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The Galerkin Method

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Abstract

This report was written in December 2019 and January 2020 under supervision of Dr. Glen Wheeler. The report looks at results involving the existence and uniqueness of solutions to (potentially non-linear) evolution equations in Banach spaces, these results are more thoroughly discussed in [3] and [4]. Finally, concluding by applying these results to Sobolev spaces and using them to prove existence and uniqueness of weak solutions as well as convergence of Galerkin schemes for said equations.

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1 Introduction

This report focuses on the convergence of Galerkin schemes for variational formulations of partial differential equations (PDE) with several results regarding the existence and uniqueness of solutions to these problems. When looking at variational formulations of PDE, which are functional equations, any potential solutions must be in your function space - which in the case of PDE, is infinite dimensional. The Galerkin method involves taking a finite dimensional subspace of the function space, projecting the variational formulation onto said subspace, solving the projected problem then concluding by taking a limit and verifying it is in-fact a unique solution to the variational formulation. This process is standard in the case of linear problems, but much less so in the non-linear case.

Initially in the project I investigated linear parabolic equations in non-divergence form, using the Galerkin method to prove existence and uniqueness of a weak solution for both second and fourth order equations. These results carry on directly from [1] where they look at second order parabolic equations in divergence form. It also made use of estimates from [2]. This work has been excluded from the report due to page limitations.

Later into the project there was a large shift in focus towards existence and uniqueness theory for evolution equations in Banach spaces. This followed the work outlined in [3] and [4] to prove results on conditions required to ensure existence and uniqueness of solutions to a more general, and potentially non-linear, evolution equation. We will discuss this in the report as it is necessary for us to then later prove convergence of Galerkin sequences for these evolution equations if our space is Hilbert.

Then the report concludes by applying the existence and uniqueness theory in Banach spaces to Sobolev spaces where we can then prove convergence of Galerkin sequences to non-linear PDE. The idea is to apply the existence and uniqueness theorems to each projected problem allowing us to successfully take a limit.

2 Statement of Authorship

Wheeler decided upon the topic. Kato and Kastsatos developed the theory regarding abstract evolution equations and their weak solutions. Kartsatos proved the convergence of more general Galerkin schemes which Noble then used to examine a simpler equation. Noble collated these results and into a common frame, replaced certain hypothesis to simplify results and theory, and wrote the report supervised by Wheeler.

3 Evolution Equations in Banach Spaces

3.1 Introduction

Here we aim to prove existence and uniqueness to evolution equations by making assumptions on our operator and initial data. Using the results proven in this section, we can then later apply that to Sobolev Spaces and yield existence and uniqueness statements for potential non-linear PDE and their Galerkin sequences. Throughout this section we do not yet make use of the Galerkin Method, instead we follow the work done by Kato and Kartsatos [3], [4] and [5] where they make of the so called Yosida approximates to prove these results.

3.2 Crafting the Yosida Approximates

Throughout Section 3 we will be looking at equations of the form

$$u'(t) + A(t)u(t) = 0, \text{ for every } t \in [0, T], u(0) = a \tag{E}$$

where $T > 0$ is a constant, $a \in X$, $u : [0, T] \to X$, $A(t) : X \to X$ for every $t \in [0, T]$, and X is a real Banach space with uniformly convex dual X^*. We hope to prove existence and uniqueness of solutions to (E) by assuming that each operator $A(t)$ is m-accretive along side some regularity hypothesis on A.

Since X is uniformly convex, the duality map F is single valued and uniformly continuous on any bounded subset of X. For a proof of these results see [3]. We will write $\langle u, f\rangle$ for the dual pairing of $u \in X$ and $f \in X^*$. To begin, we start with a lemma.

Lemma 3.2.1.

Suppose that $u : [0, T] \to X$ is weakly differentiable at $s \in [0, T]$ with derivative $u'(s)$. That is,

$$\frac{d}{dt}\langle u(t), f\rangle\Big|_{t=s} = \langle u'(s), f\rangle, \text{ for every } f \in X^*.$$

Furthermore suppose that $t \mapsto \|u(t)\|_X$ is also differentiable at s. Then

$$\|u(t)\|_X \frac{d}{dt}\|u(t)\|_X\Big|_{t=s} = \Big\langle u'(s), F\big(u(s)\big)\Big\rangle, \tag{3.2.1}$$

where $F : X \to X^*$ is the duality map.

Proof. Fix $t_+ > s$. Since F is the duality map we have

$$\Big\langle u(t_+), F\big(u(s)\big)\Big\rangle \le \|u(t_+)\|_X \|F\big(u(s)\big)\|_{X^*} = \|u(t_+)\|_X \|u(s)\|_X, \text{ and,}$$
$$\Big\langle u(s), F\big(u(s)\big)\Big\rangle = \|u(s)\|^2_X.$$

We can subtract these two to give

$$\Big\langle u(t_+) - u(s), F\big(u(s)\big)\Big\rangle \leq \|u(s)\|_X \Big(\|u(t_+)\|_X - \|u(s)\|_X\Big).$$

Dividing through by $t_+ - s$ and taking a limit as $t_+ - s \to 0^+$ allows us to obtain the estimate:

$$\Big\langle u'(s), F\big(u(s)\big)\Big\rangle \leq \|u(s)\|_X \frac{d}{dt}\|u(t)\|_X\Big|_{t=s}.$$

Doing the same trick but by fixing some $t_- < s$ yields

$$\Big\langle u'(s), F\big(u(s)\big)\Big\rangle \geq \|u(s)\|_X \frac{d}{dt}\|u(t)\|_X\Big|_{t=s}.$$

This implies equality! □

Now we can get into defining the Yosida approximates A_n. Given an m-accretive operator A we define for $n = 1, \dots,$ the operators

$$J_n := \big(I + \frac{1}{n}A\big)^{-1}, \tag{3.2.2}$$

$$A_n := AJ_n = n\big(I - J_n\big). \tag{3.2.3}$$

Both J_n and A_n just so happen to be defined everywhere on X since A is m-accretive. We can verify the second equality on (3.2.3) by calculating

$$\begin{aligned}
A_n J_n^{-1} &= A, \text{ by definition.}\\
A_n\big(I + \frac{1}{n}A\big) &= n\big(\frac{1}{n}A\big), \text{ by definition.}\\
A_n &= n\big(\frac{1}{n}A + I - I\big)\big(I + \frac{1}{n}A\big)^{-1},\\
&= n\big(I - (I + \frac{1}{n}A)^{-1}\big),\\
&= n(I - J_n), \text{ as previously claimed.}
\end{aligned}$$

From here the goal becomes to look at several properties of these operators, then apply these properties to attain our existence and uniqueness statements for (E)! This approach follows the work done in [3] and [4].

Lemma 3.2.2.

Suppose that $A : X \to X$ is an m-accretive operator. Then then J_n and A_n defined by (3.2.2) *and* (3.2.3) *are uniformly Lipschitz in X. Furthermore, for each n, J_n is 1-Lipschitz and A_n is $2n$-Lipschitz.*

Proof. Fix $n \in \mathbb{N}\backslash\{0\}$. First we will show that the J_n are 1-Lipschitz. Since A is m-accretive we have

$$\|(I + \alpha A)x - (I + \alpha A)y\|_X \geq \|x - y\|_X, \text{ for every } x, y \in X, \text{ and every } \alpha > 0.$$

Put $U = I + \alpha A$ and $x - y = u$. We will prove that $\|U^{-1}u\| \leq \|u\|_X$ for every $u \in X$, which then would imply that J_n is 1-Lipschitz. Suppose for contradiction that there exists some $\tilde{u} \in X$ so that $\|U^{-1}\tilde{u}\| > \|\tilde{u}\|_X$. Then we have

$$\|UU^{-1}\tilde{u}\|_X \geq \|U^{-1}\tilde{u}\|_X > \|\tilde{u}\|_X,$$

since $\|Uu\|_X \geq \|u\|_X$ for every u by the fact that A is m-accretive. However $\|UU^{-1}\tilde{u}\|_X = \|\tilde{u}\|_X$. This gives our contradiction, and so J_n is 1-Lipschitz! Next to show that A_n is $2n$-Lipschitz we calculate for $x, y \in X$:

$$\begin{aligned} \|A_n x - A_n y\|_X &= \|n(I - J_n)x - n(I - J_n)y\|_X, \text{ by definition.} \\ &\leq n\|x - y\|_X + n\|J_n x - j_n y\|_X, \text{ by the triangle inequality.} \\ &\leq 2n\|x - y\|_X, \text{ since } J_n \text{ is 1-Lipschitz.} \end{aligned}$$

□

Lemma 3.2.3.

Let A be m-accretive. Then each A_n is m-accretive with

$$\|A_n u\|_X \leq \|Au\|_X, \text{ for every } u \in X. \tag{3.2.4}$$

Proof. Fix $x, y \in X$. We calculate:

$$\begin{aligned} \Big\langle A_n x - A_n y, F(x - y)\Big\rangle &= n\langle x - y, F(x - y)\rangle - n\langle J_n x - J_n y, F(x - y)\rangle, \\ &\geq n\|x - y\|_X^2 - n\|x - y\|_X\|x - y\|_X, \text{ since } J_n \text{ is 1-Lipschitz.} \\ &= 0, \end{aligned}$$

which is equivalent to A_n being m-accretive. Now if $u \in X$, we have:

$$A_n u = n(u - J_n u) = n\left[J_n\Big(I + \frac{1}{n}A\Big)u - J_n u\right].$$

And since J_n is 1-Lipschitz we have

$$\|A_n u\|_X \leq n\left\|(I + \frac{1}{n}A)u - u\right\|_X \leq \|Au\|_X.$$

□

Lemma 3.2.4.

If $u \in \overline{\mathrm{Dom}(A)}$ then $J_n u \to_X u$ as $n \to \infty$.

Proof. The proof relies on the fact that the $\|A_n u\|_X$ satisfy the estimate (3.2.4) and that the J_n are uniformly Lipschitz. For the details see [3]. □

Lemma 3.2.5. *Suppose A is m-accretive. Then:*

i) If $u_n \in \mathrm{Dom}(A)$ for every $n \in \mathbb{N}\backslash\{0\}$ with $u_n \to_X u$ for some $u \in X$, and if $\|Au_n\|_X$ is bounded for every $n \in \mathbb{N}\backslash\{0\}$, then $u \in \mathrm{Dom}(A)$ and $Au_n \rightharpoonup_X Au$ weakly with respect to the dual pairing.

ii) If each $u_n \in X$ with $u_n \to_X u$ for every $n \in \mathbb{N}\backslash\{0\}$ with $u_n \to_X u$ for some $u \in X$, and if $\|A_n u_n\|_X$ is bounded for every $n \in \mathbb{N}\backslash\{0\}$, then $u \in \mathrm{Dom}(A)$ and $A_n u_n \rightharpoonup_X Au$ weakly with respect to the dual pairing.

iii) If $u \in \mathrm{Dom}(A)$ then $A_n u \rightharpoonup_X Au$ weakly with respect to the dual pairing.

Proof. First lets prove (i). Since A is m-accretive we have

$$\langle Av - Au_n, F(v - u_n)\rangle \geq 0, \text{ for all } v \in \mathrm{Dom}(A). \tag{3.2.5}$$

Since X is reflexive with X^* and $\|Au_n\|_X$ is bounded, there is a subsequence $\{u_{n_k}\}_{k\geq 1}$ with $Au_{n_k} \rightharpoonup x$ for some $x \in X$. Since F is continuous, $F(v - u_{n_k}) \to F(v - x)$. Applying (3.2.5) gives

$$\langle Av - x, F(v - u)\rangle \geq 0.$$

Next we make use of Lemma 1.1 from [3] to see that

$$\|(u - v) + (Av - x)\|_X \geq \|v - u\|_X.$$

Then simply put $v = J_1(u + x)$ to see that $\|u - v\|_X = 0$ which implies that $Au = x$ and proves that $Au_{n_k} \rightharpoonup Au$. But $\{u_n\}_{n\geq 1}$ converges strongly to u by hypothesis, so we can apply the same argument to any given subsequence. This proves (i). Next we prove (ii). Write $u_n = J_n x_n$, so that $x_n - u_n = (I - J_n)x_n = \frac{1}{n} A_n x_n \to 0$ by (3.2.3). Furthermore, since $Au_n = A_n x_n$ and $\|Au_n\|_X$ is bounded, we can just apply (i) to prove (ii). Finally to prove (iii), we just apply (ii) but the sequence $\{x_n\}_{n\geq 1}$ defined by $x_n = u$. □

3.3 Existence and Uniqueness Results

From here on-wards we will be assuming the following conditions on A:

(a) The domain, D, of $A(t)$ is independent of t.

(b) There exists a constant $L > 0$ so that

$$\|A(t)v - A(s)v\|_X \leq L|t - s|\big(1 + \|v\|_X + \|A(t)v\|_X\big), \tag{3.3.1}$$

for every $v \in D$ and every $s, t \in [0, T]$.

(c) For every $t \in [0, T]$, $A(t)$ is m-accretive.

We aim to prove existence and uniqueness of a solution to (E) by making use of the Yosida approximates and considering the approximating equation

$$u_n'(t) + A_n(t)u_n(t) = 0, \ u_n(0) = a. \tag{E_n}$$

Lemma 3.3.1.

For every $n \geq 1$, $v \in D$, and $s, t \in [0, T]$ we have:

$$\|A_n(t)v - A_n(s)v\|_X \leq L|t-s|\Big[1 + \|v\|_X + \Big(1 + \frac{1}{n}\Big)\|A_n v\|_X\Big]. \tag{3.3.2}$$

Proof. Using the definition of $A_n(t)$ we calculate:

$$\begin{aligned} A_n(t)v - A_n(s)v &= nJ_n(s)v - nJ_n(t)v \\ &= \Big(J_n(t)\Big(I + \frac{1}{n}A(t)\Big)\Big)J_n(s)v - \Big(J_n(s)\Big(I + \frac{1}{n}A(s)\Big)\Big)J_n(t)v. \end{aligned}$$

Then using the fact that $J_n(t)$ is 1-Lipschitz we have

$$\begin{aligned} \|A_n(t)v - A_n(s)v\|_X &\leq \Big\|\Big(\Big(I + \frac{1}{n}A(t)\Big)\Big)J_n(s)v - \Big(\Big(I + \frac{1}{n}A(s)\Big)\Big)J_n(t)v\Big\|_X, \\ &\leq \Big\|\Big(A(t) - A(s)\Big)J_n(s)v\Big\|_X. \end{aligned}$$

Then using condition (b) we have

$$\begin{aligned} \|A_n(t)v - A_n(s)v\|_X &\leq L|t-s|\Big[1 + \|J_n(s)v\|_X + \|A(s)J_n(s)v\|_X\Big], \\ &= L|t-s|\Big[1 + \|J_n(s)v\|_X + \|A_n(s)v\|_X\Big]. \end{aligned}$$

Furthermore, $\|J_n(s)v\|_X \leq \|v\|_X + \frac{1}{n}\|A_n(s)v\|_X$, applying this estimate concludes the proof. □

The purpose of this lemma is that it shows $t \mapsto A_n(t)v$ is Lipschitz for each $v \in X$. This is all we need for (E_n) to have a unique solution. Now we will prove some estimates on these solutions and attempt to prove convergence.

Lemma 3.3.2.

Let $a \in D$. Then there exists a constant $K > 0$ so that

$$\|u_n(t)\|_X \leq K, \ \textit{and}, \ \|u_n'(t)\|_X = \|A_n(t)u_n(t)\|_X \leq K,$$

for all $n \geq 1$.

Proof. See Appendix 1. □

Great! We have boundedness, now we just need to take a limit.

Lemma 3.3.3.

The strong limit $u(t)$ of the sequence $\{u_n(t)\}_{n\geq 1}$ exists uniformly for $t \in [0,T]$ and is Lipschitz with $u(0) = a$.

Proof. See Appendix 2. □

Lemma 3.3.4.

The limit u satisfies $u(t) \in D$ for every $t \in [0,T]$, and $A(t)u(t)$ is bounded as well as weakly continuous.

Proof. For every time t, we have $u_n(t) \to u(t)$ as well as $\|A_n(t)u_n(t)\|_X \leq K$. Lemma 3.2.5 (ii) implies that $u(t) \in D$ and $A_n(t)u_n(t) \rightharpoonup A(t)u(t)$. Hence we must also have $\|A(t)u(t)\|_X \leq K$. Now all that remains is to prove $A(t)u(t)$ is weakly continuous.

Fix $t \in [0,T]$ and $\{t_k\}_{k\geq 1}$ be a sequence with $t_k \to t$ as $k \to \infty$. We calculate:

$$\begin{aligned} \|[A(t) - A(t_k)]u(t_k)\|_X &\leq L|t - t_k|\big[1 + \|u(t_k)\|_X + \|A(t_k)u(t_k)\|_X\big], \text{ by condition (b).} \\ &\leq L|t - t_k|(1 + 2K), \text{ by Lemma 3.3.2.} \end{aligned} \tag{3.3.3}$$

This shows that

$$\limsup_{k\to\infty} \|A(t)u(t_k)\|_X = \limsup_{k\to\infty} \|A(t_k)u(t_k)\|_X \leq K.$$

Now we can apply Lemma 3.2.5 (i) to see that $A(t)u(t_k) \rightharpoonup A(t)u(t)$. Then (3.3.3) shows that $A(t_k)u(t_k) \rightharpoonup A(t)u(t)$ weakly, which gives weak continuity. □

Now to conclude all we need for existence, we just need a statement on the differentiability of the limit.

Lemma 3.3.5.

The limit u is weakly continuously differentiable. That is, for every functional $f \in X^*$, $t \mapsto \langle u(t), f\rangle$ is continuously differentiable with $\frac{d}{dt}\langle u(t), f\rangle = -\langle A(t)u(t), f\rangle$.

Proof. See Appendix 3. □

Fantastic! Now we just need to prove uniqueness and we are happy!

Lemma 3.3.6.

Let u and v be two solutions to (E) *except with $u(0) = a$ and $v(0) = b$ for some $a, b \in D$. Then the following estimate holds for almost every $t \in [0,T]$:*

$$\|u(t) - v(t)\|_X \leq \|a - b\|_X.$$

Proof. See Appendix 4. □

Which finished the question of uniqueness!

Theorem 3.3.7. *There exists a unique* $u : [0, T] \to X$ *which solves* (E) *weakly. That is, there is a map* $u : [0, T] \to X$ *which is unique up to sets of measure zero in* $[0, T]$*, weakly continuously differentiable, has* $u(t) \in D$ *for every* $t \in [0, T]$*, and solves the weak formulation:*

$$\begin{cases} \frac{d}{dt}\langle u(t), f\rangle = -\langle A(t)u(t), f\rangle, \text{ for almost every } t \in [0, T], \\ u(0) = a. \end{cases}$$

Proof. All the hard work has been done already! See Lemmata 3.3.1-3.3.6. □

3.4 Closing Remarks

Throughout Section 3 we never made use of the Galerkin method, however the purpose of the section is to first prove this existence and uniqueness theorem . Then using this result we hope to prove convergence of Galerkin sequences in Sobolev spaces for weak formulations of non-linear PDE. This section very closely followed the work outlined in [3] and the author goes on in [4] to prove a similar Theorem but with different hypothesis. The main difference being that a different estimate in place of hypothesis (b) made in the beginning of Section 3.3. Throughout [4] Kartsatos also goes into further detail on the regularity of the solution from Theorem 3.3.7.

4 Non-linear Galerkin Methods

4.1 Introduction

In this section the goal shifts to applying the work done in Section 3 to Sobolev spaces, and using it to not only prove existence and uniqueness of solutions but also to prove convergence of Galerkin sequences. We are going to be investigating problems of the form:

$$\begin{cases} u'(t) + A(t)u(t) = 0, \text{ in } [0, T], \\ u(0) = a \in X. \end{cases} \tag{E}$$

where each $A(t) : X \to X$ (X is Banach with a uniformly convex dual), and we want to find a $u : [0, T] \to X$ to solve the equation.

We will be making some slightly different hypothesis on A than that which were imposed in Section 3 but the goal is to show that we can still apply Theorem 3.3.7 to get existence and uniqueness. Following this, we look at the case where X is Hilbert, and apply the Galerkin method by projecting E onto an n-dimensional subspace of X, X_n. This will yield the equations

$$\begin{cases} u_n'(t) + P_n A(t) u_n(t) = 0, \text{ in } [0,T], \\ u_n(0) = P_n a. \end{cases} \tag{P_nE}$$

Here P_n is an orthogonal projection from X onto X_n. Then we hope to apply Theorem 3.3.7 to each projected problem to give existence and uniqueness for each n, and then we conclude the section by proving convergence of these Galerkin approximates.

In doing this we will be following the work of [5] and [6], using variations on the hypotheses from [5]:

(I) The domain, D, of $A(t)$ is independent of t.

(II) There exists a constant $L > 0$ so that

$$\|A(t)v - A(s)v\|_X \leq L|t-s|\big(1 + \|v\|_X + \|A(t)v\|_X\big),$$

for every $v \in D$ and every $s, t \in [0,T]$.

(III) For every $t \in [0,T]$, $A(t)$ is m-accretive.

(IV) X is also a separable Hilbert space and for every $n \geq 1$ and $t \in [0,T]$ we have

$$\big(I + P_n A(t)\big) D \cap X_n = X_n, \text{ for each } t,\ A(t)P_n x \to_X A(t)x \text{ for every } x \in D, \text{ and } P_n D \subset D.$$

(V) We have that $A(0)$ maps bounded subsets of D into bounded subsets of X.

The difference between this set of assumptions and those made in Section 3.3 is simply the addition of (IV) and (V) which we will need to prove convergence of the Galerkin sequence.

4.2 The Main Result

Theorem 4.2.1.
Suppose the hypothesis (I), (II), (III), (IV) and (V) all hold. Then the Galerkin sequence, $\{u_n(t)\}_{n\geq 1}$, of solutions to (P_nE) exists and converges strongly and uniformly to the unique solution $u(t)$ from Theorem 3.3.7.

Proof. Since X is a separable Hilbert space with uniformly convex dual, we have $\langle x, F(y)\rangle = \langle x, y\rangle_X$ for all $x, y \in X$ where $\langle\cdot,\cdot\rangle$ is the dual pairing and $\langle\cdot,\cdot\rangle_X$ is the inner product in X. We can immediately apply Theorem 3.3.7 to get existence and uniqueness of a solution to (E). One can verify, as in [5], that for every $n \geq 1$ Theorem 3.3.7 also guarantee's existence and uniqueness of a solution to (P_nE). This is because $P_nA(t)$ is also m-accretive and satisfies (b). Now all that remains is to check that the solution to (P_nE) converges to that of (E).

We take an inner product in both (E) and (P_nE) and put $v_n(t) = P_nx(t)$, to yield:

$$\langle x_n'(t), x_n(t) - v_n(t)\rangle_X + \langle P_nA(t)x_n(t), x_n(t) - v_n(t)\rangle_X = 0, \tag{4.2.1}$$

$$\langle x'(t), x_n(t) - v_n(t)\rangle_X + \langle A(t)x_n(t), x_n(t) - v_n(t)\rangle_X = 0. \tag{4.2.2}$$

Since P_n is an orthogonal projection, it is self-adjoint, and we know that $P_n\big(x_n - v_n(t)\big) = x_n - v_n(t)$. Using this we can simplify (4.2.1):

$$\langle x_n'(t), x_n(t) - v_n(t)\rangle_X + \langle A(t)x_n(t), x_n(t) - v_n(t)\rangle_X = 0. \tag{4.2.3}$$

Next we calculate (4.2.3) - (4.2.2):

$$\langle x_n'(t) - x'(t), x_n(t) - v_n(t)\rangle_X + \langle A(t)x_n(t) - A(t)x(t), x_n(t) - v_n(t)\rangle_X = 0.$$

We add zero in a few crafty ways to obtain the following estimates

$$\begin{aligned}
\langle x_n'(t) - v_n'(t), x_n(t) - v_n(t)\rangle_X =& \Big\langle x'(t) - v_n'(t), x_n(t) - v_n(t)\Big\rangle_X \\
&- \Big\langle A(t)x_n(t) - A(t)v_n(t), x_n(t) - v_n(t)\Big\rangle_X \\
&- \Big\langle A(t)v_n(t) - A(t)x(t), x_n(t) - v_n(t)\Big\rangle_X \\
\leq& \Big\langle x'(t) - v_n'(t), x_n(t) - v_n(t)\Big\rangle_X \\
&- \Big\langle A(t)v_n(t) - A(t)x(t), x_n(t) - v_n(t)\Big\rangle_X \\
\langle x_n'(t) - v_n'(t), x_n(t) - v_n(t)\rangle_X \leq& \frac{1}{2}\|x'(t) - v_n'(t)\|_X^2 + \frac{1}{2}\|x_n(t) - v_n(t)\|_X^2 \qquad (4.2.4)\\
&+ \frac{1}{2}\|A(t)v_n(t) - A(t)x(t)\|_X^2 + \frac{1}{2}\|x_n(t) - v_n(t)\|_X^2.
\end{aligned}$$

Here we used the fact that $A(t)$ is m-accretive so $\Big\langle A(t)x_n(t) - A(t)v_n(t), x_n(t) - v_n(t)\Big\rangle_X \geq 0$ and then applied Young's inequality. To make further use of (4.2.4) we must first show that $\{x_n(t)\}_{n\geq 1}$

and $\{x_n'(t)\}_{n\geq 1}$ are uniformly bounded on $[0,T]$. We calculate:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|x_n(t)-P_na\|_X^2 =&\langle x_n'(t), x_n(t)-P_na\rangle_X,\\ =&-\langle A(t)x_n(t), x_n(t)-P_na\rangle_X, \text{ since } P_n \text{ is self-adjoint.}\\ =&-\langle A(t)x_n(t)-A(t)P_na, x_n(t)-P_na\rangle - \langle A(t)P_na, x_n(t)-P_na\rangle,\\ \leq&\|A(t)P_na\|_X\|x_n(t)-P_na\|_X, \text{ since } A(t) \text{ is } m\text{-accretive.}\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\|x_n(t)-P_na\|_X^2 \leq\Big[\|A(0)P_na\|_X + LT\Big(1+\|P_na\|_X+\|A(0)P_na\|_X\Big)\Big]\|x_n(t)-P_na\|_X. \quad (4.2.5)$$

In that last line we simply made use of the hypothesis (II) with $s=t$ and $t=0$, then applied the reverse triangle inequality and the fact that $t\leq T$. In-fact, we know that we can write $\frac{1}{2}\frac{d}{dt}\|x_n(t)-P_na\|_X^2 = \|x_n(t)-P_na\|_X\frac{d}{dt}\|x_n(t)-P_na\|_X$. Substituting this into (4.2.5), and dividing by $\|x_n(t)-P_na\|_X$ gives for almost every t:

$$\frac{d}{dt}\|x_n(t)-P_na\|_X \leq \|A(0)P_na\|_X + LT\Big(1+\|P_na\|_X+\|A(0)P_na\|_X\Big).$$

From here we simply integrate to find that $x_n(t)$ is uniformly bounded. Now we look back at (4.2.4). We have that $\|x'(t)-v_n'(t)\|_X \leq \|x'(t)\|_X + \|v_n'(t)\|_X \leq 2\|x'(t)\|_X \leq K$ for some $K>0$ by weak continuity of $x'(t)$ and since $v_n'(t)\to_X x'(t)$ uniformly on $[0,T]$. Furthermore we have that $A(t)v_n(t)\to_X A(t)x(t)$ and $\|A(t)v_n(t)-A(t)x(t)\|_X$ is bounded since $A(t)$ is Lipschitz and $A(0)$ is bounded. This allows us to use (4.2.5) to obtain the inequality

$$\|x_n(t)-v_n(t)\|_X^2 \leq C\int_0^T\Big[\|x'(t)-v_n'(t)\|_X^2 + \|A(t)v_n(t)-A(t)x(t)\|_X^2\, dt,$$

and apply the dominated convergence theorem to give that $x_n(t)-v_n(t)\to_X 0$ uniformly. This proves that $x_n(t)\to x(t)$ uniformly on $[0,T]$. □

4.3 Closing Remarks

The purpose here in proving that the Galerkin sequence is uniformly convergent is that it can open horizons for estimates on our solutions as well as numerical schemes. Estimates give nice information about the properties of your solution, but can also be of use in the jump from a weak solution to a strong solution. It's worth noting that we simply projected onto any n-dimensional subspace X_n, so any basis can be chosen to generate the Galerkin sequence.

The frame-work here can then simply be applied to any $H^k(U)$ (provided some mild assumptions on U) to obtain existence and uniqueness of solutions, as well as convergence of Galerkin sequences, for non-linear PDE.

In this Theorem, the hypothesis (II) and (III) are potentially the most restrictive, however they're absolutely essential to this argument. Although, if the operators $A(t)$ are independent of t, they will automatically satisfy (II). In [5] and [6] they go on to prove similar statements but with weaker estimates in place of (II) and assuming an inhomogeneity term in (E). When doing this, their arguments remain relatively unchanged except they also need to appropriately bound terms that arise due to the inhomogeneity.

5 Discussion and Conclusion

5.1 Conclusion

Throughout the report we investigated two contexts in which the Galerkin method arises. The first being for linear equations as a method for proving existence and uniqueness to weak formulations, and the second where we proved convergence of the Galerkin schemes using already proven existence and uniqueness results for non-linear evolution equations.

Then we proved existence and uniqueness theorems for more general evolution equations, and applied those to prove convergence of certain Galerkin sequences for similar evolution equations. We never assumed the operator was linear here, instead we relied on specific estimates and the function space we were working in. This theory immediately applies to variational formulations of non-linear PDE in Sobolev spaces. The hypotheses here were much stricter than in the linear case, and can in-fact be weakened.

5.2 Future Developments

A future study would be interesting into how the hypothesis throughout can be tweaked, however what would be even more interesting (for myself at least) is finding what conditions are necessary to jump from a weak solution to a strong solution. All existence, uniqueness, and Galerkin scheme convergence theorems were for solutions to the weak formulation of a PDE. I'd like to see what assumptions are necessary to extend this frame-work so that we can yield results on classical solutions to the PDE.

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Appendix

Appendix 1

Lemma 3.2.1 tells us that

$$\|x(s)\|_X \frac{d}{ds}\|x(s)\|_X = \langle x'(s), F\big(x(s)\big)\rangle.$$

We put $x_n(t) = u_n(t+h) - u_n(t)$ to yield

$$\|x_n(t)\|_X \frac{d}{dt}\|x_n(t)\|_X = -\Big\langle A_n(t+h)u_n(t+h) - A_n(t)u_n(t), F\big(x_n(t)\big)\Big\rangle, \tag{A1.1}$$

for every $t \in [0,T]$ where $\|x_n(t)\|_X$ is differentiable. The first term in the right hand side of (A1.1) can we written as

$$\begin{aligned} A_n(t+h)u_n(t+h) - A_n(t)u_n(t) =& \Big[A_n(t+h)u_n(t+h) - A_n(t+h)u_n(t)\Big] \\ &+ \Big[A_n(t+h)u_n(t) - A_n(t)u_n(t)\Big]. \end{aligned}$$

Since $A_n(t)$ and $A_n(t+h)$ are m-accretive by Lemma 3.2.3, we have

$$\Big\langle A_n(t+h)u_n(t+h) - A_n(t+h)u_n(t), F\big(x_n(t)\big)\Big\rangle \geq 0$$

which allows us to estimate

$$\|x_n(t)\|_X \frac{d}{dt}\|x_n(t)\|_X \leq -\Big\langle A_n(t+h)u_n(t) - A_n(t)u_n(t), F\big(x_n(t)\big)\Big\rangle.$$

Next we apply the inequality from Lemma 3.3.1 to find

$$\|x_n(t)\|_X \frac{d}{dt}\|x_n(t)\|_X \leq Lh\Big[1 + \|u_n(t)\|_X + \big(1+\frac{1}{n}\big)\|u_n'(t)\|_X\Big]\|x_n(t)\|_X.$$

Using the same argument as in [3] one can show that we actually have

$$\frac{d}{dt}\|x_n(t)\|_X \leq Lh\Big[1 + \|u_n(t)\|_X + \big(1+\frac{1}{n}\big)\|u_n'(t)\|_X\Big],$$

for almost every $t \in [0,T]$. The using the absolute continuity of $t \mapsto \|x_n(t)\|_X$ we estimate

$$\|x_n(0)\|_X + Lh\int_0^t \Big[1 + \|u_n(s)\|_X + \big(1+\frac{1}{n}\big)\|u_n'(s)\|_X\Big]\,ds.$$

Now dividing through by h and letting it approach 0 from above yields

$$\|u_n'(t)\|_X \leq \|u_n'(0)\|_X + L\int_0^t \Big[1 + \|u_n(s)\|_X + \big(1+\frac{1}{n}\big)\|u_n'(s)\|_X\Big]\,ds.$$

However, $\|u_n'(0)\|_X = \|A_n(0)a\| \le \|A(0)a\|_X$. This gives

$$\|u_n'(t)\|_X \le C_1 + L\int_0^t \Big[1 + \|u_n(s)\|_X + \big(1+\frac{1}{n}\big)\|u_n'(s)\|_X\Big]\,ds,$$

for some $C_1 > 0$. And since $u_n(t) = a + \int_0^t u_n'(s)\,ds$ we apply the triangle inequality, then add to the previous estimate to yield

$$\|u_n(t)\|_X + \|u_n'(t)\|_X \le C_2 + C_3\int_0^t \Big[\|u_n(s)\|_X + \big(1+\frac{1}{n}\big)\|u_n'(s)\|_X\Big]\,ds.$$

Solving this inequality proves existence of a $K > 0$ such that $\|u_n(t)\|_X < K$, and $\|u_n'(t)\|_X < K$.

Appendix 2

We apply Lemma 3.2.3 to $x_{mn}(t) = u_m(t) - u_n(t)$. As with before we obtain for almost every $t \in [0, T]$,

$$\frac{1}{2}\frac{d}{dt}\|x_{mn}(t)\|_X^2 = -\Big\langle A_m(t)u_m(t) - A_n(t)u_n(t), F\big(x_{mn}(t)\big)\Big\rangle. \tag{A2.1}$$

We use the fact that $A_m(t)u_m(t) = A(t)J_m(t)u_m(t)$ and $A(t)$ is m-accretive to estimate

$$\Big\langle A_m(t)u_m(t) - A_n(t)u_n(t), F\big(y_{mn}(t)\big)\Big\rangle \ge 0, \tag{A2.2}$$

where $y_{mn}(t) = J_m(t)u_m(t) - J_n(t)u_n(t)$. Next simply add (A2.1) and (A2.2) to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|x_{mn}(t)\|_X^2 &\le \Big\langle A_m(t)u_m(t) - A_n(t)u_n(t), F\big(y_{mn}(t)\big) - F\big(x_{mn}(t)\big)\Big\rangle,\\ &\le 2K\|F\big(y_{mn}(t)\big) - F\big(x_{mn}(t)\big)\|_X, \text{ for almost every } t \in [0, T],\end{aligned}$$

by Lemma 3.3.2. Similarly to before, we have that $\|x_{mn}(t)\|_X^2$ is absolutely continuous and $x_{mn}(0) = 0$, so we can integrate from 0 to t and find

$$\|x_{mn}(t)\|_X^2 \le 4K\int_0^t \|F\big(y_{mn}(s)\big) - F\big(x_{mn}(s)\big)\|_X\,ds.$$

So all that remains is to show that the integrand converges to 0 uniformly in s. First; we know $\|x_{mn}(s)\|_X$ is bounded by Lemma 3.3.2. Furthermore,

$$\begin{aligned}\|y_{mn}(s) - x_{mn}(s)\|_X &\le \|J_M(s)u_m(s) - u_m(s)\|_X + \|J_n(s)u_n(s) - u_n(s)\|_X,\\ &= \frac{1}{m}\|A_m(s)u_m(s)\|_X + \frac{1}{n}\|A_n(s)u_n(s)\|_X,\\ &\le \Big(\frac{1}{m} + \frac{1}{n}\Big)C, \text{ by Lemma 3.3.2.}\end{aligned}$$

This expression vanishes for any $s \in [0, T]$! And by uniform continuity of the duality map, given any $\varepsilon > 0$, we can pick m, n sufficiently large so that $\|F\big(y_{mn}(s)\big) - F\big(x_{mn}(s)\big)\|_X < \varepsilon$ for any $s \in [0, T]$. Thus the strong limit $u(t)$ exists uniformly in t. And, since the $u_n'(t)$ are bounded in X, we have that the $t \mapsto u_n(t)$ are Lipschitz continuous uniformly in t and n. From here it is easy to see that this implies u is also Lipschitz continuous with $u(0) = a$.

Appendix 3

Since u_n solves (E_n) we have

$$\langle u_n(t), f\rangle = \langle a, f\rangle - \int_0^t \langle A_n(s)u_n(s), f\rangle\, ds.$$

Since $u_n(t) \to_X u(t)$, $A_n(s)u_n(s) \rightharpoonup A(s)u(s)$ weakly and $|\langle A_n(s)u_n(s), f\rangle| \leq K\|f\|_X$ we can pass through the limit and obtain

$$\langle u(t), f\rangle = \langle a, f\rangle - \int_0^t \langle A(s)u(s), f\rangle\, ds.$$

The integrand here is continuous in s (see Lemma 3.3.4), so we have continuous differentiability.

Appendix 4

To make things simple define $x(t) := u(t) - v(t)$. By linearity, x is weakly differentiable with $x'(t) = -A(t)u(t) + A(t)v(t)$. Which, as we saw in Lemma 3.3.5 is weakly continuous and bounded. Thus $x(t)$ is Lipschitz and hence $t \mapsto \|x(t)\|_X$ is differentiable at almost every $t \in [0, T]$. We then calculate:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_X^2 = -\langle A(t)u(t) - A(t)v(t), F\big(x(t)\big)\rangle \leq 0$$

for almost every $t \in [0, T]$. Hence $t \mapsto \|x(t)\|_X$ is decreasing and thus

$$\|x(t)\|_X^2 \leq \|x(0)\|_X^2 = \|a - b\|_X, \text{ for almost every } t \in [0, T].$$