite time there must be a spatial bound on the quantity which was initially bounded.

This report begins by examining the theory of one-parameter Lie group analysis, in particular its application to partial differential equations. This includes "classical" symmetry methods and "non-classical" methods (the latter is a form of conditional symmetry). One dimensional reaction-diffusion equations are then studied with the aim of determining what forms the terms $K(u)$ and $Q(u)$ may take to ensure that u has bounded support, as motivated by observations such as depicted in figure 1. Finally, a brief discussion of the limitations of the reaction-diffusion equation at explaining the motivating phenomena is included.

2.1 Statement of Authorship

My original work completed in this project consists of

- development and implementation of Maple code to generate determining equations from a given partial differential equation,
- producing all plots and diagrams presented in this report, excluding figure 1,
- discussing the requirements and consequences of compactness of solutions developed by Arrigo and Hill (equations (7)) (Arrigo & Hill 1995).

While original, this work was regularly discussed and stimulated by discussions with my supervisor, Maureen Edwards. All other material in this report has been sourced from literature and is referenced accordingly.

3 Lie Groups and Group Transformations

This paper is intended to take the reader on an intuitive journey through the theory that I learnt in order to complete this project. To this end, proofs are in general not included and some of the details required to be completely rigorous have been omitted.

3.1 Fundamental Lie Group Theory

Definition 1. Let G denote a group with group operation $m : G \times G \to G$. G is an r-parameter *Lie group* if it has the structure of an r-dimensional manifold and the functions m and $i : G \to G$ defined by

$$i(g) = g^{-1}, \qquad g \in G$$

are smooth maps between manifolds.

We may also speak of local Lie groups, in which the group axioms and requirements of the above definition only hold in some connected, open neighbourhood of the identity element of G (the words connected, open and neighbourhood draw their meaning from the manifold structure of G, not the group structure).

Lie groups are of greatest interest when they are considered to "act" on a manifold. For the following section, let M denote an n dimensional smooth manifold and G denote a Lie group with group action and inverse functions m and i respectively. In addition, the notation $g \cdot h$ will be used to mean $m(g,h)$, for $g, h \in G$.

Definition 2. A *group of transformations* acting on M (or a *group action* on M) is a Lie group G together with a differentiable function $\star : G \times M \to M$ (which we write for brevity as $\star(g,x) = g \star x$ for $g \in G$ and $x \in M$) which satisfies the following conditions:

(a) $g \star (h \star x) = (g \cdot h) \star x, \quad \forall g, h \in G, \quad \forall x \in M$

(b) $e \star x = x, \quad \forall x \in M$, where $e \in G$ is the group identity.

The following definition is a less demanding version of that just stated, in which the specified properties only have to hold locally in G:

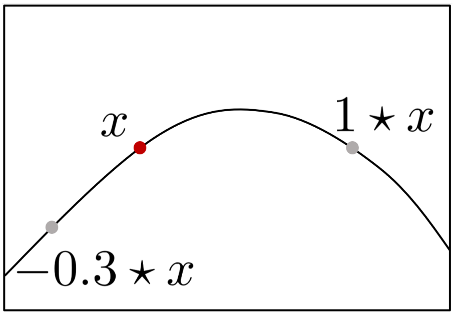
Definition 3. A *local group of transformations* acting on M is a Lie group G together with a differentiable function $\star : U \to M$, where $\{e\} \times M \subset U \subset G \times M$ and $\star$ satisfies the same two algebraic conditions stated in the previous definition.

We may think of the object $g \star x$ as the manifold element x transformed by the group element g. We may phrase this as the object g being applied to x to get $g \star x$.

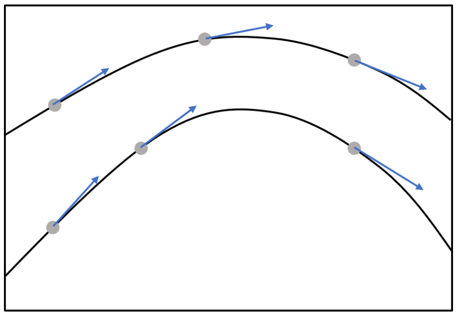
Definition 4. In the case when $G = (\mathbb{R}, +)$ acts as a group of transformations on a general manifold M, we define the *infinitesimal generator* of $(G, \star)$ on M as the differential operator $v : C^1(M;\mathbb{R}) \to C^0(M;\mathbb{R})$ defined by

$$[v(f)](x) = \lim_{\epsilon \to 0} \frac{f(\epsilon \star x) - f(x)}{\epsilon}$$

for $x \in M$. Note that this is an element of the tangent space to M at x.



(a) a possible action of a group on the manifold $\mathbb{R}^2$



(b) infinitesimal generator: a directional derivative

Figure 2: The relationship between the orbit and infinitesimal generator of a group of transformations acting on $\mathbb{R}^2$

See section 5.1 (appendices) for a visualisation of a group of transformations and a simple example of an infinitesimal generator.

Definition 5. A group action G on M is said to be *free* if for any $x \in M$ and $g \in G$, then $g \star x = x$ implies that g is the identity element of G. In other words, the identity is the only element of G that fixes any element of M under the group action.

Definition 6. The *orbit* of a group of transformations G is a subset $O^G \subset M$ which satisfies:

(a) for $x \in O^G$, $g \in G$ such that $g \star x$ is defined, we have $g \star x \in O^G$

(b) if $\tilde{O}^G \subset O^G$ and $\tilde{O}^G$ satisfies the previous requirement, then either $\tilde{O}^G = \{\}$ or $\tilde{O}^G = O^G$.

The notion of an infinitesimal generator may be extended for more general groups of transformations. One may consider an infinitesimal generator of an action of G at a point $x \in M$ to be any element of TO^G_x, the tangent space of the orbit at x. In the case of G being $\mathbb{R}$, TO^G_x is one dimensional. This is why we are able to talk about **the** infinitesimal generator of the group action - because all operators in the tangent space of the orbital are the same up to a factor.

A group of transformations is completely specified by its infinitesimal generator (just as a flow on a manifold can be completely specified by its vector field) (Olver 1994).

3.2 Symmetry Theory

A symmetry is a general name given to some function or quality that is preserved under a transformation of coordinates or variables. In this section, we aim to preserve the form of functions when coordinates are transformed according to a group of transformations.

We will refer to a system of equations of the form

$$F(x) = (F_1(x), ..., F_l(x)) = 0, \quad x \in M, \quad F : M \to \mathbb{R}^l$$

as an algebraic equation or a system of algebraic equations so long as each F_i is not a differential equation. We will soon look for transformation groups G such that solution sets $\{x \in M : F(x) = 0\}$ are *invariant*. What this means exactly is about to be defined.

Definition 7. A subset $I \subset M$ is said to be *invariant* under the action of G (or simply G*-invariant*) if for every $x \in I$ and every $g \in G$ such that $g \star x$ is defined, we have that $g \star x \in I$. In this case, we say that G is a symmetry group of I.

Definition 8. A function $f : M \to N$, where M, N are manifolds is said to be *invariant* under the action of G (or simply G*-invariant*) if for every $x \in M$ and $g \in G$ such that $g \star x$ is defined,

$$f(g \star x) = f(x).$$

Note that the above two definitions hold for both local and global (non-local) transformation groups.

Example 1. Let $f : M \to N$ be a G-invariant function and pick $x_0 \in M$. Define the subset $I_f(x_0) \subset M$ by $I_f(x_0) = \{x \in M : f(x) = f(x_0)\}$. Since f is an invariant function, for any $g \in G$ and $x \in I_f(x_0)$ such that $g \star x$ is defined, we have $f(g \star x) = f(x) = f(x_0)$, so $g \star x \in I_f(x_0)$. Hence $I_f(x_0)$ is a G-invariant subset.

As a slightly different example, do not assume anything about f other than smoothness, and suppose that $I_f(x_0) \subset M$ defined by $I_f(x_0) = \{x \in M : f(x) = f(x_0)\}$ is a G-invariant subset. This implies that $f(g \star x) = f(x)$ for $x \in I_f(x_0)$, so we may conclude that f restricted to the domain of $I_f(x_0)$ is G-invariant, but not necessarily that f is a G-invariant function on M.

Theorem 1. *Let G be a connected group of transformations acting on M, and $f : M \to \mathbb{R}$ to be a differentiable function. f is G-invariant if and only if $\boldsymbol{v}(f)$ is the zero function on M for every infinitesimal generator $\boldsymbol{v}$ of G (Olver 1994).*

For the following results, we need some notion of functional dependence:

Definition 9. Let $f_1, ..., f_k : M \to \mathbb{R}$ be smooth functions. $f_1, ..., f_k$ are said to be *functionally independent* if the only functions $\Psi : \mathbb{R}^k \to \mathbb{R}$ such that

$$\Psi(f_1(x), ..., f_k(x)) = 0$$

for all x in some nonempty, open subset $U \subset M$ are such that $\Psi(z_1, ..., z_k) \equiv 0$ on some open subset of $\mathbb{R}^k$ (which contain the image of U under $(f_1, ..., f_k)$). A set of functions is functionally dependent if it is not functionally independent.

Theorem 2. *Let G act freely on M, the dimension of M be m and the dimension of the orbits of G be s. For any $x_0 \in M$, there are exactly $m - s$ functionally independent functions $f_1, ..., f_{m-s}$ defined in a neighbourhood of x_0 which are locally invariant under G. In addition, any other function $\tilde{f}$ that is locally invariant under G near x_0 can be written in the form $\tilde{f}(x) = \phi(f_1(x), ..., f_{m-s}(x))$ near x_0, for some smooth function ϕ.*
In such a case, we call the set $\{f_1, ..., f_{m-s}\}$ a complete set of functionally independent invariants (Olver 1994).

This theorem deserves some discussion:
Fix the x_0 mentioned in Theorem 2 and define the orbit $O^G(x_0) = \{g \star x_0 \in M : g \in G\}$. Now consider the set defined by $I_{f_i}(x_0) = \{x \in M : f_i(x) = f_i(x_0)\}$, where f_i is one of the $m-s$ G-invariant functions whose existence is guaranteed by the theorem. Since f_i is a G-invariant function, for any $x \in I_{f_i}(x_0)$ and $g \in G$, we have that $f_i(g \star x) = f_i(x) = f_i(x_0)$, so $I_{f_i}(x_0)$ is a G-invariant subset of M.

In the interests of building intuition, I will temporarily restrict this discussion to the case of $M = \mathbb{R}^m$. We see that $I_{f_i}(x_0)$ is a hyperplane of dimension $m - 1$. Since f_i and f_j are functionally independent for $i \neq j$, we have that the intersection $I_{f_i}(x_0) \cap I_{f_j}(x_0)$ is of dimension $m-2$. Continuing as such, the intersection of all $I_{f_i}(x_0), \quad i = 1, ..., m-s$ is some s-dimensional subset of M. Notice that since each $I_{f_i}(x_0)$ is an invariant set containing x_0, all elements of the form $g \star x_0$ must be in $I_{f_i}(x_0)$. Hence we have $O^G(x_0) \subset I_{f_i}(x_0)$ for all i. Hence (at least locally) we have $\bigcup_{i=1}^{m-s} I_{f_i}(x_0) = O^G(x_0)$.

Theorem 3. *Let G act freely on M and $\{f_1, ..., f_{m-s}\}$ be a complete set of functionally independent invariants defined on some open subset $W \subset M$. For a system F of algebraic equations, define $I_F = \{x \in M : F(x) = 0\}$. If I_F is G-invariant, then for every $x_0 \in I_F$, there is a neighbourhood $\tilde{W} \subset W$ of x_0 and another G-invariant function $\tilde{F}$ such that $\tilde{F}(x) = \tilde{F}(f_1(x), ..., f_{m-s}(x))$ which satisfies (Olver 1994):*

$$I_F \cap \tilde{W} = I_{\tilde{F}} \cap \tilde{W} = \{x \in \tilde{W} : \tilde{F}(f_1(x), ..., f_{m-s}(x)) = 0\}.$$

The set $I_F \cap \tilde{W}$ is described by m variables (since $x \in \mathbb{R}^m$). Theorem 3 ensures that if we can find a group action that preserves I_F, we can describe the same set by $m - s$ variables. In some sense, we trade the information provided by the extra s variables for the fact that $\tilde{F}$ is G-invariant (G being an s-dimensional group action).

If we are searching for solutions to $F(x) = 0$, this result may allow us to simplify our problem by reducing the number of variables in it by the dimension s. This is the fundamental idea behind the technique of solving partial differential equations that will now be described.

3.3 Application to Partial Differential Equations

Consider a second order PDE of the form $F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0$. Instead of considering F as a function of independent variables x, t, dependent variable u and the derivatives to second order of u with respect to x and t, we consider $F : \mathbb{R}^8 \to \mathbb{R}$ and the variables $x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}$ as independent. This allows us to make use of the results from the previous sections. We aim to do this as follows:

- Construct a group transformation that makes $I_F = \{F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0\}$ invariant
- Find invariant functions of this group action
- Reformulate our equation in terms of these invariant functions to be $\tilde{F}$, a function of fewer variables

Let the infinitesimal generator of G be

$$v = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + U_{[x]}\frac{\partial}{\partial u_x} + U_{[t]}\frac{\partial}{\partial u_t} + U_{[xx]}\frac{\partial}{\partial u_{xx}} + U_{[xt]}\frac{\partial}{\partial u_{xt}} + U_{[tt]}\frac{\partial}{\partial u_{tt}},$$

where $X, T, U, U_{[x]}, U_{[t]}, U_{[xx]}, U_{[xt]}, U_{[tt]}$ are smooth functions of x, t, u. We call groups defined by such infinitesimal generators point transformations (other classes of transformations may be found by considering the coefficients of the generator to be functions of eg. x, t, u, u_x).

Although we consider $x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}$ to be independent variables, we cannot consider the perturbations due to the application of a group action to be independent. We require that perturbations due to the group action preserve the differential relationships between these variables ($u_t = \frac{\partial u}{\partial t}$, $u_{xt} = \frac{\partial^2 u}{\partial x \partial t}$ etc). We do this by placing restrictions on $X, T, U, U_{[x]}, U_{[t]}, U_{[xx]}, U_{[xt]}, U_{[tt]}$. In fact, $U_{[x]}, U_{[t]}, U_{[xx]}, U_{[xt]}, U_{[tt]}$ are completely determined by X, T, U, which themselves may be arbitrary (Edwards 2019). See section 5.2 (appendices) for details.

Theorem 1 ensures that the set

$$I_F = \{(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) \in \mathbb{R}^8 : F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0\}$$

is invariant under G if $v(F) = 0$ on I_F. We ensure that I_F is G-invariant by solving $v(F) = 0$ for $X(x, t, u), T(x, t, u), U(x, t, u)$. This leads to an equation of the form

$$\begin{aligned}
0 =&[EQN1(X,T,U)]u_x^3 \\
&+ [EQN2(X,T,U)]u_x^2 u_{xt} \\
&+ [EQN3(X,T,U)]u_x u_t \\
&+ ...
\end{aligned}$$

The determining equations $EQN1, EQN2, EQN3...$ are a system on linear PDE's. Since variables $u_x, u_t, u_{xx}, u_{xt}, u_{tt}$ may be varied independently, solving $v(F) = 0$ on I_F is equivalent to solving each determining equation equal to zero. By doing so, we may be able to then make use of theorems 2 and 3 to simplify F by making a change of variables to G-invariant variables.

This process is called one-parameter classical symmetry analysis. It may be generalised to more dimensions, more parameters or higher orders of the PDE than we have discussed, but this report will not discuss the theory of classical symmetries any further.

3.4 Application to Heat Equation

Applying classical symmetry analysis to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

gives infinitesimal generator coefficients

$$\begin{aligned}
X &= c_1 + c_2 t + c_3 x + c_4 xt, \\
T &= c_5 + 2c_3 t + c_4 t^2, \\
U &= \left[c_6 - c_4\left(\frac{x^2}{4} + \frac{t}{2}\right) - c_2\frac{x}{2}\right].
\end{aligned}$$

where $c_1, ..., c_6 \in \mathbb{R}$ are arbitrary constants (Edwards 2019).

We will consider the case where $c_2 = 1$ and $c_1 = c_3 = c_4 = c_5 = c_6 = 0$. This means that

$$v = t\frac{\partial}{\partial x} - \frac{xu}{2}\frac{\partial}{\partial u}.$$

A useful method of finding invariant functions is to solve the equations

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}$$

and take the constants of integration as the invariant functions. This will be illustrated in the case we have just described:

$$\frac{dt}{0} = \frac{dx}{t} = \frac{du}{-\frac{xu}{2}}.$$

Since $T = 0$, we know that t is an invariant function. To find the other, we solve $\frac{-x.dx}{t} = \frac{2.du}{u}$ to give $-\frac{1}{2t}x^2 = 2\ln(u) + \alpha$. Hence:

$$\ln(u) = -\frac{1}{4t}x^2 - \frac{\alpha}{2}$$
$$u = e^{-x^2/4t}\alpha,$$

where in the last line we have redefined $e^{-\frac{\alpha}{2}} \mapsto \alpha$. Now restricting one of the invariant functions to be a function of the other, we have the result that $u = e^{-x^2/4t}f(t)$, which we then substitute into the original heat equation to obtain:

$$f'(t) + \frac{f(t)}{2t} = 0,$$

which has solution $f(t) = \frac{A}{\sqrt{t}}$ for $A \in \mathbb{R}$. Hence the solution to the heat equation corresponding to this choice of constants $c_1, ..., c_6$ is

$$u(x,t) = \frac{A}{\sqrt{t}}e^{-x^2/4t}.$$

Other choices of constants $c_1, ..., c_6$ correspond to other symmetries and will yield different solutions to the heat equation.

3.5 Compactly Supported Reaction-Diffusion Solutions

Recall the motivation given at the start of this report - we aim to describe the situation in which a substance diffuses through a medium in such a way that the entirety of the diffusing substance is bounded in some finite region of space. We attempt to describe such behaviour by the one-dimensional *reaction-diffusion* equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x}\right) + Q(u). \tag{1}$$

As in the heat equation of the previous section, u is the concentration of diffusing substance and x and t represent space and time. The functions $K(u)$ and $Q(u)$ are the diffusion and reaction terms respectively.

If we allow K and Q to be arbitrary and apply Lie symmetry analysis to the reaction-diffusion equation, we find that the only solution to the determining equations:

$$X(x,t,u) = c_1,$$
$$T(x,t,u) = c_2,$$
$$U(x,t,u) = 0,$$

where c_1, c_2 are arbitrary constants. This infinitesimal generator corresponds to the transformation

$$x \mapsto x + c_1\epsilon, \qquad t \mapsto t + c_2\epsilon, \qquad u \mapsto u.$$

It is clear that two invariant functions are $\alpha := x - ct$ and u, where $c := c_1/c_2$. Since X, T, U are constants, the variables u_t, u_{xx} are also invariant. We may thus rewrite $F(x, t, u, u_t, u_{xx}) = 0$ as $\tilde{F}(\alpha, u, u_t, u_{xx}) = 0$. Together with $u = \phi(x, t)$ we may conclude that $u = f(\alpha)$ for some C^2 function f. Substituting this in to the original reaction-diffusion equation gives:

$$-cf'(\alpha) = K'(f(\alpha))f'(\alpha)^2 + K(f(\alpha))f''(\alpha) + Q(f(\alpha)).$$

Thus we have converted a second order PDE to a second order ODE. This symmetry corresponds to a travelling wave solution of u.

It should be stressed that the symmetry presented here is the **only** symmetry of this form which works for all functions $K(u)$ and $Q(u)$. But as is clear in figure 1, not all diffusive behaviour can be described as a travelling wave. So perhaps there are certain forms of K and Q which enable other symmetries to be possible, and thus other solutions to the reaction-diffusion equation. Pattle showed that the reaction-diffusion equation with $K(u) = u^n$ and $Q(u) = 0$ may yield compactly supported solutions, however did not use Lie symmetry analysis in this result (Pattle 1959).

To find particular forms of diffusion and reaction terms with possibly different properties, we follow a similar procedure as has already been described, except when we solve the determining equations, we also enable ourselves to pick particular forms of K and Q. Thus we are more restrictive of K and Q so that we may be more general with X, T, U. Effectively, we are now solving determining equations of the form

$$\begin{aligned} 0 =& [EQN1(X, T, U, K, Q)]u_x^3 \\ &+ [EQN2(X, T, U, K, Q)]u_x^2 u_{xt} \\ &+ [EQN3(X, T, U, K, Q)]u_x u_t \\ &+ \ldots \end{aligned}$$

for functions $X(x, t, u), T(x, t, u), U(x, t, u), K(u), Q(u)$.
This process reveals that more symmetries are possible when K is constant, exponential or a power law (ie. $K(u) = u^n$) (Ibragimov 1994). Each of these three results has a corresponding $Q(u)$ term which enables symmetry.

Figure 3 illustrates Pattle's solution (Pattle 1959). The left image in the figure is the true solution. As we wish to find solutions with compact support, we consider the solutions of the left to represent

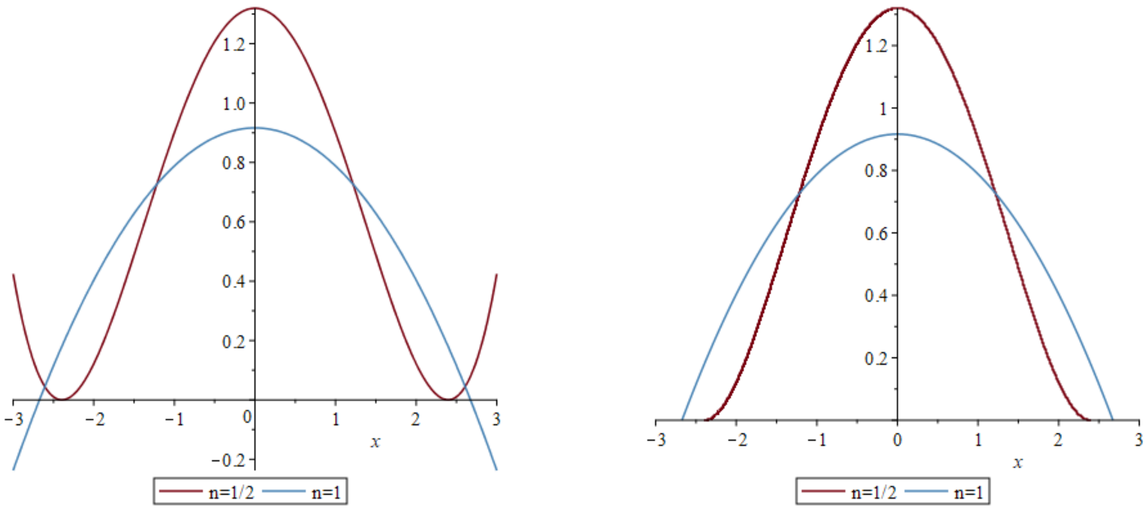


Figure 3: Solution for two different power law diffusivities. Concentration u is plotted as a function of position x at a fixed time t.

physical situations as depicted to the right, where concentration is identically zero outside the plotted area. This also avoids the question of physically interpreting a negative valued concentration.

One way that we may further extend our class of known diffusion and reaction terms that enable greater symmetry is through a slightly different method, termed the "non-classical" method. As one may guess, the method described up to now is often referred to as the "classical" method, and the symmetries it yields "classical" symmetries.

3.6 Non-Classical Symmetries

The "classical" symmetry method as described up to this point is a useful tool in simplifying PDE's, but it may be expanded upon in various ways. A commonly used expansion is the "non-classical" method (a type of conditional symmetry). The theory and methodology behind this approach is very similar to what has been described so far in this report, but one additional assumption is incorporated into the process.

In the classical symmetry approach, the requirement that $u = \phi(x,t)$ for some C^2 function ϕ was used to describe the infinitesimal generator coefficients $U_{[x]}, U_{[t]}, U_{[xx]}, U_{[xt]}, U_{[tt]}$ as functions of X, T, U so that the coordinates (x,t,u) can be changed to (gx, gt, gu) and derivatives of the new coordinates remain meaningful. The non-classical approach also uses this assumption to restrict the set which we aim to make invariant, ie.

$$I_F = \{F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0\} \cap \{u - \phi(x,t) = 0\},$$

instead of just $\{F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0\}$.

The difference between classical and non-classical may be summed up as this: in the classical approach, we required symmetry of a set (defined by $F = 0$) that is greater than what is strictly necessary. In the non-classical approach, we only solve what is strictly necessary, which enables us to find more solutions, but also makes the determining equations non-linear and consequently more difficult to solve.

The extra requirement of I_F in this non-classical approach can be satisfied by ensuring that the equation $u = \phi(x,t)$ is invariant under the group transformation G, ie.

$$gu = \phi(gx, gt).$$

This is equivalent to saying that

$$\begin{aligned} u + \epsilon U + \mathcal{O}(\epsilon^2) &= \phi(x + \epsilon X + \mathcal{O}(\epsilon^2), t + \epsilon T + \mathcal{O}(\epsilon^2)) \\ &= \phi(x,t) + \epsilon(X\frac{\partial \phi}{\partial x} + T\frac{\partial \phi}{\partial t}) + \mathcal{O}(\epsilon^2) \\ &= \phi(x,t) + \epsilon(Xu_x + Tu_t) + \mathcal{O}(\epsilon^2). \end{aligned}$$

By equating the first-order ϵ terms, we have:

$$U = Xu_x + Tu_t \tag{2}$$

The non-classical symmetry method makes use of this relation in the following way:

- without loss of generality, it is assumed that $T = 1$, and hence equation (2) becomes

$$u_t = U - Xu_x. \tag{3}$$

  This assumption is made purely to simplify the determining equations. The case of $T = 0$ ought to be considered for completeness, but will not be done in this report.

- By applying D_x and D_t to equation (3), two "differential consequences" are produced:

$$u_{xt} = \frac{\partial U}{\partial x} + u_x\left(\frac{\partial U}{\partial u} - \frac{\partial X}{\partial x} - u_x\frac{\partial X}{\partial u}\right) - Xu_{xx} \tag{4}$$

$$u_{tt} = \frac{\partial U}{\partial t} + u_t\frac{\partial U}{\partial u} - u_x\left(\frac{\partial X}{\partial t} + u_t\frac{\partial X}{\partial u}\right) - Xu_{xt} \tag{5}$$

- The equation $v(F) = 0$ is solved as in classical case (and assuming $T = 1$), giving a series of generating equations which are functions of $x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}$. The original differential equation (for which we are trying to find symmetries), as well as equations (5), (4) and (3) are used to eliminate as many variables as possible, and the remaining form of the generating equations are solved for $X(x, t, u)$ and $U(x, t, u)$.

The non-classical symmetry method generally gives rise to non-linear determining equations for the generating coefficients, where the classical method gave linear equations.

3.7 Non-Classical Analysis of Reaction-Diffusion Equation

The reaction-diffusion equation (1) has been extensively studied using the non-classical symmetry analysis as described above. Many symmetries have been found for particular forms of K and Q, most of which do not lead to solutions for u with compact support.

After setting the infinitesimal generator components $T = 1$, $X = 0$ and $U = U(u)$ (as inspired by observations of previous symmetry calculations), Arrigo and Hill found that a symmetry is present for arbitrary K so long as Q has the form

$$Q(u) = (1 + c_3 K(u)) \left(\frac{c_1 \int K(u)du + c_2}{K(u)} \right), \tag{6}$$

where c_1, c_2, c_3 are arbitrary constants (Arrigo & Hill 1995). In this case, the U component of the infinitesimal generator is

$$U(u) = \frac{c_1 \int K(u)du + c_2}{D(u)}.$$

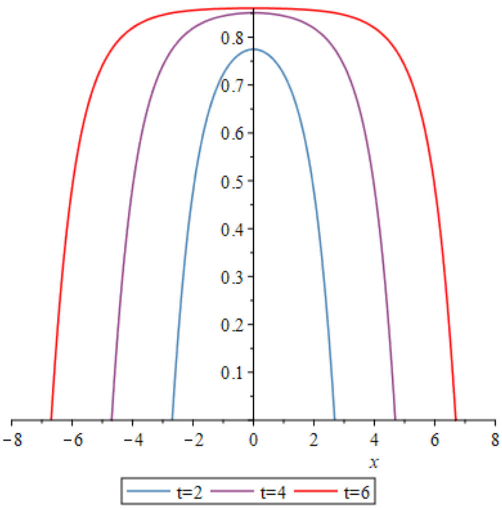
With an eye out for compactly supported solutions, we restrict to the cases when $c_1 \neq 0$, $c_3 \neq 0$.The families of solutions are

$$u(x, t) = \begin{cases} F^{-1}(A \cos(\omega x) \exp(c_1 t) - c_2/c_1), & \text{for } c_1 c_3 > 0 \\ F^{-1}(A \cosh(\omega x) \exp(c_1 t) - c_2/c_1), & \text{for } c_1 c_3 < 0, \end{cases} \tag{7}$$

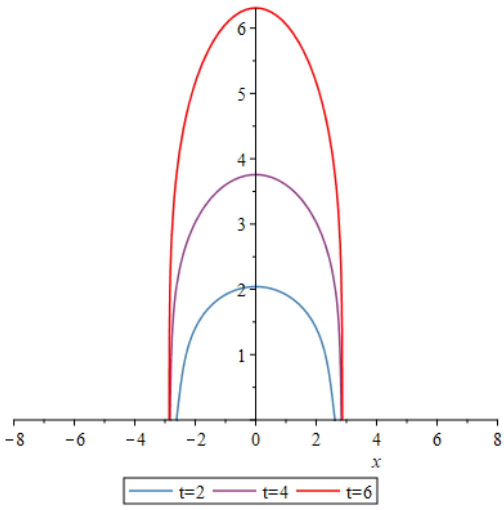
where A is a constant, $\omega = \sqrt{|c_1 c_3|}$ and F^{-1} denotes the inverse of the function $F(u) := \int_0^u K(p)dp$ (assuming that all these objects exist).

One example that results in compactly supported solution is taking $K(u) = 1 + u^3$ (and thus specifying $Q(u) = (1 + c_3(1 + u^3))\frac{c_1(u+u^4/4)+c_2}{1+u^3}$). Corresponding solutions (7) are illustrated in figure 4.

What an observant reader may notice about this case is that in general, $Q(0) \neq 0$. This means that our physicality assumption that the solution is simply zero outside its restricted domain is problematic.



(a) $c_1 < 0$ and $c_3 > 0$



(b) $c_1 > 0$ and $c_3 > 0$

Figure 4: Profile of equation (7) for $K(u) = 1 + u^3$

Q being non-zero in this region would imply that this infinite region where u is zero has a non-zero source term Q and the term $\frac{\partial}{\partial x}(K(u)\frac{\partial u}{\partial x})$ is zero. Hence we may expect the $u = 0$ region of our piecewise solution to not remain at zero, but to increase with time. One inelegant way of getting around this is to define Q to be a piecewise function:

$$Q(u) = \begin{cases} (1 + c_3 K(u)) \left(\frac{c_1 \int K(u) du + c_2}{K(u)} \right), & \text{for } u > 0 \\ 0, & \text{for } u = 0. \end{cases} \tag{8}$$

You are not alone if you find this incredibly unsatisfying. This work-around would not be necessary if Q as determined by equation (6) satisfied $Q(0) = 0$. This can be achieved if $c_2 = 0$ or if $c_3 = -1/K(0)$. To discuss this further, notice that $F(0) = 0$ and so any domain in which F^{-1} exists will satisfy $F^{-1}(p) \neq 0$ for any $p \neq 0$ (due to bijectivity of F on this domain). Thus the endpoints of support that we seek are defined by

$$A \cos(\omega x) \exp(c_1 t) - c_2/c_1 = 0 \text{ for } c_1 c_3 > 0,$$
$$A \cosh(\omega x) \exp(c_1 t) - c_2/c_1 = 0 \text{ for } c_1 c_3 < 0.$$

Fixing $c_2 = 0$ means that these endpoints are either at fixed x defined by $\cos(\omega x)$, or do not exist (and hence we do not have compact support). It is assumed that K is a strictly positive function, so fixing $c_3 = -1/K(0)$ ensures that c_3 is negative. This leads to the solutions in equation (7) having either compact support for only a finite time, or having shrinking support. Thus there are no solutions of form (7) with growing compact support which lasts for arbitrary time and with Q defined by equation (6) for all u.

Non-classical symmetries of the reaction-diffusion equation have not been exhaustively studied and so it is still possible to find reaction and diffusion terms that may explain observed behaviour corresponding to compactly supported concentration. Another possibility is that the reaction-diffusion equation is simply too great a simplification of the observed physical processes which we are trying to describe. A more complicated model that may be analysed using the same technique is one that describes the reaction of the medium to the diffusive substance by means of a second diffusive substance. In our example of figure 1, we might model both the diffusion of the virus and that of the plant RNA which reacts with it. A commonly used model of such a situation is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big(K_u(u)\dfrac{\partial u}{\partial x}\Big) + Q_u(u,v) \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}\Big(K_v(v)\dfrac{\partial v}{\partial x}\Big) + Q_v(u,v), \end{cases}$$

where u is the virus concentration and v the RNA concentration (Cherniha & Davydovych 2012). An exploration of coupled systems of reaction-diffusion equations is a topic for possible future work.

4 Conclusion

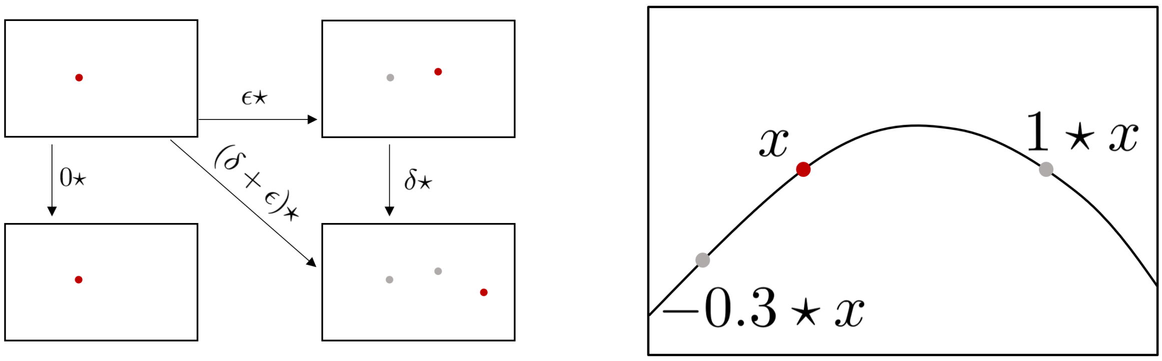
In this report, Lie symmetry theory has been presented and developed into a tool to analyse PDEs. In order to explain physically observed phenomena such as those in figure 1, reaction-diffusion equations were analysed using symmetry techniques in order to find solutions with compactly supported concentration. Using results developed by Arrigo and Hill on reaction-diffusion equations (Arrigo & Hill 1995), compactly supported solutions were sought. It is concluded that solutions of the form described in equation (7) only have compact support if the reaction term is defined in a piecewise manner. Further exploration of compactly supported diffusion may require a different model of diffusion, and is briefly discussed.

5 Appendices

5.1

Figure 5 depicts a particular transformation of $\mathbb{R}^2$ by the group $(\mathbb{R}, +)$. Figure 5a gives a visual representation of the requirements of definition 2: when applied to any element of the manifold, the identity element of the group (0 in this case) does not change the manifold element. The result of applying group element $\delta + \epsilon$ to an arbitrary manifold element is identical to applying ϵ to the original manifold element then δ to the result.

The requirement of differentiability of $\star$ leads to a situation such as that depicted in figure 5b. A differentiable curve may be defined by $\star$ and parametrised by ϵ.



(a) (b)

Figure 5: A possible action of the group $(\mathbb{R}, +)$ on the manifold $\mathbb{R}^2$

Example 2. Take $M = \mathbb{R}^2$ and consider the group action of $\mathbb{R}$ on this manifold as defined by the following $\cdot$ and $\star$ functions:

$$a \cdot b = a + b \quad a, b \in \mathbb{R} \quad a \star (x, y) = (x - a, y + a) \quad (x, y) \in \mathbb{R}^2.$$

One can easily check that $\mathbb{R}$ forms a group under $\cdot$ and that $\star$ makes it into a group of transformations acting on $\mathbb{R}^2$. Whenever $\mathbb{R}$ forms a group of transformations, we call it a one parameter transformation group.

Since $\star : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable function, we can define the infinitesimal generator of the group of transformations $(\mathbb{R}, \star)$ to be $v(x, y) = \frac{d}{da}[a \star (x, y)]|_{a=0}$. In this case, this is $v(x, y) = \frac{d}{da}[x - a, y + a]|_{a=0} = (-1, 1)$.

5.2

This section will demonstrate explicitly the requirements on $X, T, U, U_{[x]}, U_{[t]}, U_{[xx]}, U_{[xt]}, U_{[tt]}$ to ensure that the differential relationships $u_t = \frac{\partial u}{\partial t}$, $u_{xt} = \frac{\partial^2 u}{\partial x \partial t}$, etc. are preserved by the group action.

Suppose that we are considering the manifold $\mathbb{R}^8 = (x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$ and a one-parameter local group of transformations G with elements $g(\epsilon)$, where $g(0)$ is the identity of G. Using the notation $g(\epsilon) \star (x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = (gx, gt, gu, gu_x, gu_t, gu_{xx}, gu_{xt}, gu_{tt})$ (ϵ is suppressed for brevity), we have:

$$
\begin{aligned}
gx &= x + \epsilon X + \mathcal{O}(\epsilon^2),\\
gt &= t + \epsilon T + \mathcal{O}(\epsilon^2),\\
gu &= u + \epsilon U + \mathcal{O}(\epsilon^2),\\
gu_x &= u_x + \epsilon U_{[x]} + \mathcal{O}(\epsilon^2),\\
gu_t &= u_t + \epsilon U_{[t]} + \mathcal{O}(\epsilon^2),\\
gu_{xx} &= u_{xx} + \epsilon U_{[xx]} + \mathcal{O}(\epsilon^2),\\
gu_{xt} &= u_{xt} + \epsilon U_{[xt]} + \mathcal{O}(\epsilon^2),\\
gu_{tt} &= u_{tt} + \epsilon U_{[tt]} + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

for small ϵ. As already suggested, G may perturb the x, t, u coordinates in a truly independent manner and preserve the differential relationships between the eight parameters, which means that X, T, U may be any smooth functions. The action of G on the coordinates $u_x, u_t, u_{xx}, u_{xt}, u_{tt}$ will be determined by the action on x, t, u. The benefit of the above ϵ formulation is that we may describe this relationship by expressing $U_{[x]}, U_{[t]}, U_{[xx]}, U_{[xt]}, U_{[tt]}$ as functions of X, T, U.
We must encode the differential relationships between $gx, gt, gu, gu_x, gu_t, gu_{xx}, gu_{xt}$ and gu_{tt} into the components of the infinitesimal generator v.
In effect, we restrict ourselves to looking at the cases where the differential relationships $u_t = \frac{\partial u}{\partial t}$, $u_{xt} = \frac{\partial^2 u}{\partial x \partial t}$, etc. are preserved by the application of $g(\epsilon)$. That is, we must have $gu_t = \frac{\partial gu}{\partial gt}$, $gu_{xt} = \frac{\partial^2 gu}{\partial gx \partial gt}$, etc. These requirements completely define the functions $U_{[x]}, U_{[t]}, U_{[xx]}, U_{[xt]}, U_{[tt]}$.

To do this, we first consider the solution space of our function

$$I = \{(x, t, u) \in \mathbb{R}^3 : u = \phi(x, t)\},$$

where ϕ is the solution of the PDE we wish to solve. We are only interested in transformation groups G which satisfy $gu = \phi(gx, gt)$. This is equivalent to requiring that I is G-invariant. We will make a further assumption on ϕ which is that I can locally be described by using any two of the three coordinates x, t, u. Due to its G-invariance, we may also describe it using any two of the coordinates gx, gt, gu. We will make use of this information to use standard change-of-coordinate derivative results: Recall that we wish to construct $U_{[x]}$ such that $gu_t = \frac{\partial gu}{\partial gt}$. We must make the observations that gx and gt are independent, giving $\frac{\partial gx}{\partial gx} = 1$ and $\frac{\partial gx}{\partial gt} = 0$. From this we see that

$$\frac{\partial gu}{\partial gt} = det\begin{pmatrix} \frac{\partial gu}{\partial gt} & \frac{\partial gu}{\partial gx} \\ \frac{\partial gx}{\partial gt} & \frac{\partial gx}{\partial gx} \end{pmatrix} = \frac{\partial(gu, gx)}{\partial(gt, gx)}$$

is the Jacobian corresponding to a change in basis from (gu, gx) to (gt, gx). We may of course relate this to the Jacobians corresponding to a basis change from (gu, gx) to (x, t) and then to (gt, gx):

$$\frac{\partial gu}{\partial gt} = \frac{\partial(gu, gx)}{\partial(gt, gx)} = \frac{\partial(gu, gx)}{\partial(t, x)}\frac{\partial(t, x)}{\partial(gt, gx)} = \frac{\frac{\partial(gu,gx)}{\partial(t,x)}}{\frac{\partial(gt,gx)}{\partial(t,x)}} = \frac{\frac{\partial gu}{\partial t}\frac{\partial gx}{\partial x} - \frac{\partial gu}{\partial x}\frac{\partial gx}{\partial t}}{\frac{\partial gt}{\partial t}\frac{\partial gx}{\partial x} - \frac{\partial gx}{\partial t}\frac{\partial gt}{\partial x}},$$

where the components of the final quantity can be easily calculated to be:

$$\begin{aligned}
\frac{\partial gu}{\partial t} &= \frac{\partial}{\partial t}[u + \epsilon U(x, t, u(x, t)] + \mathcal{O}(\epsilon^2) = u_t + \epsilon\Big(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial u}u_t\Big) + \mathcal{O}(\epsilon^2), \\
\frac{\partial gt}{\partial t} &= \frac{\partial}{\partial t}[t + \epsilon T(x, t, u(x, t)] + \mathcal{O}(\epsilon^2) = 1 + \epsilon\Big(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial u}u_t\Big) + \mathcal{O}(\epsilon^2), \\
\frac{\partial gx}{\partial t} &= \frac{\partial}{\partial t}[x + \epsilon X(x, t, u(x, t)] + \mathcal{O}(\epsilon^2) = \epsilon\Big(\frac{\partial X}{\partial t} + \frac{\partial X}{\partial u}u_t\Big) + \mathcal{O}(\epsilon^2), \\
\frac{\partial gu}{\partial x} &= \frac{\partial}{\partial x}[u + \epsilon U(x, t, u(x, t)] + \mathcal{O}(\epsilon^2) = u_x + \epsilon\Big(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial u}u_x\Big) + \mathcal{O}(\epsilon^2), \\
\frac{\partial gt}{\partial x} &= \frac{\partial}{\partial x}[t + \epsilon T(x, t, u(x, t)] + \mathcal{O}(\epsilon^2) = \epsilon\Big(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial u}u_x\Big) + \mathcal{O}(\epsilon^2), \\
\frac{\partial gx}{\partial x} &= \frac{\partial}{\partial x}[x + \epsilon X(x, t, u(x, t)] + \mathcal{O}(\epsilon^2) = 1 + \epsilon\Big(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial u}u_x\Big) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

At this point, I introduce the differential operators D_x, D_t defined on functions of $x, t, u, u_x, u_t, \ldots$ by:

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \ldots, \\
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \ldots
\end{aligned}$$

Hence we may rewrite the above to be:

$$\frac{\partial gu}{\partial t} = D_t(gu) = u_t + \epsilon D_t(U) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial gt}{\partial t} = D_t(gt) = 1 + \epsilon D_t(T) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial gx}{\partial t} = D_t(gx) = \epsilon D_t(X) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial gu}{\partial x} = D_x(gu) = u_x + \epsilon D_x(U) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial gt}{\partial x} = D_x(gt) = \epsilon D_x(T) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial gx}{\partial x} = D_x(gx) = 1 + \epsilon D_x(X) + \mathcal{O}(\epsilon^2).$$

Putting all this together, we have:

$$\frac{\partial gu}{\partial gt} = \frac{(u_t + \epsilon D_t(U) + \mathcal{O}(\epsilon^2))(1 + \epsilon D_x(X) + \mathcal{O}(\epsilon^2)) - (u_x + \epsilon D_x(U) + \mathcal{O}(\epsilon^2))(\epsilon D_t(X) + \mathcal{O}(\epsilon^2))}{(1 + \epsilon D_t(T) + \mathcal{O}(\epsilon^2))(1 + \epsilon D_x(X) + \mathcal{O}(\epsilon^2)) - (\epsilon D_t(X) + \mathcal{O}(\epsilon^2))(\epsilon D_x(T) + \mathcal{O}(\epsilon^2))}$$

$$\begin{aligned}\frac{\partial gu}{\partial gt} &= \frac{(u_t + \epsilon D_t(U) + \mathcal{O}(\epsilon^2))(1 + \epsilon D_x(X) + \mathcal{O}(\epsilon^2)) - (u_x + \epsilon D_x(U) + \mathcal{O}(\epsilon^2))(\epsilon D_t(X) + \mathcal{O}(\epsilon^2))}{(1 + \epsilon D_t(T) + \mathcal{O}(\epsilon^2))(1 + \epsilon D_x(X) + \mathcal{O}(\epsilon^2)) - (\epsilon D_t(X) + \mathcal{O}(\epsilon^2))(\epsilon D_x(T) + \mathcal{O}(\epsilon^2))} \\ &= \frac{(u_t + \epsilon D_t(U) + \epsilon D_x(X)u_t + \mathcal{O}(\epsilon^2)) - (\epsilon D_t(X)u_x + \mathcal{O}(\epsilon^2))}{(1 + \epsilon D_t(T) + \epsilon D_x(X) + \mathcal{O}(\epsilon^2)) - \mathcal{O}(\epsilon^2)} \\ &= \frac{u_t + \epsilon(D_t(U) + D_x(X)u_t - D_t(X)u_x) + \mathcal{O}(\epsilon^2)}{1 + \epsilon(D_t(T) + D_x(X)) + \mathcal{O}(\epsilon^2)}.\end{aligned}$$

Since $\frac{1}{1+\epsilon\alpha+\mathcal{O}(\epsilon^2)} = 1 - \epsilon\alpha + \mathcal{O}(\epsilon^2)$, we have:

$$\begin{aligned}\frac{\partial gu}{\partial gt} &= u_t + \epsilon(D_t(U) + D_x(X)u_t - D_t(X)u_x) + \mathcal{O}(\epsilon^2)][1 - \epsilon(D_t(T) + D_x(X)) + \mathcal{O}(\epsilon^2)] \\ &= u_t + \epsilon(D_t(U) + D_x(X)u_t - D_t(X)u_x - D_t(T)u_t - D_x(X)u_t) + \mathcal{O}(\epsilon^2) \\ &= u_t + \epsilon(D_t(U) - D_t(X)u_x - D_t(T)u_t) + \mathcal{O}(\epsilon^2).\end{aligned}$$

Hence we have calculated that $U_{[t]} = D_t(U) - D_t(X)u_x - D_t(T)u_t$. The other prolongation coefficients can be calculated in a similar way to be

$$\begin{aligned}U_{[x]} &= D_x(U) - D_x(X)u_x - D_x(T)u_t, \\ U_{[xx]} &= D_x(U_{[x]}) - D_x(X)u_{xx} - D_x(T)u_{xt}, \\ U_{[xt]} &= D_x(U_{[t]}) - D_x(X)u_{xt} - D_x(T)u_{tt}, \\ U_{[tt]} &= D_t(U_{[t]}) - D_t(X)u_{xt} - D_t(T)u_{tt}.\end{aligned}$$

For PDE's of higher order, we may define functions such as $U_{[xxx]}$ in a similar fashion to those above.

We may take the perturbations of x, t, u to be arbitrary, but then the perturbations of $u_x, u_t, u_{xx}, u_{xt}, u_{tt}$ are entirely determined by those of x, t, u.

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