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Determining the Expected Metric Dimension of Random 3-Regular Graphs

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1 Abstract

The metric dimension of a connected graph G is defined the minimum cardinality of a subset $S \subset V(G)$ such that all vertices in G are uniquely determined by their distances to vertices in S. Formally, for an ordered subset $S = \{v_1, \ldots, v_n\}$, with $v_i \in V(G)$ distinct for all $i \in [n]$, the representation of any vertex $v \in V(G)$ with respect to S is defined as $r(v|S) = (d(v, v_1), \ldots, d(v, v_n))$. The set S is then a locating set of G if every vertex in $V(G)$ has a unique representation with respect to S. We present the groundwork for establishing an upper bound on the metric dimension of a connected 3-regular graph. This is approached by probabilistic methods, specifically, this paper sets out to bound the number of vertices in an arbitrary 3-regular graph that failed to distinguish two arbitrary vertices with respect to the metric dimension, hence establishing an upper bound for the metric dimension.

2 Introduction

The notation of the metric dimension of a connected graph was first defined by Blumenthal in his monograph [Blumenthal, L. M. (1953)] then reintroduced independently by Slater in 1975 [Slater, P. J. (1975)] then Harary and Melter in 1976 [Melter, R., & Harary, F. (1976)]. Since, only moderate progress has been made towards understanding the typical metric dimension of a graph.

The metric dimension is a property of connected graphs that describes the minimum number of vertices required to uniquely label each vertex of the graph by their distances to the selected vertices. The metric dimension of a connected graph is moderately understood with strong upper bounds on the metric dimension. In this paper, we make progress towards determining an upper bound on the metric dimension of a connected 3-regular graph by probabilistic methods.

3 Statement of Authorship

The workload was divided as follows:

- Will Veenman developed theory for the project, produced the mathematical results and wrote the report.
- Nick Wormald developed theory for the project, supervised the work and proofread the report.

4 Uniform Model for Random Regular Graphs

First stated explicitly by Bollobas [Bollobás, B. (1979)], the pairing model is a uniformly random model for regular graphs. For an r-graph G with n vertices, we can model this graph G as a perfect pairing P of rn objects, which we will denote as "points" for clarity, with points partitioned into n cells of r points each. These cells correspond to vertices in G and points in each cell correspond to individual places for "incoming" edges to the corresponding vertex. By this, each pairing P corresponds to a pseudograph (multigraph with loops permitted).

For our purposes, we need to generate simple graphs (no loops or multiple edges) uniformly at random (u.a.r.), which can be done by choosing a pairing P u.a.r and rejecting the result if it has loops or multiple edges. We can donate the uniform probability space of pairings by $\mathcal{P}_{n,d}$. Pairings can be selected u.a.r by many methods but for our purposes, we will choose the points in pairs sequentially. This method will select pairings u.a.r. as long as the second point in every pair is chosen u.a.r., regardless of how the first pair in every pair is selected. This principal is essential for many of the results we obtain, hence we donate it as the *independence property* of the pairing model.

5 Motivation for Studying S_i Sets

Upon observing 3-regular graphs, we found it useful to define $S_i^v : \{u \in V(G) : d(u,v) = i\}$ and for two different arbitrary $u, v \in G$, find where these S_i sets intersect. We define $C = C(u,v)$ as the union of $S_i^u \cap S_i^v$ over all i. We note that any vertex in C would be unable to distinguish u, v, meaning that a locating set could not be contained within C. This motivates estimating the size of C, for which we need to study S_i.

6 S_i^v Sets Developing From a Single Vertex

We consider a 3-regular graph G selected u.a.r. by a corresponding pairing P_G selected u.a.r. We "discover" G by considering an arbitrary vertex $v \in V(G)$ and finding subsequent S_i^v defined as $S_i^v : \{u \in V(G) : d(u,v) = i\}$. In our pairing model, S_1^v is determined by pairing each of the points in the cell corresponding to v. Then, S_{i+1}^v is generated from S_i^v by pairing the unpaired points of cells corresponding to vertices in S_i^v. The vertices, not already in a prior S_i^v set, with corresponding points that are paired up with points from S_i^v form S_{i+1}^v.

We define the following, let T_i be the set of unpaired (or "free") points in cells corresponding to vertices in S_i^v at stage i, that is, after all edges between S_{i-1}^v and S_i^v have been determined but prior to any pairs between S_i^v and S_{i+1}^v being made. Let R_i be the set of all free points at stage i. We note that when we pair points corresponding to vertices from S_{i-1}^v into R_{i-1} forming S_i^v, vertices in S_i^v may have only a single edge emanating from S_{i-1}^v but may also be receiving two or three edges from different vertices in S_{i-1}^v. For $k = 0, 1, 2, 3$, Let $H_i^{(k)}$ be the set of vertices with corresponding points in R_{i-1}/T_{i-1} that have k edges leading from S_{i-1}^v. We then also define $s_i := |S_i^v|$, $t_i := |T_i|$, $r_i := |R_i|$ and $h_i^{(k)} := |H_i^{(k)}|$.

We note that each vertex in $H_i^{(1)}$ contributes two free points to the vertices in S_i^v. Likewise, each vertex in $H_i^{(2)}$ contributes one free point and $H_i^{(3)}$ contributes none, hence

$$t_i = 2h_i^{(1)} + h_i^{(2)}$$

We note that the vertices that form S_i^v are those that have at least one edge leading back into S_{i-1}^v, then

$$s_i = h_i^{(1)} + h_i^{(2)} + h_i^{(3)}$$

Lastly, we can see from inspection that

$$r_{i+1} = r_i + t_{i+1} - t_i - 3s_{i+1}$$

Thus we have the equations

$$t_i = 2h_i^{(1)} + h_i^{(2)}$$

$$s_i = h_i^{(1)} + h_i^{(2)} + h_i^{(3)}$$

$$r_{i+1} = r_i + t_{i+1} - t_i - 3s_{i+1}$$

Considering $h_{i+1}^{(k)}$, we can, instead of pairing up points from T_i, we can pair up points from R_i/T_i and consider how many "hit" T_i and how many times. We first note that there are $\frac{r_i - t_i}{3}$ total vertices free to form S_{i+1}^v in R_i/T_i at stage i. The probability that a particular vertex joins into S_i^v exactly k times is a product of the probabilities that a given paired up point of a vertex "hits" T_i vs. it not hitting T_i, as expressed by

$$\frac{\left(\prod_{n=0}^{k} t_i + 1 - n\right)\left(\prod_{m=0}^{3-k} r_i - t_i - m\right)}{\prod_{l=1}^{3}(r_i - l)}$$

Taking into account the number of different ways these probabilities can be achieved as a binomial and recalling that we are testing a total of $\frac{r_i - t_i}{3}$ free vertices, we have that the expected value for $h_{i+1}^{(k)}$ is

$$\mathbb{E}\left(h_{i+1}^{(k)}\right) = \frac{\left(\frac{r_i - t_i}{3}\right)}{(t_i + 1)(r_i - t_i)}\binom{3}{k}\frac{\left(\prod_{n=0}^{k} t_i + 1 - n\right)\left(\prod_{m=0}^{3-k} r_i - t_i - m\right)}{\prod_{l=1}^{3}(r_i - l)}$$

Then,

$$\frac{\left(\frac{r_i-t_i}{3}\right)}{(t_i+1)(r_i-t_i)}\binom{3}{k}\frac{\left(\prod_{n=0}^{k} t_i+1-n\right)\left(\prod_{m=0}^{3-k} r_i-t_i-m\right)}{\prod_{l=1}^{3}(r_i-l)}$$
$$=\frac{1}{3(t_i+1)}\binom{3}{k}\frac{\left(\prod_{n=0}^{k} t_i+1-n\right)\left(\prod_{m=0}^{3-k} r_i-t_i-m\right)}{\prod_{l=1}^{3}(r_i-l)}$$

We note that

$$\frac{1}{\prod_{l=1}^{3}(r_i-l)}=\frac{1}{r_i^3+O(r_i^2)}=\frac{1}{r_i^3}\left(\frac{1}{1+O(\frac{1}{r_i})}\right)=\frac{1}{r_i^3}\left(1+O\left(\frac{1}{r_i}\right)\right)$$

Likewise

$$\frac{1}{t_i+1}=\frac{1}{t_i}+O\left(\frac{1}{t_i^2}\right)=\frac{1}{t_i}\left(1+O\left(\frac{1}{t_i}\right)\right)$$

So we can simplify our expression to

$$h_k^{i+1}=\frac{1}{3r_i^3t_i}\binom{3}{k}\left(\prod_{n=0}^{k} t_i+1-n\right)\left(\prod_{m=0}^{3-k} r_i-t_i-m\right)\left(1+O\left(\frac{1}{t_i}\right)+O\left(\frac{1}{r_i}\right)+O\left(\frac{1}{t_ir_i}\right)\right)$$
$$h_k^{i+1}=\frac{1}{3r_i^3t_i}\binom{3}{k}\left(\prod_{n=0}^{k} t_i+1-n\right)\left(\prod_{m=0}^{3-k} r_i-t_i-m\right)\left(1+O\left(\frac{1}{t_i}\right)\right)$$

But we also have

$$\prod_{n=0}^{k} t_i-1-n=t_i^{k+1}+O(t_i^k)$$

And

$$\prod_{m=0}^{3-k} r_i-t_i-m=(r_i-t_i)^{4-k}+O((r_i-t_i)^{3-k})$$

Hence

$$h_k^{i+1}=\frac{1}{3r_i^3t_i}\binom{3}{k}\left(t_i^{k+1}+O(t_i^k)\right)\left((r_i-t_i)^{4-k}+O((r_i-t_i)^{3-k})\right)\left(1+O\left(\frac{1}{t_i}\right)\right)$$
$$=\frac{1}{3r_i^3}\binom{3}{k}\left(t_i^{k}+O(t_i^{k-1})\right)\left((r_i-t_i)^{4-k}+O((r_i-t_i)^{3-k})\right)\left(1+O\left(\frac{1}{t_i}\right)\right)$$

We note that since $T_i \subseteq R_i$, we $r_i \geq t_i$, we have

$$\left(t_i^{k}+O(t_i^{k-1})\right)\left((r_i-t_i)^{4-k}+O(r_i^{3-k})\right)=t_i^k(r_i-t_i)^{4-k}+O(r_i^3)$$

So,

$$h_k^{i+1}=\frac{1}{3r_i^3}\binom{3}{k}\left(t_i^k(r_i-t_i)^{4-k}+O(r_i^3)\right)\left(1+O\left(\frac{1}{t_i}\right)\right)$$
$$=\frac{1}{3r_i^3}\binom{3}{k}\left(t_i^k(r_i-t_i)^{4-k}+O(r_i^3)\right)$$

We assume that $t_i << r_i$, hence

$$h_k^{i+1} = \frac{1}{3r_i^3}\binom{3}{k}\left(t_i^k(r_i)^{4-k} + O(r_i^3)\right)$$

Then, for $k = 1$

$$\begin{aligned} h_1^{i+1} &= \frac{1}{3r_i^3}\binom{3}{1}\left(t_i^1(r_i)^3 + O(r_i^3)\right) \\ &= t_i + O(1) \end{aligned}$$

For $k = 2$

$$\begin{aligned} h_2^{i+1} &= \frac{1}{3r_i^3}\binom{3}{2}\left(t_i^2(r_i)^2 + O(r_i^3)\right) \\ &= t_i^2 r_i^{-1} + O(1) \end{aligned}$$

For $k = 3$

$$\begin{aligned} h_k^{i+1} &= \frac{1}{3r_i^3}\binom{3}{3}\left(t_i^3(r_i)^1 + O(r_i^3)\right) \\ &= \frac{1}{3}t_i^3 r_i^{-2} + O(1) \end{aligned}$$

Hence,

$$t_{i+1} = 2h_1^{i+1} + h_2^{i+1} = 2t_i + t_i^2 r_i^{-1} + O(1)$$

$$s_{i+1} = h_1^{i+1} + h_2^{i+1} + h_3^{i+1} = t_i + t_i^2 r_i^{-1} + \frac{1}{3}t_i^3 r_i^{-2} + O(1)$$

$$\begin{aligned} r_{i+1} = r_i + t_{i+1} - t_i - 3s_{i+1} &= r_i + (2t_i + t_i^2 r_i^{-1}) - t_i - 3t_i - 3t_i^2 r_i^{-1} - t_i^3 r_i^{-2} + O(1) \\ &= r_i - 2t_i - 2t_i^2 r_i^{-1} - t_i^3 r_i^{-2} + O(1) \end{aligned}$$

We take $s_i = a_i n$, $r_i = d_i n$ and $t_i = e_i n$. Beginning with

$$h_k^{i+1} = \frac{1}{3r_i^3}\binom{3}{k}\left(t_i^k(r_i - t_i)^{4-k} + O(r_i^3)\right)$$

Which becomes

$$\begin{aligned} h_k^{i+1} &= \frac{1}{3(d_i n)^3}\binom{3}{k}\left((e_i n)^k(d_i n - e_i n)^{4-k} + O(n^3)\right) \\ &= \frac{1}{3(d_i)^3}\binom{3}{k}\left((e_i)^k(d_i - e_i)^{4-k} n + O(1)\right) \end{aligned}$$

Then,

$$h_1^{i+1} = \frac{1}{3(d_i)^3}\binom{3}{1}\left((e_i)^1(d_i - e_i)^3 n + O(1)\right) = \frac{e_i(d_i - e_i)^3}{d_i^3}n + O(1)$$

$$h_2^{i+1} = \frac{1}{3(d_i)^3}\binom{3}{2}\left((e_i)^2(d_i - e_i)^2 n + O(1)\right) = \frac{e_i^2(d_i - e_i)^2}{d_i^3}n + O(1)$$

$$h_3^{i+1} = \frac{1}{3(d_i)^3}\binom{3}{3}\left((e_i)^3(d_i - e_i)^1 n + O(1)\right) = \frac{e_i^3(d_i - e_i)}{3d_i^3}n + O(1)$$

Then

$$e_{i+1}n = t_{i+1} = 2h_1^{i+1} + h_2^{i+1} = 2\frac{e_i(d_i - e_i)^3}{d_i^3}n + \frac{e_i^2(d_i-e_i)^2}{d_i^3}n + O(1)$$

$$= \left(-\frac{e_i^4}{d_i^3} + \frac{4e_i^3}{d_i^2} - \frac{5e_i^2}{d_i} + 2e_i\right)n + O(1) = e_i\left(-\left(\frac{e_i}{d_i}\right)^3 + 4\left(\frac{e_i}{d_i}\right)^2 - 5\left(\frac{e_i}{d_i}\right) + 2\right)n + O(1)$$

$$a_{i+1}n = s_{i+1} = h_1^{i+1} + h_2^{i+1} + h_3^{i+1} = \frac{e_i(d_i-e_i)^3}{d_i^3}n + \frac{e_i^2(d_i-e_i)^2}{d_i^3}n + \frac{e_i^3(d_i-e_i)}{3d_i^3}n + O(1)$$

$$= \left(-\frac{e_i^4}{3d_i^3} + \frac{4e_i^3}{3d_i^2} - \frac{2e_i^2}{d_i} + e_i\right)n + O(1) = e_i\left(-\left(\frac{e_i}{d_i}\right)^3 + 2\left(\frac{e_i}{d_i}\right)^2 - 2\left(\frac{e_i}{d_i}\right) + 1\right)n + O(1)$$

$$d_{i+1}n = r_{i+1} = r_i + t_{i+1} - t_i - 3s_{i+1} = d_in + 2\frac{e_i(d_i-e_i)^3}{d_i^3}n + \frac{e_i^2(d_i-e_i)^2}{d_i^3}n - e_in$$

$$-3\frac{e_i(d_i-e_i)^3}{d_i^3}n - 3\frac{e_i^2(d_i-e_i)^2}{d_i^3}n - \frac{e_i^3(d_i-e_i)}{d_i^3}n + O(1)$$

$$= d_in - e_in - \frac{e_i(d_i-e_i)^3}{d_i^3}n - 2\frac{e_i^2(d_i-e_i)^2}{d_i^3}n - \frac{e_i^3(d_i-e_i)}{d_i^3}n + O(1)$$

$$= \left(\frac{(d_i-e_i)^2}{d_i}\right)n + O(1)$$

Or simply

$$e_{i+1} = e_i\left(-\left(\frac{e_i}{d_i}\right)^3 + 4\left(\frac{e_i}{d_i}\right)^2 - 5\left(\frac{e_i}{d_i}\right) + 2\right) + O\left(\frac{1}{n}\right)$$

$$a_{i+1} = e_i\left(-\left(\frac{e_i}{d_i}\right)^3 + 2\left(\frac{e_i}{d_i}\right)^2 - 2\left(\frac{e_i}{d_i}\right) + 1\right) + O\left(\frac{1}{n}\right)$$

$$d_{i+1} = \left(\frac{(e_i-d_i)^2}{d_i}\right) + O\left(\frac{1}{n}\right) = e_i\left(\left(\frac{e_i}{d_i}\right) - 2 + \left(\frac{d_i}{e_i}\right)\right) + O\left(\frac{1}{n}\right)$$

Let $b_i = \frac{e_i}{d_i}$, then

$$e_{i+1} = e_i\left(-b_i^3 + 4b_i^2 - 5b_i + 2\right) + O\left(\frac{1}{n}\right) = e_i(2-b_i)(b_i-1)^2 + O\left(\frac{1}{n}\right)$$

$$a_{i+1} = e_i\left(-b_i^3 + 2b_i^2 - 2b_i + 1\right) + O\left(\frac{1}{n}\right)$$

$$d_{i+1} = e_i\left(b_i - 2 + \frac{1}{b_i}\right) = \frac{e_i}{b_i}(b_i-1)^2 + O\left(\frac{1}{n}\right)$$

Then,

$$b_{i+1} = \frac{e_{i+1}}{d_{i+1}} = \frac{e_i(2-b_i)(b_i-1)^2 + O\left(\frac{1}{n}\right)}{\frac{e_i}{b_i}(b_i-1)^2 + O\left(\frac{1}{n}\right)} = b_i(2-b_i) + O\left(\frac{1}{n}\right)$$

This motivates the study of the recursion

$$b_{i+1} \approx b_i(2-b_i)$$

in order to understand the evolution of the equations above. Initially, b_i will begin small for small i and roughly double from b_{i-1} but soon approaches 1. Once $b_i \geq 1 - \frac{1}{n}$, say firstly at $i = \hat{i}$, then all free remaining points must also be free points and we have found the last S_i^v set to be $S_{\hat{i}}^v$.

Let $1 - \epsilon$ with ϵ small and let $b_i \approx 1 - \epsilon$, we then claim that $b_{i+j} = 1 - \epsilon^{2^j}$ for $j \geq 1$. To show this, we note the base case $j = 1$

$$b_{i+1} = b_i(2 - b_i) = (1 - \epsilon)(1 + \epsilon) = 1 - \epsilon^{2^1}$$

Then, we do induction on j by assuming $j = k$ holds for some natural number k and noting,

$$b_{i+k+1} = b_{i+k}(2 - b_{i+k}) = \left(1 - \epsilon^{2^k}\right)\left(1 + \epsilon^{2^k}\right) = 1 - \left(\epsilon^{2^k}\right)^2 = 1 - \epsilon^{2^{k+1}}$$

Which completes our induction.

We can also ask, given some ϵ, how long will it take before $b_i \geq 1 - \frac{1}{n}$? Well, since S_i sets when $b_i \geq 1 - \frac{1}{n}$, we have

$$b_i = 1 - \epsilon^{2^i} \geq 1 - \frac{1}{n}$$
$$\epsilon^{2^i} \leq \frac{1}{n}$$
$$\log_\epsilon\left(\frac{1}{n}\right) = 2^i$$
$$\log_2\left(\log_\epsilon\left(\frac{1}{n}\right)\right) = i$$

So if $b_i = 1 - \epsilon$ for some i, then we expect it will take $\log_2\left(\log_\epsilon\left(\frac{1}{n}\right)\right)$ more stages before the whole graph is searched.

7 S_i^v and S_i^u Sets Developing From Two Vertices

We can extend our prior work of developing S_i^v sets from a vertex v to that of simultaneously developing two groups of sets S_i^v and S_i^u from two arbitrary vertices $v, u \in G$. Letting u, v be arbitrary vertices in G, we define S_i^v and S_i^v in the typical way but we also define the following sets:

Let C_i be the set of vertices that are distance i from both u and v. That is $C_i = \{w \in G : d(w,u) = d(w,v)\}$. Also define $c_i := |C_i|$.

Let T_i^u be the set of free points in S_i^u at stage i, likewise, let T_i^v be the set of free points in S_i^v at stage i, and, let T_i^C be the set of free points in C_i at stage i. Let T_i be the set of all free points at stage i in $S_i^u \cup S_i^v \cup C_i$. Note that T_i^u is expected to have the same size as T_i^v, so define $t_i^S := |T_i^u| \approx |T_i^v|$ and $t_i^C := |T_i^C|$. Also let $T_i = T_i^u \cup T_i^v \cup T_i^C$, then $t_i = |T_i|$.

Let R_i is the set of unpaired points at stage i.

Let $H_i^{(u,k)}$ to be the subset of vertices in S_i at stage i that join exactly k times to points in only T_{i-1}^u, likewise, let $H_i^{(v,k)}$ to be the subset of vertices in S_i at stage i that join exactly k times to points in only T_{i-1}^v and let $H_i^{(C,k)}$ be the subset of vertices in C_i at stage i that join exactly k times to points in $T_{i-1}^u \cup T_{i-1}^v \cup T_{i-1}^C$ with at least one join into T_{i-1}^C or at least one join into each of T_{i-1}^u and T_{i-1}^v. Also let $H_i^k = H_i^{(u,k)} \cap H_i^{(v,k)} \cap H_i^{(C,k)}$. Then, define $h_i^k = |H_i^k|$, $h_i^{(S,k)} = |H_i^{(u,k)}| \approx |H_i^{(v,k)}|$, and $h_i^{(C,k)} = |H_i^{(C,k)}|$.

By the same justifications as prior, we have the equations

$$t_i^S = 2h_i^{(S,1)} + h_i^{(S,2)}$$

$$t_i^C = 2h_i^{(C,1)} + h_i^{(C,2)}$$

$$s_i = h_i^{(S,1)} + h_i^{(S,2)} + h_i^{(S,3)}$$

$$c_i = h_i^{(C,1)} + h_i^{(C,2)} + h_i^{(C,3)}$$

$$\begin{aligned} r_{i+1} &= r_i + \left(2t_{i+1}^S + t_{i+1}^C\right) - \left(2t_i^S + t_i^C\right) - 3\left(2s_{i+1} + c_{i+1}\right) \\ &= r_i + t_{i+1} - t_i - 6s_{i+1} - 3c_{i+1} \end{aligned}$$

Recalling that

$$\mathbb{E}\left(h_{i+1}^{(k)}\right) = \frac{\left(\frac{r_i - t_i}{3}\right)}{(t_i+1)(r_i-t_i)} \binom{3}{k} \frac{\left(\prod_{n=0}^{k} t_i + 1 - n\right)\left(\prod_{m=0}^{3-k} r_i - t_i - m\right)}{\prod_{l=1}^{3}(r_i - l)}$$

By realizing that a vertex with points in R_i joins into C_i if and only if at least one of its edges lead from C_{i-1} or if there is a least one edge leading from each of S_{i-1}^v and S_{i-1}^u. Identically, a vertex with points in R_i does not lead into C_{i+1} if all of its edges lead into either S_{i-1}^v or S_{i-1}^u. Hence we have,

$$h_i^{(C,k)} \approx \left(1 - 2\left(\frac{t_i^S}{t_i}\right)^k\right) h_i^k$$

And

$$h_i^{(S,k)} \approx \frac{1}{2}\left(h_i^k - h_i^{(C,k)}\right) = \frac{1}{2}h_i^k - \left(\frac{1}{2} - \left(\frac{t_i^S}{t_i}\right)^k\right) h_i^k$$

Note that to obtain these approximations, we have to assume independency. (NOTE TO NICK: I haven't been able to show error rigorously here yet but I suspect that it's just as small as all the other error we've been getting)

We, let $t_i = \hat{t}_i n$, $t_i^S = \hat{t}_i^S n$, $t_i^C = \hat{t}_i^C n$, $s_i = \hat{s}_i n$, $c_i = \hat{c}_i n$ and $r_i = \hat{r}_i n$, then we can rewrite the equations above by

$$t_i^S = 2h_i^{(S,1)} + h_i^{(S,2)} = h_i^1 - \left(1 - 2\left(\frac{t_i^S}{t_i}\right)^1\right) h_i^1 + \frac{1}{2}h_i^2 - \left(\frac{1}{2} - \left(\frac{t_i^S}{t_i}\right)^2\right) h_i^2$$

$$= \frac{1}{3}\binom{3}{1}\left((t_i)^1(r_i)^0 + O(1)\right) - \left(1 - 2\left(\frac{t_i^S}{t_i}\right)^1\right)\left(\frac{1}{3}\binom{3}{1}\left((t_i)^1(r_i)^0 + O(1)\right)\right)$$

$$+\frac{1}{2}\left(\frac{1}{3}\binom{3}{2}\left((t_i)^2(r_i)^{-1} + O(1)\right)\right) - \left(\frac{1}{2} - \left(\frac{t_i^S}{t_i}\right)^2\right)\left(\frac{1}{3}\binom{3}{2}\left((t_i)^2(r_i)^{-1} + O(1)\right)\right)$$

Then by letting $a_i = \frac{t_i^S}{t_i}$ and $b_i = \frac{t_i}{r_i}$, we get

$$t_i^S = (t_i + O(1)) - (1 - 2a_i)(t_i + O(1)) + \frac{1}{2}(t_i b_i + O(1)) - \left(\frac{1}{2} - a_i^2\right)(t_i b_i + O(1))$$

$$= 2t_i a_i + t_i a_i^2 b_i + O(1)$$

Then,

$$\hat{t}_i^S n = \hat{t}_i n\left(2a_i + a_i^2 b_i\right) + O(1)$$

Or,

$$\hat{t}_i^S = \hat{t}_i\left(2a_i + a_i^2 b_i\right) + O\left(\frac{1}{n}\right)$$

Likewise for the other equations, so from

$$t_i^S = 2h_i^{(S,1)} + h_i^{(S,2)}$$

$$t_i^C = 2h_i^{(C,1)} + h_i^{(C,2)}$$

$$s_i = h_i^{(S,1)} + h_i^{(S,2)} + h_i^{(S,3)}$$

$$c_i = h_i^{(C,1)} + h_i^{(C,2)} + h_i^{(C,3)}$$

$$r_{i+1} = r_i + t_{i+1} - t_i - 6s_{i+1} - 3c_{i+1}$$

We obtain the equations

$$\hat{t}_i^S = \hat{t}_i\left(2a_i + a_i^2 b_i\right) + O\left(\frac{1}{n}\right)$$

$$\hat{t}_i^C = \hat{t}_i\left(2 - 4a_i + b_i - 2a_i^2 b_i\right) + O\left(\frac{1}{n}\right)$$

$$\hat{s}_i = \hat{t}_i\left(a_i + a_i^2 b_i + \frac{1}{3}a_i^3 b_i^2\right) + O\left(\frac{1}{n}\right)$$

$$\hat{c}_i n = \hat{t}_i\left(1 - 2a_i + b_i - 2a_i^2 b_i + \frac{1}{3}b_i^2 - \frac{2}{3}a_i^3 b_i^2\right) + O(1)$$

$$\hat{r}_{i+1} = \hat{r}_i + \hat{t}_{i+1} - \hat{t}_i - 6\hat{t}_i a_i \left(1 + a_i b_i + \frac{1}{3}a_i^2 b_i^2\right) - 3\hat{t}_i \left(1 - 2a_i + b_i - 2a_i^2 b_i + \frac{1}{3}b_i^2 - \frac{2}{3}a_i^3 b_i^2\right) + O\left(\frac{1}{n}\right)$$

When we obtained these same equations for one vertex, we were able to factorise some to obtain a useful recursion. These equations have not shown themselves to be as easy to manipulate into some number of recursions and require future work.

8 Discussion

Our work on the growth of S_i sets emanating from a single vertex was found to be successful. We developed a recursion on the ratio of free points to remaining points at stage i which also allowed us to determine an expected j for which S_j sets terminate with respect to the graph size n and proportionb_i. However, analysing S_i sets from two vertices was found to be more difficult, with us unsuccessfully determining a set of recursions that would illuminate the behavior of variables we defined. With further work however, this problem may easily be circumvented and either an approximate recursion may be found or an alternative method may be used to analyse the evolution of S_i^u, S_i^v and C_i sets. If these are understood well enough, they should give a bound on the size of a locating set, since such a set must contain at least one vertex outside of $C(u,v)$ for all $u, v \in V(G)$.

9 Conclusion

In this report, we begun work towards to finding the typical metric dimension of random connected 3-regular graphs, which may provide an upper bound in the deterministic case. Our basis for determining this upper bound was to estimate the size of $C(u,v)$ given a 3-regular graph of size n. Studying $C(u,v)$ is key since all locating sets must have one vertex outside $C(u,v)$ for all $u, v \in V(G)$. To understand $C(u,v)$, we observed S_i sets, first developing a recursion on one vertex, then partially completing the work towards understanding S_i^u and S_i^v in the two vertex case. For the future, we hope to complete our work on the two vertex case and estimate the size of $C(u,v)$, thus providing us with an upper bound on the metric dimension in the deterministic case.

10 References

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