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Interface Stability and Interplay of Surface Tension and Inertial Stabilisation Mechanisms

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1 Abstract

Interfacial dynamics is a modern, highly applicable branch of physical mathematics that is developing at a rapid rate. The aim of the project is to advance knowledge in interfacial dynamics by investigating how surface tension couples with other stabilisation mechanisms (eg. the inertial stabilisation mechanism). It was confirmed that an increasing value of surface tension T had stabilising effects while decreasing T sufficiently far lead to destabilisation. The effects of surface tension were investigated in 3 key cases which were named conservative dynamics, Rayleigh-Taylor dynamics, and Landau-Darrieus dynamics.

2 Introduction

Recent progress in the study of interfacial dynamics has been made by applying group theory and the general matrix method [1-4], resulting in the discovery that the dynamics can be stabilised by inertial effects (motion and displacement of the heavy and light fluid). The effects of surface tension have not yet been considered in details in this context. Surface tension is a force acting between two fluids at their interface. The magnitude and direction of the tension force are dependent on the properties of the fluids. To summarise, the main goal of this research project was to develop a mathematical, and physical understanding of interfacial dynamics and in particular, investigate the mechanisms of both stabilisation and destabilisation.

2.1 The Set-up

The set up basically consists of a heavy fluid with mass density ρ_h on top of a lighter fluid with mass density ρ_l as pictured in Figure 1.

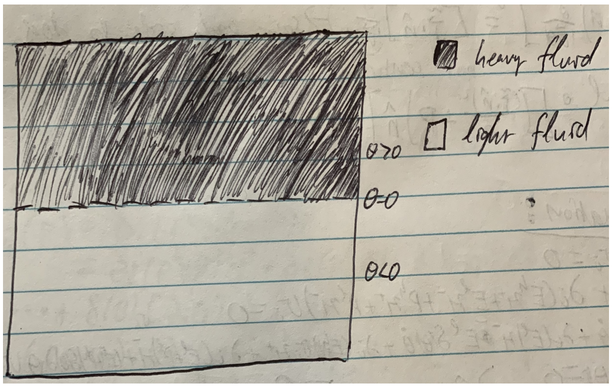

Figure 1

We call the boundary where the two fluids (can be plasmas or other materials) meet, the interface. At the interface, many interesting phenomena are observed. Interfacial mixing and transport across the interface are very complex nonlinear processes it occurs in systems that exist in a state that is far from equilibrium. This interfacial dynamical behaviour is observed across a huge range of length scales (nanometers in plasma fusion to gigametres in supernovae) in a myriad of different kinds of systems.

2.2 Why is This Important?

While applications in the real world were not the goal or motivation for this project, it should still be noted that there are many valuable applications. Some examples of these applications include: inertial confinement fusion, supernovae, mantle-lithosphere tectonics in geology, and liquefied natural gas stability during transportation [1].

2.3 The Report

Only one boundary condition will be derived as they are all derived using the same ideas and techniques. Other results were attained by using Mathematica 12 software. In terms of background for understanding the project, resources [1-4] should be sufficient but to my knowledge there is no text that describes the subject matter in detail.

Statement of Authorship

Results, Mathematica code, and the report were written by me and proofread by Snezhana I. Abarzhi who also provided a lot of assistance in conceptual understanding.

3 The Physical Scenario

Recall the set-up depicted in Figure 1. The notable physical quantities are: mass density ρ, veclocity $\vec{v}$, pressure P, specific internal energy e. These can then be combined to calculate total energy $E = \rho\big(e + \frac{\vec{v}\cdot\vec{v}}{2}\big)$ and specific enthalpy $W = e + \frac{P}{\rho}$. We can then introduce a differentiable scalar valued function $\theta(x, y, z, t)$ such that $\theta = 0$ along the interface between the heavy fluid ($\theta > 0$) and the lighter fluid ($\theta < 0$). Note that in regions of $\theta > 0$, the physical quantities take on an h subscript as they are in the bulk of the heavy fluid and the same goes for the light fluid with $\theta < 0$. The mass flux across the interface is given by $\tilde{j} = \rho\big(\frac{\dot{\theta}}{|\vec{\nabla}\theta|}\vec{n} + \vec{v}\big)$, where $\vec{n}$ is a unit vector normal to the interface.

The system must obey the conservation of mass, momentum, and energy. The equations of motion are therefore given by the Euler equations. Note also that it is assumed that the fluids are incompressible $\vec{\nabla} \cdot \vec{v} = 0$. Using the Einstein summation convention, the equations are:

$$\frac{\partial \rho}{\partial t} + \partial_i(\rho v_i) = 0 \qquad \text{(Mass Continuity)}$$

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j) + \partial_i(P) = 0 \quad \text{i = 1, 2, 3} \qquad \text{(Momentum Conservation)}$$

$$\partial_t(E) + \partial_i(E + P)v_i = 0 \qquad \text{(Energy Conservation)}$$

If we consider a flow field across the entire domain $(\rho, \vec{v}, P, E)$, then it makes sense to then write this out as $(\rho, \vec{v}, P, E)_h H(\theta) + (\rho, \vec{v}, P, E)_l H(-\theta)$ where $H(\theta)$ is the Heaviside function. You can then substitute the appropriate expressions into the conservation laws and consider a jump across the interface denoted [...]. Using this technique, the 4 boundary conditions can be derived. An example using the mass continuity equation is given below:

From mass continuity,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

Use $\rho = \rho_h H(\theta) + \rho_l H(-\theta)$, $\rho \vec{v} = \rho_h \vec{v}_h H(\theta) + \rho_l \vec{v}_l H(-\theta)$, and $\frac{dH(\theta)}{d\theta} = \delta(\theta)$ (Dirac-δ distribution).

$$\frac{\partial \rho}{\partial t} = H(\theta)\frac{\partial \rho_h}{\partial t} + H(-\theta)\frac{\partial \rho_l}{\partial t} + (\rho_h - \rho_l)\delta(\theta)\frac{\partial \theta}{\partial t}$$

$\vec{\nabla} \cdot (\rho \vec{v}) = \vec{\nabla} \cdot (\rho_h \vec{v}_h H(\theta) + \rho_l \vec{v}_l H(-\theta)) = H(\theta)\vec{\nabla} \cdot (\rho_h \vec{v}_h) + H(-\theta)\vec{\nabla} \cdot (\rho_l \vec{v}_l) + \rho_h \vec{v}_h \cdot \vec{\nabla} H(\theta) + \rho_l \vec{v}_l \cdot \vec{\nabla} H(-\theta)$

$\vec{\nabla} H(\pm\theta) = \pm\vec{\nabla}\theta\delta(\theta) = \pm\delta(\theta)\left|\vec{\nabla}\theta\right|\hat{n}$.

Then

$$H(\theta)\left(\frac{\partial \rho_h}{\partial t} + \vec{\nabla} \cdot (\rho_h \vec{v}_h)\right) + H(-\theta)\left(\frac{\partial \rho_l}{\partial t} + \vec{\nabla} \cdot (\rho_l \vec{v}_l)\right) + (\rho_h - \rho_l)\delta(\theta)\frac{\partial \theta}{\partial t} + \delta(\theta)\left|\vec{\nabla}\theta\right|\hat{n} \cdot (\rho_h \vec{v}_h - \rho_l \vec{v}_l) = 0$$

Mass is conserved in the bulk of both the heavy and light fluid individually, so the continuity equation is satisfied. It is also true that $\vec{v}_h \cdot \hat{n} = \vec{v}_l \cdot \hat{n} = \vec{v} \cdot \hat{n}$. Therefore

$$(\rho_h - \rho_l)\delta(\theta)\left|\vec{\nabla}\theta\right|\left(\frac{\dot{\theta}\hat{n} \cdot \hat{n}}{\left|\vec{\nabla}\theta\right|} + \vec{v} \cdot \hat{n}\right) = 0 \implies \rho_h\left(\frac{\dot{\theta}\hat{n}}{\left|\vec{\nabla}\theta\right|} + \vec{v}\right) \cdot \hat{n} - \rho_l\left(\frac{\dot{\theta}\hat{n}}{\left|\vec{\nabla}\theta\right|} + \vec{v}\right) \cdot \hat{n} = 0$$

$\tilde{j}_h \cdot \hat{n} - \tilde{j}_l \cdot \hat{n} = 0 \implies [\tilde{j} \cdot \hat{n}] = 0$

After applying this method to each conservation law:

$[\tilde{j} \cdot \hat{n}] = 0$, $[\big(P + \frac{(\tilde{j} \cdot \hat{n})^2}{\rho}\big)\hat{n}] = 0$, $[(\tilde{j} \cdot \hat{n})\big(\frac{\tilde{j} \cdot \hat{\tau}}{\rho}\big)\hat{\tau}] = 0$, and $[(\tilde{j} \cdot \hat{n})\big(W + \frac{\tilde{j} \cdot \tilde{j}}{2\rho^2}\big)] = 0$

4 Linearisation, Solution, Results

4.1 Linearisation

In order to linearise the dynamics we perturb the flow fields as in [2]:
$\vec{v} = \vec{V} + \vec{u}$, $|\vec{u}| \ll \left|\vec{V}\right|$, $\theta = z^*(x,t) - z$, $P = P_0 + p$, $|p| \ll |P_0|$, $W = W_0 + w$, $|w| \ll |W_0|$,
$\tilde{j} = \vec{J} + \vec{j}$, $\left|\vec{j}\right| \ll \left|\vec{J}\right|$, $\vec{n} = \vec{n}_0 + \vec{n}_1$, $|\vec{n}_1| \ll |\vec{n}_0|$, $\vec{\tau} = \vec{\tau}_0 + \vec{\tau}_1$, $|\vec{\tau}_1| \ll |\vec{\tau}_0|$, $\left|\frac{\partial z^*}{\partial x}\right| \ll 1$, $\left|\frac{\dot{\theta}}{|\vec{\nabla}\theta|}\right| \ll \left|\vec{V}\right|$.

To leading order:
$\theta = -z$, $\vec{n}_0 = (0,0,-1)$, $\vec{\tau}_0 = (1,0,0)$, $J_n = \vec{J} \cdot \vec{n}_0$,
$[J_n] = 0$, $[\big(P_0 + \frac{J_n^2}{\rho}\big)\vec{n}_0] = 0$, $[J_n\big(\frac{(\vec{J}\cdot\vec{\tau}_0)}{\rho}\big)\vec{\tau}_0] = 0$, $[J_n\big(W_0 + \frac{\vec{J}\cdot\vec{J}}{2\rho^2}\big)\vec{n}_0] = 0$
To first order:
$\vec{n}_1 = (\frac{\partial z^*}{\partial x}, 0, 0)$, $\vec{\tau}_1 = (0, 0, \frac{\partial z^*}{\partial x})$, $j_n = \vec{j} \cdot \vec{n}_0$,
$[j_n] = 0$, $[\big(p + \frac{2J_n j_n}{\rho}\big)\vec{n}_0] = 0$, $[J_n\big(\frac{\vec{J}\cdot\vec{\tau}_1 + \vec{j}\cdot\vec{\tau}_0}{\rho}\big)] = 0$, $[J_n\big(w + \frac{\vec{J}\cdot\vec{j}}{\rho^2}\big)] = 0$

4.2 Solution

During the project, three different scenarios were considered. They are referred to as: Conservative dynamics (CD), Rayleigh-Taylor dynamics (RT), and Landau-Darrieus dynamics (LD). Each case has it's own characteristic set of conditions. In the CD case, the mass, normal component of linear momentum, tangential component of linear momentum, and total energy are all conserved quantities. This is the scenario where most attention was focused. In the RT case, there is a special condition where the mass flux across the boundary is 0 for all time. This is different to the boundary condition stated earlier which says that the **jump** in mass flux is 0. In the LD case, it just so happens that energy cannot be conserved at the boundary. This can manifest itself physically during chemical reactions.

The solution method is exactly as followed in [4]. By experimental observation and the construction by Landau, it is true that $\vec{u}_h = \vec{\nabla}\Phi_h$ and $\vec{u}_l = \vec{\nabla}\Phi_l + \vec{\nabla} \times \vec{\Psi}_l$. All of $\vec{\Psi}_l$, Φ_h, Φ_l are called fluid potentials. The heavy fluid has a potential velocity field but the light fluid has both a potential component and a vortical component to its velocity field. The equations describing the system can be reduced to $M\vec{r} = \vec{0}$, with $\vec{r} = (\Phi_h, \Phi_l, V_h z^*, \Psi_l)^T$ (Ilyin, Goddard, Anisimov, Abarzhi, 2019) being a vector and $M = (S - \omega P)$ is a 4×4 matrix dependent on the boundary conditions at the interface. The relationship between M, P, and S comes from the fact that the actual equation describing the system is $P\vec{\dot{r}} = S\vec{r}$ and solutions $\vec{v}\exp(\omega t)$ are sought. i.e $\vec{r} \sim \exp(\omega t)$. Note that ω describes a system eigenvalue which is very important when we get to looking at the stability of these fundamental solutions. The eigenvalues ω_i of each fundamental solution $\vec{r_i}$ can be found by solving either $Det(M) = 0$ or $Det(P^{-1}S - \omega I) = 0$. The net solution is then a linear combination of the relevant fundamental solutions. $\vec{r} = \sum_i c_i\vec{r_i}$. Also note that the range of i may differ between each scenario.

4.2.1 The Conservative Dynamics

Considering the conservative dynamics case, the matrix M is given by

$$M = \begin{pmatrix} -R & -1 & R\omega - \omega & i \\ 1 & -1 & 1-R & \frac{i\omega}{R} \\ R - R\omega & R+\omega & GR - G - RT & -2iR \\ \omega & -\omega & T + \omega - R\omega & iR \end{pmatrix}$$

Note: $R = \frac{\rho_h}{\rho_l}$, G is a dimensionless gravitational term and T is a dimensionless surface tension term.

Solve $Det(M) = 0$ for $\omega_i(G, R, T)$:

$\implies \omega_1 = \frac{\sqrt{G+R+GR-R^2-RT}}{\sqrt{R-1}}, \omega_2 = -\frac{\sqrt{G+R+GR-R^2-RT}}{\sqrt{R-1}}, \omega_3 = R, \omega_4 = -R$. ω_4 is non-physical as the velocity field grows away from the interface, ω_3 has zero fields of velocity and pressure. The solution is unstable when the eigenvalue is real and greater than 0. When the term under the square root is less than 0, ω_1 and ω_2 are complex conjugates and the fundamental solutions corresponding to them are stable. If the term under the square root is greater than 0, ω_1 and ω_2 are purely real, the solution corresponding to ω_1 is unstable and the solution corresponding to ω_2 is stable. So the term which dictates the stability of the first fundamental solution is $G + R + GR - R^2 - RT$.

First consider the case where gravity is absent ($G = 0$). Then the stability term is $R(1-R)-RT$. The first part is called an inertial term and the other part is the tension term. Note that since $R = \frac{\rho_h}{\rho_l}$, it follows that $R \geq 1$ and so the inertial term is less than or equal to 0. The sign of the tension term is dependent on what direction the surface tension acts. A positive value of T will work to keep the stability term less than 0 and thus the entire solution remains stable. A negative value of T will attempt to "counteract" the inertial stabilisation, eventually leading to the solution becoming unstable. This means that surface tension greater than 0 acts as a stabilisation mechanism and surface tension less than 0 acts as a destabiliser. There exists a critical surface tension such that if the tension is any lower, the solution is unstable, and if the tension is any higher, the solution is stable.

Bringing gravity back into the equation, the stability term is $G(R+1) + R(1-R) - RT$. G is prescribed to be greater than or equal to 0 so it follows that the gravitational part of the stability term is always greater than or equal to 0. What does this mean? To achieve a stable solution, the stability term must be less than 0. Therefore the inertial term acts to stabilise and the gravitational term acts to destabilise the solution. As mentioned previously, the tension term is capable of doing either. Can solve for the critical tension T_{cr} by setting the stability term to 0 and solving for T. See Figure 2 for a plot of $T_{cr}(G,R) = \frac{G(R+1)+R(1-R)}{R}$

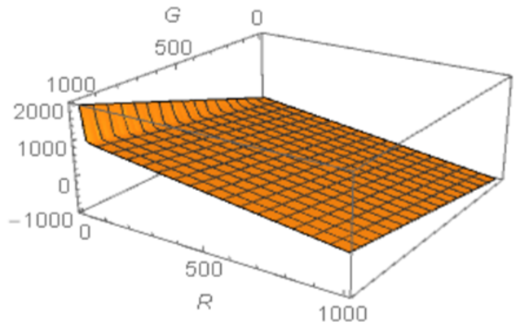

Figure 2

Note that significant variation in T_{cr} occurs around $R = 1$, this is due to the disappearance of the inertial stabilisation mechanism as R goes toward 1. See Figure 3 for a set of plots of $T_{cr}(R)$ at particular values of G. The region above each curve is a region of stability and the region below is a region of instability. The shape of $T_{cr}(R)$ appears to change as G increases and once again there is some wild behaviour around $R = 1$. In all instances T_{cr} is monotonically decreasing with R at fixed G. i.e as the density ratio increases, a smaller surface tension term is required to stabilise the system. When $G = 0$, T_{cr} is strictly less than 0 for all possible values of R which is $(1,\infty)$. This means that as long as the surface tension points in a particular direction (positive direction), the solution will be stable. T_{cr} is of the form linear $+\frac{G}{R}$. So if G is large compared to R it appears as $\frac{1}{R}$ but otherwise it takes on a linear shape. As we expect, a larger acceleration G requires a larger tension term in order for the solution to be stable.

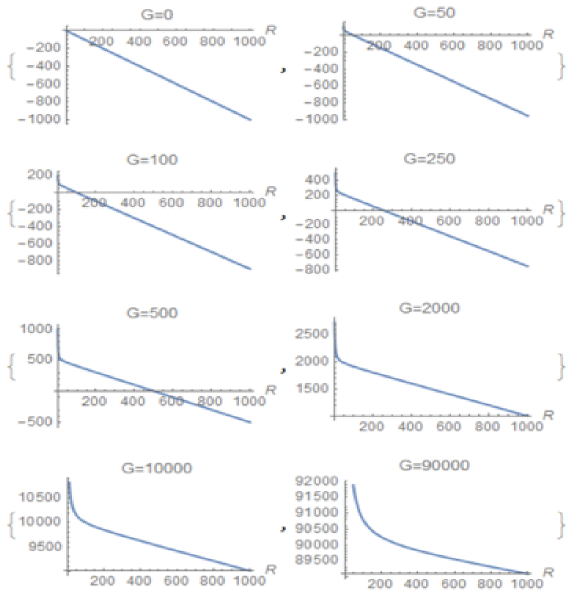

Figure 3

As was stated previously, surface tension can stabilise or destabilise the solution depending on whether it is above or below some threshold value. Solving the equation $T_{cr} = 0$, a function $G(R) = \frac{R(R-1)}{(R+1)}$ is obtained. This is a curve in the parameter space of the system.The curve is plotted in Figure 4 and its physical significance is that at any point along the curve, the stability of the fundamental solution is determined purely by the direction that the surface tension acts (i.e magnitude independent).

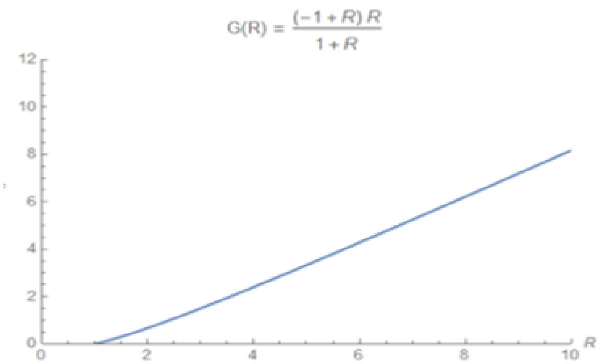

Figure 4

4.2.2 Comparison with RT and LD dynamics

Not all of the details of the RT and LD case will be considered here. The solution method is very similar to the CD case. There are significant differences in what is observed in these cases.

For RT dynamics (no mass flux at boundary), there is actually no inertial stabilisation term, just a gravitational destabilising term and a tension term which plays an identical role to previously. In the CD case, the matrix M was a 4×4 matrix which gave 4 independent solutions with 4 distinct eigenvalues. However in the RT case M is a 3×3 matrix that gives only 2 independent solutions. This indicates that there is a degeneracy. $T_{cr} = \frac{G(R-1)}{R}$ derived from the solution with eigenvalue $\omega_1 = \frac{\sqrt{R-1}\sqrt{GR-G-RT}}{\sqrt{R^2-1}}$. The other eigenvalue is either the complex conjugate or the negative of the first.

For LD dynamics (no energy conservation at the boundary) there is also no inertial stabilisation term. The eigenvalues are given by $\omega_1 = \frac{-R+\sqrt{-G-R+R^2+GR^2+R^3-RT-R^2T}}{R+1}$, $\omega_2 = \frac{-R-\sqrt{-G-R+R^2+GR^2+R^3-RT-R^2T}}{R+1}$, and $\omega_3 = R$. The matrix M for the LD dynamics is also 4×4 but there are only 3 independent solutions with 3 distinct eigenvalues. So once again there is degeneracy. The problem concerning this degeneracy is tackled in [2].

5 Discussion and Conclusion

The end goal and result of this project was the investigation of the effects of surface tension as a means of stabilisation and how it interacts with the inertial stabilisation mechanism in a system where a heavy fluid is placed on top of a light fluid in the presence of gravity. Stability of the system was analysed by using fundamental solutions along with their eigenvalues. Issues of degeneracy were encountered along the way. Three different cases (conservative dynamics, Rayleigh-Taylor, Landau-Darrieus) were considered and analysed. In all of the systems it was found that surface tension had the same effect. An increasingly positive value of T was found to act to stabilise the system.

There are plenty of ways to go with this in the future. Firstly, the RT and LD dynamics can be considered in a more complete way. Another future addition is to actually calculate the eigenvectors and use them to form plots of pressure and velocity fields. Another possible direction is to perform analysis using group theory and the method of moment expansions. The field of interfacial dynamics is young and there is much research to be done in it.

6 Acknowledgments

I wish to acknowledge Snezhana I. Abarzhi and Daniil V. Ilyin for their valuable previous research in interfacial dynamics which provided the foundations for this project. I'd also like to acknowledge AMSI for providing such a valuable opportunity and financial support.

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