AMSI VACATION RESEARCH SCHOLARSHIPS 2019–20

EXPLORE THE MATHEMATICAL SCIENCES THIS SUMMER

Exploring Toposes

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Abstract

The notion of elementary toposes is powerful as it unites logic and geometry via many significant aspects. It is a fundamental element in modern category theory. A natural next step would be identifying and exploring some concrete toposes. This project focuses on the category of directed graphs and proves that it is a topos in a hands-on manner.

1 Introduction

The notion of a topos is of great importance in modern category theory. "Topos" is a Greek word meaning "place", and loosely speaking, we can think of a topos as a place for one to do mathematics [2]. Therefore, we may suspect that a topos is equipped with general structures and properties shared across different areas of mathematics. In fact, the notion of a topos plays a significant role in geometry as well as logic. In early 1960s, Alexander Grothendieck first introduced the concept of a topos in his work on algebraic geometry. It is later generalised by William Lawvere when he was trying to view sets in the categorical sense, and hence the birth of 'elementary toposes', also as the toposes in logic [1]. An elementary topos is also what we will be examine for the purpose of this paper, and we will simply call it a topos from this point onwards.

Meanwhile, a directed graph is a graph with nodes and directed edges between the nodes. One of the famous applications is the Google PageRank Algorithm, where we use a directed graph to map out the relationships between links (Figure 1 is an example of such mapping). The attraction and usefulness of each page can then be estimated based on the number of links to and out of the page. Other implementations of directed graphs include representing networks, family trees, etc.

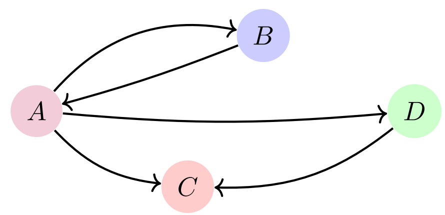


Figure 1: An example of mapping of the relationship between pages of the internet. In particular, A, B, C, D represent pages, and there is an edge from one page to another if the former page links to the latter.

Interestingly, the idea of toposes and directed graphs are deeply connected. To spell it out, the category of directed graphs is, in fact, a topos. This is also the main result of this research project. We will start by introducing some basic category languages and the definition of a topos. We use the category **Set** as an illustration of a topos. The next part of this paper proves that there is a category $\mathscr{G}$ of directed graphs and that this particular category is a topos.

Statement of authorship

Under the direction of Professor Lárusson, Jiang developed the structure and proofs presented in this report. Definitions employed in this report are adopted from Leinster's *Basic Category Theory*.

2 Preamble

Before we get into the topic of topos, we should first give the definition of a category following that in Leinster's book *Basic Category Theory* [3].

Definition 1. A category $\mathscr{A}$ consists of:

- a collection $\text{ob}(\mathscr{A})$ of **objects**;
- for each $A, B \in \text{ob}(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of **arrows** from A to B;
- for each $A, B, C \in \text{ob}(\mathscr{A})$, a function

$$\begin{aligned} \mathscr{A}(B,C) \times \mathscr{A}(A,B) &\to \mathscr{A}(A,C) \\ (g,f) &\mapsto g \circ f, \end{aligned}$$

  called **composition**;
- for each $A \in \text{ob}(\mathscr{A})$, an element 1_A of $\mathscr{A}(A, A)$, called the **identity** on A,

satisfying the following axioms:

- **associativity**: for each $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$ and $h \in \mathscr{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$;
- **identity laws**: for each $f \in \mathscr{A}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

For example, **Set** is a category whose **objects** are sets. Given sets A and B, **arrows** from A to B are the usual functions from A to B. Then, the **composition** of maps would be the composition of functions in the usual sense, thus **associativity** holds; and the **identity maps** are just the identity functions.

Now, what is a topos? As it turns out, there are many equivalent ways to define a topos, and we will adopt the following definition for the purpose of this project.

Definition 2. A topos is a category with

- a terminal object;
- products of any two objects (and hence all finite products);
- all exponentials;
- a subobject classifier.

Detailed definitions of the bullet points above will be given later.

Loosely speaking, a topos is a category with certain properties that make it behave like the category of sets. So, in order to get an understanding of these properties, we will start by examining the category of sets.

Theorem 3. *The category* **Set** *is a topos.*

Proof. To show that **Set** is a topos, we need to show that it is equipped with a terminal object, all products, all exponentials and a subobject classifier.

- Firstly, an object T is **terminal** in a category $\mathscr{A}$ if for every $A \in \mathscr{A}$, there is exactly one map $A \to T$. Moreover, such an object satisfies universality laws. That is, if a terminal object exist in a category, then it is unique up to unique isomorphism.

In the category **Set**, one-element sets are the terminal objects, and they are, of course, mutually canonically isomorphic. To see this, let $A \in \mathbf{Set}$ and $T = \{t\}$. Then there is a unique map $f : A \to T$ by sending every element in A to the single element $t \in T$.

- Secondly, let $\mathscr{A}$ be a category, and I be a set, and let $(A_i)_{i\in I}$ be a family of objects in $\mathscr{A}$. A **product** of $(A_i)_{i\in I}$ consists of an object P and a family of maps

$$(P \xrightarrow{p_i} A_i)_{i\in I}$$

with the property that for all families

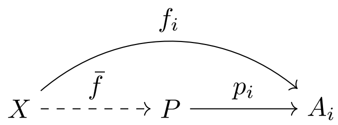
$$(X \xrightarrow{f_i} A_i)_{i\in I},$$

there exists a unique arrow $X \xrightarrow{\bar{f}} P$ such that $p_i \circ \bar{f} = f_i$ for all $i \in I$.

Now in the category **Set**, the object $P = \prod_{i\in I} A_i$ is the set of all families $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$, and the projection maps are such that $p_i(P) = A_i$. That is

$$\prod_{i\in I} A_i \xrightarrow{p_i} A_i.$$

To see this, we need to prove that for all objects and maps $X \xrightarrow{f_i} A_i$. there exists a unique map $\bar{f} : X \to P$ such that



commutes. To see the existence, let $x \in X$, and define $\bar{f} : X \to P$ by $\bar{f}(x) = (f_i(x))_{i\in I}$. Then $(p_i \circ \bar{f})(x) = p_i(\bar{f}(x)) = p_i((f_i(x))_{i\in I}) = f_i(x)$, as required. To see the uniqueness, suppose $\hat{f} : X \to P$ is another map such that $p_i \circ \hat{f} = f_i$. Now let $x \in X$, and $\hat{f}(x) = (a_i)_{i\in I}$ for some $a_i \in A_i, i \in I$. Then $(p_i \circ \hat{f})(x) = p_i(\hat{f}(x)) = p_i((a_i)_{i\in I}) = a_i = f_i(x)$. Thus, $\hat{f} = \bar{f}$.

- Thirdly, we want to show **Set** is equipped with exponential objects. We will start by giving the definition of an exponential.

Definition 4. Let $\mathscr{A}$ be a category and $A, C \in \mathscr{A}$. For each $B \in \mathscr{A}$, an **exponential** C^B is an object of $\mathscr{A}$ such that $\mathscr{A}(A \times B, C) \cong \mathscr{A}(A, C^B)$ naturally in A and C.

Remark 5. The naturality in the definition is set up so that for each $B \in \mathscr{A}$, it is possible to construct exactly one exponential C^B of $C \in \mathscr{A}$ (up to a unique isomorphism). That is, constructions of exponentials satisfy universality axiom.

Now, let $A, B \in \mathbf{Set}$, an exponential B^A of B is the set of all functions from A to B. To see that such construction satisfies the definition of exponential objects, we want to show that $\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$ naturally in $A \in \mathbf{Set}$ and $C \in \mathbf{Set}$.

Let $f : A \times B \to C$ be an arrow in $\mathbf{Set}$ for some $a \in A, b \in B$, define $\bar{f} : A \to C^B$ by

$$(\bar{f}(a))(b) = f(a, b).$$

In the other direction, let $g : A \to C^B$ be an arrow for some $a \in A$, define $\bar{g} : A \times B \to C$ by

$$\bar{g}(a, b) = (g(a))(b)$$

for some $b \in B$. Then there is a canonical bijection between g and $\bar{g}$, and f and $\bar{f}$.

- Finally, a map $\mu : A \to B$ in a category $\mathscr{A}$ is **a monomorphism** (or **monic**) if for all objects $X \in \mathscr{A}$ and maps

$$X \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{\mu} B,$$

we have $\mu \circ f = \mu \circ g$ imply $f = g$.

For example, a map in $\mathbf{Set}$ is monic if and only if it is injective. Clearly, being injective implies being monic; the "only if" part can be proved by taking A to be a one-element set.

Suppose $i : X \to A$ and $j : Y \to A$ are two monomorphisms in $\mathscr{A}$. Clearly, we can define an equivalence relation $\cong$ as follows. We have $i \cong j$ if and only if there exists an isomorphism $k : X \to Y$ such that $j \circ k = i$.

Definition 6. A **subobject** of an object A in a category $\mathscr{A}$ is an equivalence class (with respect to $\cong$) of monomorphisms

$$i : X \hookrightarrow A$$

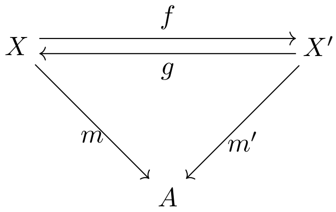
into A, and a **subobject classifier** for $\mathscr{A}$ is an object $\Omega \in \mathscr{A}$ such that

$$\{\text{subobject}\} \cong \text{map}(A, \Omega).$$

We claim that a subobject classifier of $\mathbf{Set}$ is any two-element set such as $\{0, 1\}$. Again, any two such sets are isomorphic.

To see this, let $A \in \mathbf{Set}$, and let $X \xrightarrow{m} A$ and $X' \xrightarrow{m'} A$ be monic in $\mathbf{Set}$, we need to prove that m and m' are equivalent if and only if they have the same image, and thus, the subobjects of A are in a canonical one-to-one correspondence with the subsets of A.

We divide the proof into two parts. We first show the "only if" part. Suppose m and m' are equivalent. Then there exists an isomorphism $f : X \to X'$ such that $m' \circ f = m$. That is, for every $x' \in X'$, there exists precisely one $x \in X$ such that $f(x) = x'$. Let $x' \in X'$ and $x \in X$ be the corresponding element with respect to f. Then $m'(x') = m'(f(x)) = (m' \circ f)(x) = m(x)$. Thus, m and m' have the same image. We now proceed to show the "if" part. Suppose m and m' have the same image $A' \in A$. Then since m and m' are monics, they are injective. That is, for each $a \in A'$, there is precisely one $x \in X$ and one $x' \in X'$ such that $m(x) = a$ and $m'(x') = a$. This gives a bijective relationship between x and x'. We can therefore find $f : X \to X'$ and $g : X' \to X$ with $fg = 1_{X'}$ and $gf = 1_X$ such that the triangles



commute. Thus, $m \cong m'$.

Since for each set $A \in$ **Set**, {subsets of A} $\cong \text{map}(A, \{0,1\})$, $\{0,1\}$ is a subobject classifier in **Set.** □

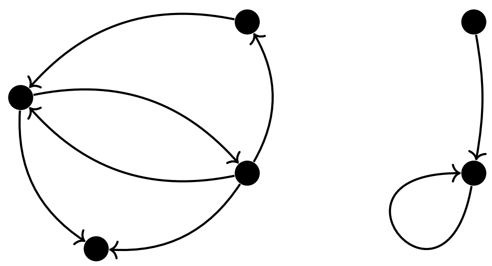
3 Directed graphs

3.1 Definitions

We now proceed to the main focus of this paper, and we will start by giving the definition of directed graphs.

Definition 7. A **directed graph** is a graph with a set of nodes (or vertices) V and a set of directed edges E between the nodes. It is worth noting that V and E do not need to be finite sets.

For example,



is a directed graph.

We now claim that there is a category $\mathscr{G}$ of directed graphs described as follows. Its objects are directed graphs. Given two directed graphs $A, B \in \mathscr{G}$, an arrow from A to B consists of a pair (f, ψ) of maps: map: nodes $\xrightarrow{f}$ nodes and map: edges $\xrightarrow{\psi}$ edges such that if an edge e goes from node a to node b in $A \in \mathscr{G}$, then $\psi(e)$ goes from node $f(a)$ to node $f(b)$ in $B \in \mathscr{G}$. Compositions of arrows would be compositions of the maps respectively. That is, if $(f, \psi) \in \mathscr{G}(A, B)$ and $(g, \phi) \in \mathscr{G}(B, C)$, then

$$\begin{aligned} \mathscr{G}(B,C) \times \mathscr{G}(A,B) &\to \quad \mathscr{G}(A,C) \\ (g,\phi) \times (f,\psi) \quad &\mapsto \quad (g \circ f, \phi \circ \psi). \end{aligned}$$

For each object $G \in \mathscr{G}$, the identity arrow $1_G = (1_{\text{node}}, 1_{\text{edge}})$ sends every node in G to itself and every edge to itself. Clearly, $\mathscr{G}$ satisfies associativity and identity laws.

There is one interesting observation about directed graphs, and we will dig a little deeper into this by defining a map between categories, called a functor.

Definition 8. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **functor** $F : \mathscr{A} \to \mathscr{B}$ consists of:

- a function
$$\text{ob}(\mathscr{A}) \to \text{ob}(\mathscr{B}),$$
written as $A \mapsto F(A)$;

- for each $A, A' \in \mathscr{A}$, a function
$$\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A')),$$
written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$;
- $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Remark 9. Let

$$e \underset{\tau}{\overset{\sigma}{\rightrightarrows}} v$$

be a category with two objects and two non-identity maps from the first to the second. Then we can think of a directed graph as a functor

$$F : \; e \underset{\tau}{\overset{\sigma}{\rightrightarrows}} v \longrightarrow \mathbf{Set}.$$

Proof. We can see this by considering a set E of edges and a set V of vertices, so $E, V \in \mathbf{Set}$. Then F sends e to $E \in \mathscr{G}$ and v to $V \in \mathscr{G}$. Moreover, $F(\sigma)$ is an arrow that sends the sources of edges in E to the set of vertices V, $F(\tau)$ is an arrow that sends the targets of edges in E to V. Clearly, F would preserve composition and identity maps. □

Remarks 10. (a) Natural transformations are maps between the functors. In this case, they are precisely the morphisms of directed graphs.

(b) A **presheaf** on a category $\mathscr{A}$ is a functor $\mathscr{A}^{op} \to \mathbf{Set}$. Furthermore, reversing the arrows in the first category, we can view a directed graph as a presheaf. It may be proved that every presheaf category is a topos (see Chapter 6 in Leinster's *Basic Category Theory* [3]).

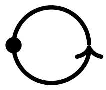
3.2 The category of directed graphs as a topos

We now reach the part where we prove the main result of this paper.

Theorem 11. *The category $\mathscr{G}$ of directed graphs is a topos.*

Proof. Again, we will divide the proof into four parts based on the definition of a topos. That is, we will prove that $\mathscr{G}$ is equipped with a terminal object, all products, exponentials and a subobject classifier.

- Firstly, we claim that a terminal object T of $\mathscr{G}$ is a node with a self-loop. Clearly, any two such directed graphs are isomorphic.

T = 

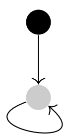
This is a terminal object because we can send every node in a directed graph $G \in \mathscr{G}$ to the only node in $T \in \mathscr{G}$ and every edge to the only edge in $T \in \mathscr{G}$, and that is a unique map from G to T.

- Secondly, let $A, B \in \mathscr{G}$ be two directed graphs. A product of A and B consists of an object in $\mathscr{G}$ and projection maps, and they can be defined as follows.

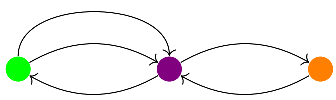
Suppose A has nodes $a_1, a_2, \cdots$, and B has nodes $b_1, b_2, \cdots$. Notice that A and B can have infinitely many nodes. Then each node in $A \times B$ can be labeled as a pair (a_i, b_j), where $a_i \in A$ and $b_j \in B$. For instance, if A has m nodes and B has n nodes, $m, n \in \mathbb{N}$, then $A \times B$ has mn nodes. Moreover, let $a_i, a_k \in A$ and $b_j, b_l \in B$. Then there is an edge e from (a_i, b_j) to (a_k, b_l) if and only if there is an edge e_A from a_i to a_k in A and an edge e_B from b_j to b_l in B, i.e. $e = (e_A, e_B)$.

Suppose (a_i, b_j) and (a_k, b_l) are two nodes in P and there is an edge e that goes from (a_i, b_j) to (a_k, b_l) in P. This means that there is an edge e_A that goes from a_i to a_k in A and an edge e_B that goes from b_j to b_l in B. Then the projection maps $p_A : P \to A$ and $p_B : P \to B$ would send nodes by $p_A(a_i, b_j) = a_i \in A$ and $p_B(a_i, b_j) = b_j \in B$, and would send edges by $p_A(e) = e_A \in A$ and $p_B(e) = e_B \in B$.

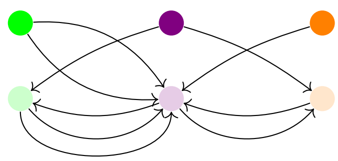
For example, suppose we have

$A =$ 

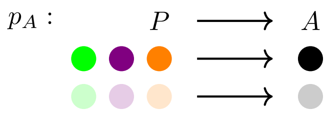
and

$B =$ 

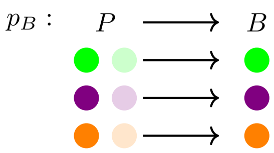
Then the product of A and B would be:

$P = A \times B =$ 

together with projection maps such that



and



and the projection maps that send edges follow from that.

Now we show that the product thus defined really is the product in the categorical sense. Suppose there is a directed graph X and arrows $f_A : X \to A$ and $f_B : X \to B$ that satisfy the properties of $\mathscr{G}$. Then we claim that there is a unique arrow $\bar{f} : X \to P$ such that $p_A \circ \bar{f} = f_A$ and $p_B \circ \bar{f} = f_B$. We will show both the existence and the uniqueness of such an arrow $\bar{f}$.

To show the existence, let x_1, x_2 be two nodes in F and there is an edge e from x_1 to x_2. We define $\bar{f} : X \to P$ by $\bar{f}(x_1) = (f_A(x_1), f_B(x_1))$ and $\bar{f}(e) = (f_A(e), f_B(e))$. Then

$$(p_A \circ \bar{f})(x_1) = p_A(\bar{f}(x_1)) = p_A(f_A(x_1), f_B(x_1)) = f_A(x_1),$$

and

$$(p_B \circ \bar{f})(x_1) = p_B(\bar{f}(x_1)) = p_B(f_A(x_1), f_B(x_1)) = f_B(x_1).$$

Also, we have

$$(p_A \circ \bar{f})(e) = p_A(\bar{f}(e)) = p_A(f_A(e), f_B(e)) = f_A(e),$$

and

$$(p_B \circ \bar{f})(e) = p_B(\bar{f}(e)) = p_B(f_A(e), f_B(e)) = f_B(e).$$

To show the uniqueness, suppose now $\hat{f} : F \to P$ is another arrow that satisfies $p_A \circ \hat{f} = f_A$ and $p_B \circ \hat{f} = f_B$. Let $\hat{f}(x_1) = (a, b)$ for some $a \in A$ and $b \in B$, and $\hat{f}(e) = (e_a, e_b)$ for some $e_a \in A$ and $e_b \in B$. Then

$$(p_A \circ \hat{f})(x_1) = p_A(\hat{f}(x_1)) = p_A(a, b) = a = f_A(x_1),$$

and

$$(p_B \circ \hat{f})(x_1) = p_B(\hat{f}(x_1)) = p_B(a, b) = b = f_B(x_1).$$

That is $\hat{f}(x_1) = (f_A(x_1), f_B(x_1)) = \bar{f}(x_1)$. Moreover,

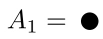
$$(p_A \circ \hat{f})(e) = p_A(\hat{f}(e)) = p_A(e_a, e_b) = e_a = f_A(e),$$

and

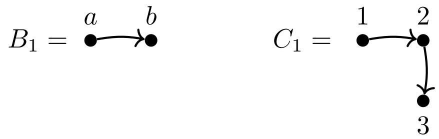
$$(p_B \circ \hat{f})(e) = p_B(\hat{f}(e)) = p_B(e_a, e_b) = e_b = f_B(e).$$

That is $\hat{f}(e) = (f_A(e), f_B(e)) = \bar{f}(e)$.

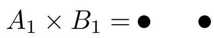
- Thirdly, suppose we are given directed graphs B and C. Since a directed graph is made up of nodes and edges, we will first use the graph



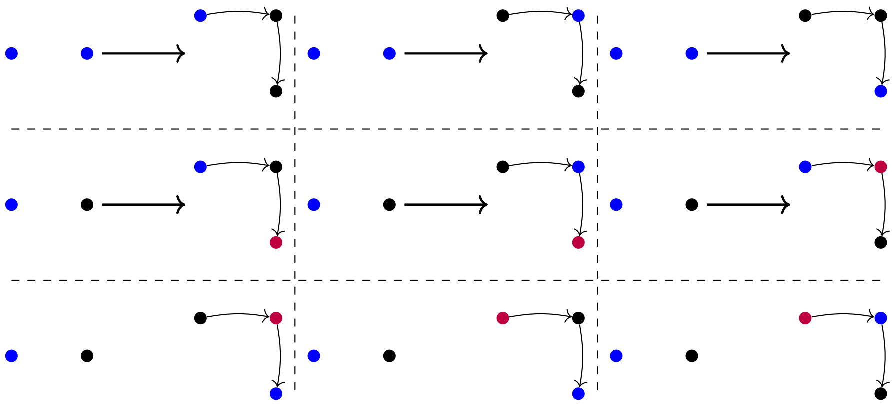
to find the set of nodes in C^B. Suppose we have another two directed graphs B_1 and C_1.



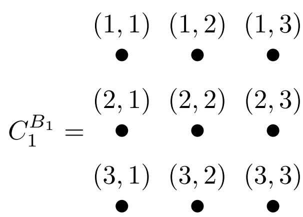
Then we would have $A_1 \times B_1$ as follows.



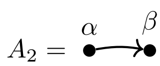
Since there is no edge in A_1, maps $\mathscr{G}(A_1, C_1^{B_1})$ are equivalent to choosing one node in $C_1^{B_1}$. Also, no edge exists in $A_1 \times B_1$. So maps from $A_1 \times B_1$ to C_1 really boil down to choosing ordered pairs of nodes from the nodes in C_1. As each node in $A_1 \times B_1$ have three choices (the three nodes in C_1), there are $3 \times 3 = 9$ different maps in $\mathscr{G}(A_1 \times B_1, C_1)$.



Therefore, there should be 9 nodes in $C_1^{B_1}$, and we can label them in the form of (a, b) as shown below, where the first number represents the choice of the node in C_1 that is mapped with $a \in B_1$, and similar for the second number with $b \in B_1$.

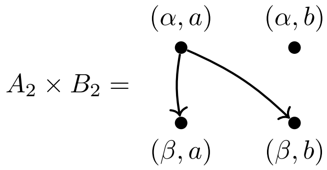


Now we use the directed graph

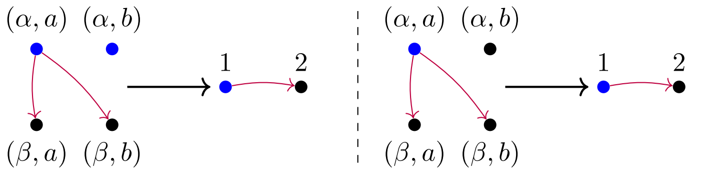


to find the set of edges in C^B. Suppose we have another two directed graphs B_2 and C_2.

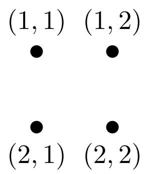
Then we have $A_2 \times B_2$ as shown below.



There are two possible ways to map from $A_2 \times B_2$ to C_2. That is, both edges have to be mapped to the single edge in C_2, and the node singled out can be mapped to either one of the nodes in C_2.



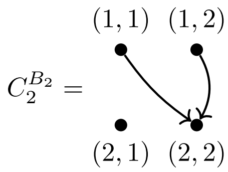
Moreover, according to the discussion on nodes in C^B, the set of nodes in $C_2^{B_2}$ should be as follows.



An edge in $C_2^{B_2}$ can be described as going from (a_s, b_s) to (a_t, b_t). Since the two edges in $A_2 \times B_2$ have to be mapped to the single edge in C_2, we need $a_s = 1, a_t = 2$ and $b_t = 2$. Meanwhile, b_s can be either 1 or 2. Hence, we get the table below.

a_s	b_s	a_t	b_t
1	1	2	2
1	2	2	2

This means, one of the two edge in $C_2^{B_2}$ should go from (1,1) to (2,2), and the other goes from (1,2) to (2,2).



Now we consider the general case. Suppose $A \in \mathscr{G}$ has nodes $V_A = \{a_1, a_2, \cdots\}$ and edges $E_A = \{\alpha_1, \alpha_2, \cdots\}$, $B \in \mathscr{G}$ has nodes $V_B = \{b_1, b_2, \cdots\}$ and edges $E_B = \{\beta_1, \beta_2, \cdots\}$, and $C \in \mathscr{G}$ has nodes $V_C = \{c_1, c_2, \cdots\}$ and edges $E_C = \{\gamma_1, \gamma_2, \cdots\}$. Again, $A, B, C \in \mathscr{G}$ can have infinitely many nodes and edges. However, if B has p nodes and C has q nodes , $p, q \in \mathbb{N}$, then C^B would have q^p nodes in the form of p-tuples $(b_1, b_2, \cdots, b_p)$, where $b_i \in V_C$ for $i \in \{1, 2, \cdots, p\}$. Generally, nodes and edges in C^B can be defined as follows. If $g : A \times B \to C$ is an arrow in $\mathscr{G}$, and $g(a_i, b_j) = c_k$ for some $a_i \in V_A, b_j \in V_B, c_k \in V_C$, then we can think of C^B as sending b_j to c_k. If an edge goes from b_i to b_j in B, and it can be mapped to an edge going from c_k to c_l in C by checking arrows in $\mathscr{G}(A \times B, C)$, then there is an edge in C^B going from $(b_1, b_2, \cdots, b_i, \cdots)$ to $(b_1, b_2, \cdots, b_j, \cdots)$ with $b_i = c_k$ and $b_j = c_l$. That is, if $g(\alpha_i, \beta_j) = \gamma_k$, where $\alpha_i \in E_A$, $\beta_j \in E_B$ and $\gamma_k \in E_C$, we can think of C^B as sending β_j to γ_k.

To see that this satisfies the definition of an exponential, we want to prove that $\mathscr{G}(A \times B, C) \cong \mathscr{G}(A, C^B)$ naturally in $A, C \in \mathscr{G}$. Let $f : A \times B \to C$ be an arrow in $\mathscr{G}$ for some nodes a, b and edges e_A, e_B in A, B respectively, define $\bar{f} : A \to C^B$ by

$$(\bar{f}(a))(b) = f(a, b),$$

and

$$(\bar{f}(e_A))(e_B) = f(e_A, e_B).$$

In the other direction, let $g : A \to C^B$ be an arrow in $\mathscr{G}$ for some nodes a and edges e_A in A, define $\bar{g} : A \times B \to C$ by

$$\bar{g}(a, b) = (g(a))(b),$$

and

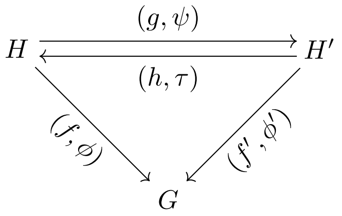
$$\bar{g}(e_A, e_B) = (g(e_A))(e_B)$$

for some nodes b and edges e_B in B. Then there is a canonical bijection between g and $\bar{g}$, and f and $\bar{f}$.

- Finally, we claim that a subobject of $\mathscr{G}$ consists of a selection of nodes and a selection of edges between the selected nodes.

To see this, let $G \in \mathscr{G}$ be a directed graph, and let $H \xrightarrow{(f,\phi)} G$ and $H' \xrightarrow{(f',\phi')}$ be monic in $\mathscr{G}$. We need to show that f and f', ϕ and ϕ' are isomorphic respectively if and only if they map H and H' to the same nodes and edges in G. It follows that we can identify the subobject of G with selected nodes and edges of G.

We first show the"only if" part. Suppose f and f' are isomorphic. Then there exists a bijective function $g : H \to H'$ that sends nodes in H to nodes in H' such that $f' \circ g = f$. That is for each node $v' \in H'$, there exists a unique $v \in H$ such that $g(v) = v'$. Let $v \in H$ and $v' \in H'$ be the corresponding node with respect to g. Then we have $f(v) = (f' \circ g)(v) = f'(g(v)) = f'(v')$. Thus, f and f' map nodes in H and H' to the same nodes in G. Similarly, suppose $\phi \cong \phi'$, then ϕ and ϕ' would map edges in H and H' to the edges between those nodes in G according to the definition of a directed graph. We now show the "if" part. Suppose (f, ϕ) and (f', ϕ') map H and H' to the same nodes and edges in G. Then since f and f', ϕ and ϕ' are monic, for each node $v \in G$ and each edge $e \in G$, there is precisely one corresponding node and one corresponding edge in H with respect to (f, ϕ) and in H' with respect to (f', ϕ'). This gives a bijective relationship between nodes and edges respectively in H and H'. We can therefore find $(g,\psi) \in \mathscr{G}(H,H')$ and $(h,\tau) \in \mathscr{G}(H',H)$ with $(gh,\psi\tau) = 1_H$ and $(hg,\tau\psi) = 1'_H$ such that the triangles



commutes. Thus, $f \cong f'$ and $\phi \cong \phi'$.

Now, a subobject classifier Ω of $\mathscr{G}$ would be described as follows.

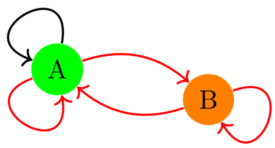
$\Omega =$ 

Figure 2: A subobject classifier $\Omega \in \mathscr{G}$ of the category **Directed Graph**.

As shown in Figure 2, it consists of two nodes A and B with two self-loops on node A, one self loop on node B, one edge from A to B, and one edge from B to A.

Let $G \in \mathscr{G}$. Since a subobject of G consists of selected nodes and edges between those nodes, an arrow $f : G \to \Omega$ sends selected nodes to the green nodes $A \in \Omega$ and selected edges to its black self-loop, and f sends the other nodes (unselected) to the orange node $B \in \Omega$. If an unselected edge goes from a selected node to a selected node, then it will be mapped to the orange self-loop on A. If an unselected edge goes from a selected node to an unselected node, it will be mapped to the orange edge from A to B. If an unselected edge goes from an unselected node to a selected node, it will be mapped to the orange edge from B to A. Lastly, if an unselected edge goes from an unselected node to an unselected node, it will be mapped to the orange self-loop on B. Clearly, $\{\text{subobjects of } G\} \cong \text{maps}\{G,\Omega\}$. □

Notice that we are not assuming that our graphs are finite. So it gives rise to the following question: does the full subcategory (consisting of a subclass of objects in $\mathscr{G}$ and all arrows between the selected objects) of finite directed graphs also form a topos? Easy to see, following the exact proof as above, we can prove that this question has a positive answer.

Theorem 12. *The category of finite directed graphs is a topos.*

As we pointed out in the introduction, directed graphs are extremely useful in our everyday life. Meanwhile, there are other notions of a graph of equal importance. There is a category of undirected graphs whose objects and arrows are similar to those in the category of directed graphs except the edges have no direction. Despite great similarities, the category of undirected graphs are significantly different from the category of directed graphs in the categorical sense. However, this is beyond the scope of this paper, although worth further investigation.

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