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C^*-algebras of self-similar groupoid actions on higher rank graphs

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Abstract

C^*-algebras are complex algebras equipped with an involution operator that is conjugate linear, the prototypical example being the complex numbers $\mathbb{C}$. They can be generated in many cases, such as with groups, étale groupoids and directed graphs. In particular, k-graphs, higher dimensional generalisations of vertex-edge graphs, can generate a universal C^*-algebra through a representation of the k-graph as partial isometries. In a separate direction, k-graphs can be viewed as a purely combinatorial object, where a groupoids elements permute the paths of a k-graph. One can enforce a self-similarity condition on the action of a groupoid on a k-graph, from which algebraic structures can naturally arise. Of particular interest is the fact that we can associate a C^*-algebra to a self-similar groupoid action. Previous research have found two conditions on a k-graph - aperiodicity and cofinality - which guarantee the simplicity of the k-graph C^*-algebra and it is an open question whether a similar characterisation of the simplicity of the groupoid action C^*-algebra exists. This report looks at the required theory and the progress made in generalising the above conditions to the self-similar groupoid case.

Introduction

This report is the culmination of a six week research project held over the summer of 2019-2020, funded by the Australian Mathematical Sciences Institute (AMSI) Vacation Research Scholarship (VRS). The aim of this report is to summarise the theory of self-similar groupoid actions on k-graphs and their associated C^*-algebras. We will also present our findings and proofs on generalising conditions which guarantee the simplicity of the generated C^*-algebra.

This field originated with the study of self-similar group actions on sets. One associates a permutation of a set to each element in a group and imposes a self-similarity condition on the group (Nekrashevych (2005)). These actions themselves can generate a group structure, creating interesting examples such as the Grigorchuk group, the first example of a finitely generated infinite torsion group of intermediate growth. Such examples are not trivially found through other methods, yet have comparatively simple descriptions in the theory of self-similar actions.

A higher rank graph, or a k-graph, is a category which models a graph and employs a notion of k-dimensional length. Analogous to self-similar group actions acting on sets, a groupoid action on a k-graph associates a partial isomorphism of a k-graph to a groupoid element and imposes a self-similar condition on it. The seminal paper by Kumjian & Pask (2000) showed that a C^*-algebra can be associated with a k-graph. Extending their work, Afsar et al. (2019)) define two C^*-algebras $C(G, \Lambda)$ and $\mathcal{O}(G, \Lambda)$ generated by a self-similar groupoid action (G, Λ), and it is an open question whether these two C^*-algebras coincide. One useful invariant of C^*-algebras that can distinguish them is their simplicity, where a C^*-algebra is simple if and only if it has only trivial ideals.

Previous work (Robertson & Sims (2006)) has shown that the simplicity of a k-graph C^*-algebra is characterised by two conditions - aperiodicity and cofinality - on the k-graph. The goal of supervisors Brownlowe and Afsar is to find the groupoid-generalisations of the cofinality and aperiodic conditions and characterise the simplicity of groupoid action C^*-algebras. This project will focus on making progress on proving a required lemma for this generalisation to hold: That if a self-similar groupoid action on a k-graph is G-aperiodic, then it satisfies what we call condition G-(B). We have formulated Condition G-(B) to mirror Condition (B), which was proven to be equivalent to aperiodicity for k-graphs in Robertson & Sims (2006).

This report will begin with an introduction on self-similar group actions on sets, then discuss k-graphs before moving onto self-similar groupoid actions on k-graphs. C^*-algebras will be defined before looking at how we associate C^*-algebras to k-graphs and self-similar groupoid actions on k-graphs. Finally, we define the research problem and detail the progress we have made during this summer project, including proofs that show certain conditions, generalised to the groupoid case, are equivalent.

Acknowledgements

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Statement of Authorship

As part of the project, the author read through the required theory from multiple sources, summarised the necessary information and wrote up this report. The author also worked to prove the conjectured theorem. Supervisors Brownlowe and Afsar provided sources, guidance throughout the project and also worked on the conjecture.

1 Self-similar group actions

We begin this report with an overview of self-similar group actions on sets as a prototype of self-similar groupoid actions on k-graphs. For a comprehensive overview on self-similar group actions, the source Nekrashevych (2005) was very helpful in preparing this section.

Definition 1.1 *Let $X = \{a_1, \dots, a_n\}$ be a finite set of n elements. Define the set*

$$X^* = \{\textit{Set of all finite combinations of elements of } X\}$$

We call X an alphabet and X^* the set of finite words of X.

Note that X^* includes the empty word as an empty combination of elements of X. We can naturally illustrate X^* as an infinite graph, where the vertices are words and vertices are connected by an edge if and only if they are of the form v and vx for some letter $x \in X$ and word $v \in X^*$. (see illustration). This is a rooted tree, where the root is the empty set $\emptyset$. We are interested in studying the functions on the rooted tree which preserve its structure:

Definition 1.2 *$f : X^* \to X^*$ is called an endomorphism of X^* if it preserves the root of X^* and the adjacency of vertices. That is, $f(\emptyset) = \emptyset$ and if $v \in X^*$ and $x \in X$, then $f(v)$ and $f(vx)$ are adjacent. Automorphisms are the bijective endomorphisms of X^*.*

We write $\mathrm{Aut}(X^*)$ for the group of automorphisms of X^*.

Definition 1.3 *Faithful self-similar group action. An action of a group G of order n on a set X^* is said to be faithful if the action is an injection into the symmetric group S_n. A faithful action of a group G on X is said to be self-similar if for every $g \in G$ and every $x \in X$, there exists $h \in G$ and $y \in X$ such that*

$$g(xw) = yh(w)$$

That is, when we have a self-similar action, we can break down the action of g into its action letter by letter. The faithfulness assumption on the action of G forces y and h to be uniquely determined by g and x.

1.1 Odometer

The odometer, or the adding machine, is a simple example of a self-similar group action. Let $X = \{0, 1\}$. For $0, 1 \in X$ and $w \in X^*$, define the automorphism $a : X^* \to X^*$ by

$$a(0w) = 1w$$
$$a(1w) = 0a(w)$$

For example,

$$a(0011100) = 1011100$$
$$a(1111001) = 0000101$$

Hence, the odometer, true to its name, acts as a counter - for example, in a car where the counter for number of kilometres travelled is in binary.

1.2 Grigorchuk group

The Grigorchuk group is a well-known example of a group generated by self-similar actions. Let $X = \{0, 1\}$. For $0, 1 \in X$ and $w \in X^*$, define automorphisms on X^* as follows:

$$\begin{array}{cc} \mathrm{a(0w) = 1w} & \mathrm{a(1w) = 0w} \\ \mathrm{b(0w) = 0a(w)} & \mathrm{b(1w) = 1c(w)} \\ \mathrm{c(0w) = 0a(w)} & \mathrm{c(1w) = 1d(w)} \\ \mathrm{d(0w) = 0w} & \mathrm{d(1w) = 1b(w)} \end{array}$$

See figure 1 for the Moore diagram of the Grigorchuk group. The Grigorchuk group is the group generated by these four automorphisms $\langle a, b, c, d\rangle$. It was the first example of an infinite group with finitely many generators, each having finite order - or an infinitely generated torsion group. It is also an example of a group of intermediate growth - the number of elements that can be written as a word of n letters, up to equivalence classes, grows strictly faster than

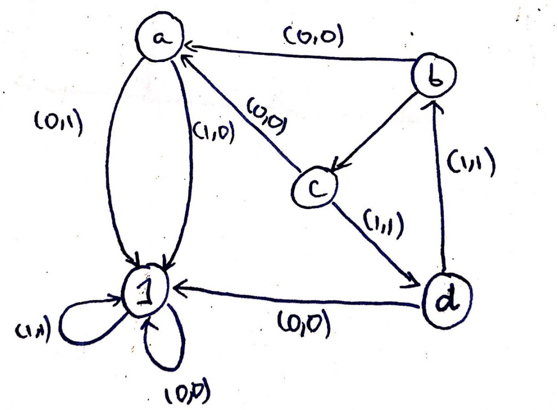

Figure 1: Grigorchuk Moore diagram with generators $\{a, b, c, d\}$. Arrows represent change in state. If a is the input and b is the output, then we write (a, b) next to the corresponding arrow. From Nekrashevych (2005), figure 6.

polynomial speed but slower than exponential speed. This demonstrates the power of the self-similarity phenomenon in producing simple descriptions of otherwise difficult-to-define algebraic structures.

2 Higher rank k-graphs

This section will describe k-graphs, for which we assume the reader has the basic vocabulary of category theory. Leinster (2016) was a useful source for this, and should be consulted for the central definitions. For resources on k-graphs, see Sims (2010). For a category A we write its collection of objects as $\mathrm{Obj}(A)$ and the set of morphisms, or paths, on $\mathrm{Obj}(A)$ as $\mathrm{Mor}(A)$. We write $a \in A$ as shorthand for $a \in \mathrm{Mor}(A)$.

We can redefine familiar algebraic structures in category theoretic language. For example, the category-theoretic group is a single object category with the only morphism being the identity morphism, and a groupoid is a generalisation of a group where you can have multiple identities:

Definition 2.1 *A groupoid is a category with all morphisms having a corresponding inverse morphism. When a groupoid is a single object category, it is a group.*

This is the definition that we will use, following Afsar et al. (2019).

Notation-wise, we follow the convention of Sims (2010). Whenever we discuss a directed graph E, we refer to the 4-tuple $E = (E^0, E^1, r, s)$, where E^0 is the set of vertices of the graph, E^1 is the set of edges of E and $r, s : E^1 \to E^0$ are the range and source functions respectively. A sequence of edges $e_1 \dots e_n$, $e_i \in E^1$ denotes a *path* of length n, and E^n denotes the set of paths of length n. We write $\mathbb{N}^k$ for the k-tuple of natural numbers and, for $a, b \in \mathbb{N}^k$, we write $a \leq b$ if and only if the inequality holds coordinate-wise. A directed graph is *row-finite* if every vertex has a finite number of edges emanating from it, and it has *no sources* if every vertex is the range of some $e \in E^1$. We write $p \vee q$ for $p, q \in \mathbb{N}^k$ to denote the coordinate-wise maximum and $p \wedge q$ for the coordinate-wise minimum.

We begin with an example of a category which motivates using category theory to describe higher dimensional graphs.

Definition 2.2 *Let $E = (E^0, E^1, r, s)$ be a directed graph. The path category $P(E)$ is defined to be the category with its objects being the collection of vertices E^0 and morphisms being paths between the vertices. Composition of morphisms is given by concatenation of paths, as long as the range and source of adjacent paths match up. We can assign a 1-dimensional length functor $d : P(E) \to \mathbb{N}$ to the path category which returns the number of edges traversed by each path.*

Now, what happens if we want the notion of a k-dimensional graph? Category theory gives a good framework for describing graph-like structures with paths, and in fact, what we need is a category on which we have a notion of k-dimensional distance, and a way of ensuring the morphisms on such a category are actually 'path-like':

Definition 2.3 *A k-graph is a countable category Λ along with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the following factorisation property: If $\lambda \in \Lambda$, $m, n \in \mathbb{N}^k$ and we have $d(\lambda) = m + n$, then there exists unique $u, v \in \Lambda$ such that $\lambda = uv$ and $d(u) = m$, $d(v) = n$. We call d the degree functor on the k-graph Λ.*

The degree functor gives us a notion of k-dimensional length. The unique factorisation property seems to be an odd requirement, but we give an example to illustrate the intuition: In a 1-graph, given any one dimensional path of length m, there is exactly one way to factorise that path into paths of length p and degree q in that order, where $p + q = m$. Requiring the unique factorisation property forces our arbitrary countable category and the morphisms on that category to resemble paths on a graph.

For $n \in \mathbb{N}^k$, we write Λ^n to denote the set of morphisms $\{\lambda \in \Lambda : d(\lambda) = n\}$.

Lemma 2.1 *There is a well-defined identification of Obj(Λ) with the identity morphisms in Λ.*

We give the proof, since it demonstrates some of the considerations one takes when dealing with k-graphs and its factorisation property.

Proof: For any identity morphism $\lambda \in \Lambda$ with domain dom and codomain cod, $d(\lambda) = d(\lambda\lambda) = 2d(\lambda)$, where the last equality holds as d is a functor. Hence, $d(\lambda) = 0$. Conversely, if $d(\lambda) = 0$, then $d(\lambda) = 0 + 0$ and we can write $\lambda = \mathrm{id}_{\mathrm{cod}}\lambda = \lambda\mathrm{id}_{\mathrm{dom}}$. The factorisation property forces $\lambda = \mathrm{id}_{\mathrm{dom}}$. This shows that the set of identity morphisms in Λ is exactly the set of degree 0 morphisms. In particular, we identify each object in Λ with its identity morphism, and write Λ^0 to denote this set.

Example 2.1 *Let Ω_k be the countable category with the objects Obj(Λ) being the lattice points $\mathbb{N}^k$ and morphisms Mor(Λ) $= \{(m, n)|m, n \in \mathbb{N}^k, m \leq n\}$. The range is given by $r(m, n) = (m, m)$ and $s(m, n) = (n, n)$. The composition of morphisms $(a, b)(c, d)$ is defined if $r(c, d) = s(a, b)$, and is given by $(a, b)(c, d) = (a, d)$. Let $d : \mathbb{N}^k \to \mathbb{N}^k$ be given by $d(m, n) = n - m$. The factorisation property holds: Suppose $m, n \in \mathbb{N}^k$ and that we can write $d(m, n) = n - m = a + b$ with $a, b \in \mathbb{N}^k$. Then $(m, n) = (m, m + a)(m + a, m + a + b)$, and this is a unique factorisation. Hence, Ω_k is a k-graph.*

Due to the factorisation property, explicitly depicting a k-graph, even in the two-dimensional case, involves interesting technicalities with unique factorisations. An easier way of representing a k-graph is through the use of skeletons - that is, a k-coloured directed graph, from which we can derive a k-graph.

Definition 2.4 *Let E be a directed graph of edges and vertices, and let $c : E^1 \to \{e_1, \ldots, e_k\}$ be the colour map on E. Then E is a k-coloured graph.*

Let $\mathbb{F}_k$ represent the free group generated by the set $\{e_1, \ldots, e_k\}$ and E^* be the set of paths in E - that is, sequences of edges such that the range and sources of the edges match up in the

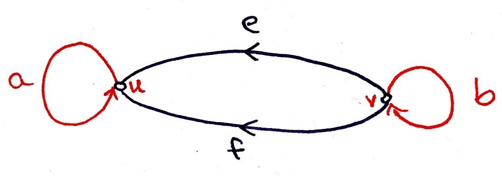

Figure 2: A 2-coloured graph skeleton with blue edges e and f, and red edges a and b. This skeleton generates the 2-graph with the collection of squares in figure 3.

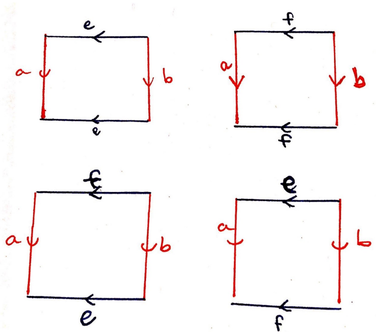

Figure 3: Each row defines a 2-graph where the unique factorisation property is given by each square. For example, the first row has the factorisations $ea = ba$ and $fa = bf$.

correct way. Then we can extend the colour map c to a functor $c : E^* \to \mathbb{F}_k$ by defining

$$\text{for } \alpha = \alpha_1\alpha_2 \cdots \in E^*$$
$$c(\alpha) = c(\alpha_1)c(\alpha_2)\ldots$$

Theorem 2.1 *Let Λ be a k-graph. Then there exists a k-coloured graph, or skeleton, of Λ, from which Λ can be derived by specifying its factorisation properties. Conversely, given a k-coloured graph, specifying the unique factorisations of paths defines a unique k-graph Λ.*

See [AidanSims], particularly chapter 1, for a rigorous proof. We give an example to illustrate this theorem in figures 2 and 3.

Terminology. Let Λ be a k-graph. We say Λ is *row-finite* if, for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $|v\Lambda^n| < \infty$. We say that Λ has *no sources* if, for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $v\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n \text{ and } r(\lambda) = v\} \neq \emptyset$. Given two k-graphs Λ and Γ, we say $x : \Lambda \to \Gamma$ is a *graph morphism* if it preserves connectivity and degree. Given a k-graph Λ, we define an *infinite path* to be a graph morphism $x : \Omega_k \to \Lambda$. For any $n \in \mathbb{N}^k$, we write $x(n)$ for the morphism $x(n, n) \in \Lambda$. Write the *range* of an infinite path x as $r(x) := x(0)$. Given $n \in \mathbb{N}^k$, the *shift map* $\sigma^n : \Lambda^\infty \to \Lambda^\infty$ is defined by $\sigma^n(x)(p, q) = x(p + n, q + n)$. We call Λ *aperiodic* if for every $n \neq m \in \mathbb{N}^k$ and for every $v \in \Lambda^0$, there exists $x \in v\Lambda^\infty$ such that $\sigma^n(x) \neq \sigma^m(x)$. We call Λ *cofinal* if for every $v \in \Lambda^0$ and $x \in \Lambda^\infty$, there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n) \neq \emptyset$. We say that Λ satisfies Condition (B) if for every $x \in v\Lambda^\infty$, there exists $\mu \neq \nu \in \Lambda v$ such that $\mu x \neq \nu x$.

3 Self-similar groupoid actions on k-graphs

Having defined the k-graph, we will now define a self-similar groupoid action on a k-graph. The source for this section is Afsar et al. (2019), particularly chapters 2, 3, 6 and 9.

Definition 3.1 *Let Λ be a k-graph. A partial isomorphism of Λ consists of vertices $v, w \in \Lambda^0$ and a bijection $g : v\Lambda \to w\Lambda$ satisfying*

1. *For all $n \in \mathbb{N}^k$, the restriction $g|_{v\Lambda^n}$ is a bijection of $v\Lambda^n$ onto $w\Lambda^n$*
2. *For $\lambda \in v\Lambda$ and $e \in s(\lambda)\Lambda$, we have $g(\lambda e) \in g(\lambda)\Lambda$.*

We call v the domain of g, denoted dom$(g) = v$, and w the codomain of g, denoted cod$(g) = w$. Denote by PIso(Λ) the set of all partial isomorphisms of Λ.

A partial isomorphism of a k-graph is a way to generalise permutations on sets, but it acts locally in the sense that it only permutes the paths starting with a specific vertex to the paths ending in another vertex.

Example 3.1 *Let $id : \Lambda \to \Lambda$ be the identity map on Λ. Then the restriction $id_v : v\Lambda \to v\Lambda$ for any $v \in \Lambda^0$ is a partial isomorphism with domain and codomain v.*

Lemma 3.1 *Let Λ be a k-graph. Then PIso(Λ) is a groupoid with unit space $\{id_v : v \in \Lambda^0\}$. We identify this space with Λ^0.*

We can now define the groupoid action on a k-graph by associating a partial isomorphism with a groupoid element, analogous to how one associates group elements with a permutation:

Definition 3.2 *Let G be a groupoid. We say that G acts on Λ if there exists a groupoid homomorphism $\varphi : G \to PIso(\Lambda)$. We denote the image of $g \in G$ under φ by $\varphi_g : dom(g)\Lambda \to cod(g)\Lambda$, and for $\lambda \in \Lambda$, we write the action of g on λ as $g \cdot \lambda := \varphi_g(\lambda)$.*

Immediately following is the definition of a self-similar groupoid action:

Definition 3.3 *A self-similar groupoid action is a pair (G, Λ) consisting of a k-graph Λ, a groupoid G with unit space Λ^0 and a faithful action of G on Λ such that for all $g \in G$ and paths $e \in dom(g)\Lambda$, there exists $h \in G$ such that*

$$g \cdot (e\lambda) = (g \cdot e)(h \cdot \lambda)$$

By faithfulness, h is unique, so we denote $h := g|_e$ and call it the restriction of g to e.

A groupoid can also act on infinite paths in the following way:

Definition 3.4 *Let (G, Λ) be a self-similar groupoid action and $x \in dom(g)\Lambda^\infty$. The action of $g \in G$ on x is defined by*

$$(g \cdot x)(p, q) = g|_{x(0,p)} \cdot x(p, q)$$

As a natural generalisation of the aperiodicity condition (defined in Kumjian & Pask (2000)), we have G-aperiodicity:

Definition 3.5 *Let Λ be a strongly connected finite k-graph. Let (G, Λ) be a self-similar groupoid action on Λ. We say Λ is G-aperiodic if for all $g \in G$ there exists $x \in dom(g)\Lambda^\infty$ such that for all $n \neq m \in \mathbb{N}^k$, we have $\sigma^n(x) \neq \sigma^m(g \cdot x)$.*

For the construction of the C^*-algebra of a k-graph Λ and a self-similar groupoid action (G, Λ), Afsar et al. (2019), particularly chapters 5 and 6, define the Nica-Toeplitz algebra $\mathcal{T}(G, \Lambda)$ and the Cuntz-Pimsner algebra $\mathcal{O}(G, \Lambda)$, and it is proven that $\mathcal{O}(G, \Lambda)$ is universal for families of unitaries satisfying certain properties, in a way very similar to the Cuntz-Krieger families.

4 C^*-algebras

For a comprehensive overview of C^*-algebras, see Putnam (2019). In particular, chapter 1 is essential for students who have not encountered C^*-algebras before. Chapter 1.11 on representations of C^*-algebras is essential for chapters 2 and 3. Finally, chapter 3.3 on the construction of a C^*-algebra from an étale groupoid is summarised in the appendix as a useful resource for understanding (algebraic) groupoids and the construction of C^*-algebras from groupoids.

The prototypical examples of C^*-algebras are the complex numbers $\mathbb{C}$ and the set of $n \times n$ matrices over the complex field $M_n(\mathbb{C})$, where the $*$-operation is the complex conjugate and the conjugate transpose respectively. Another is the set of bounded linear operators over a Hilbert space $\mathcal{H}$. A bounded linear operator f is an operator on $\mathcal{H}$ which is linear and $\|f\|_{\mathcal{B}(\mathcal{H})} := \sup_{h \in H}\{\|fh\|_{\mathcal{H}}\}$ is finite, where $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$ denotes the operator norm, and $\|\cdot\|_{\mathcal{H}}$ denotes the norm derived from the inner product on $\mathcal{H}$. In fact, the C^*-condition is very restrictive, to the point that the *only* C^*-algebras are of this form:

Theorem 4.1 *(Gelfand-Naimark-Segal) GNS construction. Every C^*-algebra is isomorphic to some $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$.*

The commutative unital C^*-algebras have an exact description as well:

Theorem 4.2 *Every commutative unital C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X*

See Putnam (2019) 1.4 and 1.12 for their statements and proofs.

4.1 The C^*-algebra of a k-graph

This section will define the universal C^*-algebra of a k-graph. Kumjian & Pask (2000) is the original paper which introduced the notion of k-graphs and their C^*-algebras. The appendix contains a brief write-up of the construction of the C^*-algebra of a directed graph, which will be required to understand how a k-graph generates a C^*-algebra.

We will need a way of representing a k-graph as partial isometries and projections:

Definition 4.1 *Let Λ be a row-finite k-graph with no sources. Let B be a C^*-algebra. A Cuntz-Krieger Λ-family in B is a set of partial isometries $s = \{s_\lambda : \lambda \in \Lambda\}$ satisfying the following axioms:*

1. *$\{s_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections in B.*

2. *$s_\lambda s_\mu = s_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$*

3. $s_\lambda^* s_\lambda = s_{s(\lambda)}$ *for all* $\lambda \in \Lambda$

4. $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$ *for arbitrary* $v \in \Lambda^0$ *and* $n \in \mathbb{N}^k$.

$C^*(\Lambda)$ is defined to be the universal C^*-algebra generated by the family s. It is universal in the sense that if $t = \{t_\lambda : \lambda \in \Lambda\}$ is another Cuntz-Krieger Λ-family in B generating the C^*-algebra $C^*(t)$, then there exists a unique isomorphism $\pi : C^*(\Lambda) \to C^*(t)$ sending s_λ to t_λ. For the full construction, see Sims (2010).

5 Conditions for the simplicity of $C^*(G, \Lambda)$

One of the major results of Robertson & Sims (2006) is the discovery of two conditions which imply that the ideals of $C^*(\Lambda)$ are trivial:

Theorem 5.1 *Let* Γ *be a row-finite k-graph with no sources. Then* $C^*(\Lambda)$ *is simple if and only if* Λ *is aperiodic and cofinal.*

A key result in proving this theorem is that Λ being aperiodic is if and only if Λ satisfying Condition (B) (see Robertson & Sims (2006) for the proof and the statement of Condition (B)). This summer project was aimed at discovering a way to generalise Condition (B) to the self-similar groupoid case such that it is equivalent to G-aperiodicity. We again list the definition of G-aperiodicity in Definition 5.1 and we also define our Condition G-(B) in Definition 5.2 as the groupoid analogue of Condition (B).

Definition 5.1 *Let* Λ *be a strongly connected finite k-graph. Let* (G, Λ) *be a self-similar groupoid action on* Λ. *We say* Λ *is G-aperiodic if for every* $g \in G$ *and every* $p \neq q \in \mathbb{N}^k$ *there exists* $x \in dom(g)\Lambda^\infty$ *such that* $\sigma^p(x) \neq \sigma^q(g \cdot x)$.

Definition 5.2 *Let* Γ *be a strongly connected finite k-graph and* (G, Λ) *a self-similar groupoid action on* Λ. *We say that* Λ *satisfies Condition G-(B) if, for all* $h \in G$, *there exists* $y \in dom(h)\Lambda^\infty$ *such that for all paths* $\mu \neq \nu$, *we have* $\mu(h \cdot y) \neq \nu(y)$.

Theorem 5.2 *Let* Λ *be a strongly connected finite k-graph and* (G, Λ) *be a self-similar groupoid action. Then* Λ *is G-aperiodic if* Λ *satisfies Condition G-(B).*

Statement: By the contrapositive, we assume Λ is not G-aperiodic and show that Λ does not satisfy Condition G-(B). That is, suppose there exists $g \in G$ and $p \neq q \in \mathbb{N}^k$ such that for all $x \in \text{dom}(g)\Lambda^\infty$, we have $\sigma^p(x) = \sigma^q(g \cdot x)$. We want to prove that there exists $h \in G$ such that for all $y \in \text{dom}(g)\Lambda^\infty$, there exists paths $\mu_y \neq \nu_y$ with $\mu(h \cdot y) \neq \nu(y)$.

Let $\xi \in \text{dom}(g)\Lambda^p$ and $\alpha \in s(\xi)\Lambda^{p \vee q - p}$. Let $\eta = (\xi\alpha)(0, q)$ and $\beta = (\xi\alpha)(q, p \vee q)$. Define $h := g|_{\eta\beta}$ and let $y \in \text{dom}(h)\Lambda^\infty$ be arbitrary. Then define $x = \xi\alpha y \in \text{dom}(g)\Lambda^\infty$. From the definition, we have $x = \xi\alpha y = \eta\beta y$. Claim:

1. $\alpha \neq g|_\eta \cdot \beta$
2. $\alpha y = (g|_\eta \cdot \beta)(h \cdot y)$

The first follows immediately by the disparity in the degrees of the morphisms α and $g|_\eta \cdot \beta$, since groupoid actions are degree preserving. For the second, we calculate:

$$\begin{aligned}
\alpha y = \sigma^p(\xi\alpha y) &= \sigma^q(g \cdot (\eta\beta y)) \quad (\text{Since } \xi\alpha y = x \in \text{dom(g)}\Lambda^\infty) \\
&= \sigma^q((g \cdot \eta)(g|_\eta \cdot \beta)(g|_{\eta\beta} \cdot y)) \\
&= (g|_\eta \cdot \beta)(g|_{\eta\beta} \cdot y) \quad (\text{Since } (g \cdot \eta) \text{ has degree } q) \\
&= (g|_\eta \cdot \beta)(h \cdot y) \quad (\text{By definition of } h)
\end{aligned}$$

proving the second claim.

This proves the contrapositive, since we have found paths $\mu = \alpha$ and $\nu = g|_\eta \cdot \beta$ and $h \in G$ such that for all $y \in \text{dom}(h)\Lambda^\infty$, $\mu y = \nu(h \cdot y)$ with $\mu \neq \nu$, which is exactly the negation of Condition G-(B). □

Unfortunately, due to the time constraint of this summer project, the reverse implication was not able to be proved. We have, however, produced a weaker result:

Theorem 5.3 *Λ satisfies Condition G-(B) if Λ has the following property: For all $g \in G$, there exists $x \in dom(g)\Lambda^\infty$ such that for all $p \neq q \in \mathbb{N}^k$, $\sigma^p(x) \neq \sigma^q(g \cdot x)$. Any k-graph which satisfies the above property is called locally G-aperiodic.*

Statement: Assume that Λ does not satisfy Condition G-(B). We want to prove that Λ does not satisfy the above condition. That is, we assume that there exists $h \in G$ such that for all $y \in \mathrm{dom}(h)\Lambda^\infty$, there exist paths $\mu_y \neq \nu_y$ satisfying

$$\mu_y(h \cdot y) = \nu_y(y)$$

and we want to show that there exists $g \in G$ and $p \neq q \in \mathbb{N}^k$ such that for all $x \in \mathrm{dom}(g)\Lambda^\infty$, we have

$$\sigma^{p_x}(x) = \sigma^{q_x}(g \cdot x)$$

Proof: Fix $y \in \mathrm{dom}(h)\Lambda^\infty$ arbitrary. Then there exists $\mu_y \neq \nu_y$ such that $\mu_y(h \cdot y) = \nu_y(y)$. Using the properties of the shift map,

$$\begin{aligned}\sigma^{d(\mu_y)\vee d(\nu_y)-d(\mu_y)}(h \cdot y) &= \sigma^{d(\mu_y)\vee d(\nu_y)}(\mu_y(h \cdot y)) \\ &= \sigma^{d(\mu_y)\vee d(\nu_y)}(\nu_y(y)) \\ &= \sigma^{d(\mu_y)\vee d(\nu_y)-d(\nu_y)}(y)\end{aligned}$$

To negate the statement, take $g = h \in G$, $x = y \in \mathrm{dom}(h)\Lambda^\infty$, and $p_x = d(\mu_y) \vee d(\nu_y) - d(\mu_y)$ and $q_x = d(\mu_y) \vee d(\nu_y) - d(\nu_y)$.

It remains to show that $p_x \neq q_x$. To do this, we show that $d(\mu_y) \neq d(\nu_y)$, which would prove our statement. Suppose by contradiction that $d(\mu_y) = d(\nu_y)$. By assumption, we have $\nu_y \neq \mu_y$ and $\mu_y(h \cdot y) = \nu_y(y)$. Then

$$(\mu_y(h \cdot y))(0, d(\mu_y)) = (\nu_y(y))(0, d(\mu_y)) = (\nu_y(y))(0, d(\nu_y))$$

That is, we are taking the first section of degree $d(\mu_y)$ of both infinite paths. Since these infinite paths are equal, we get that

$$\mu_y = (\mu_y(h \cdot y))(0, d(\mu_y)) = (\nu_y(y))(0, d(\nu_y)) = \nu_y$$

which is a contradiction with our assumption. □

At face value, local G-aperiodicity in Theorem 5.3 is stronger than G-aperiodicity; however, it is hoped that the two conditions can be proven equivalent. Indeed, Robertson & Sims (2006) proves that local aperiodicity and aperiodicity are equivalent in k-graphs, and their proof methods generalise readily in the cases that we have studied, so we remain hopeful that such a proof can be found.

6 Future directions

To prove that Condition G-(B) and G-aperiodicity are equivalent, future work would look at proving the equivalence of G-aperiodicity and local G-aperiodicity. Other useful directions to consider would be formulating an equivalent finite-path version of the infinite-path-reliant conditions G-aperiodicity and Condition G-(B), since finite paths are much easier to work with and create examples of. Such a formulation would allow researchers to generate counter-examples and proofs more easily.

7 Conclusion

In this report, we summarised the main background information on C^*-algebras of self-similar groupoid actions on higher rank graphs, including the theory on constructing C^*-algebras on some common algebraic structures (see appendix) and the main definitions and theorems. This culminated in us defining Condition G-(B) and local G-aperiodicity, proving that Condition G-(B) implies G-aperiodicity and that local G-aperiodicity implies Condition G-(B). We have also given interesting directions to look at for future projects.

Appendix

The purpose of this appendix is to give some of the definitions that we have left out for the sake of brevity, and detail constructions of C^*-algebras from different algebraic structures. These constructions were useful to read

The definition of a C^*-algebra originates from Putnam (2019).

Definition 7.1 *A C^*-algebra is a non-empty set A along with the addition and multiplication operations such that*

- *Addition is commutative and associative.*
- *Multiplication is associative.*
- *There exists a field of scalars (most commonly $\mathbb{C}$) and a multiplication defined on scalars and elements of A.*
- *There exists an operation $*$, called involution, such that $(a^*)^* = a$. It is conjugate linear (in the case the field is $\mathbb{C}$) and $(ab)^* = b^*a^*$.*
- *Addition and multiplication distribute over each other in the usual way.*

A must have a norm on it making it a Banach algebra. More precisely, the norm must satisfy

- $\|\lambda a\| = |\lambda|\|a\|$
- $\|a + b\| \leq \|a\| + \|b\|$
- $\|ab\| \leq \|a\|\|b\|$

with A being complete under the induced metric $|\cdot - \cdot|$. Finally, the C^*-condition must be satisfied: For all $a \in A$,

$$\|a^*a\| = \|a\|^2$$

7.1 Etale groupoids and the C^*-algebra of an étale groupoid

We include this section as a way to familiarise oneself with working with (étale) groupoids and the generic construction of a C^*-algebra, which usually involves looking at the bounded linear operators on some space and defining a norm on those operators.

The construction of a C^*-algebra from a groupoid relies heavily on the groupoid being étale:

Definition 7.2 *A topological groupoid G is called étale if the range and source functions $r, s : G \times G \to G$ are local homeomorphisms.*

For a locally compact, Hausdorff étale groupoid G, we denote by $C_c(G)$ the set of compactly supported continuous functions on G. $C_c(G)$ turns out to be very important in the construction of a groupoid C^*-algebra:

Theorem 7.1 *Let G be a locally compact, Hausdorff étale groupoid. Then $C_c(G)$ is a $*$-algebra with addition being the pointwise addition of functions, multiplication given by the formula*

$$(ab)(g) = \sum_{r(h)=r(g)} a(h)b(h^{-1}g)$$

for $a, b \in C_c(G)$ and $h, g \in G$, and involution given by

$$a^*(g) = \overline{a(g^{-1})}$$

Theorem 3.3.4 of Putnam (2019) gives a sketch of the main parts of the proof, including well-definedness and distributivity. Associativity and the properties of the involution are short calculations. In order to extend the $*$-algebra into a C^*-algebra, we need to define a norm in which the extension of $C_c(G)$ is complete, and the norm satisfies the C^*-property. Hence, we introduce the (indirect) notion of a left-regular representation of a groupoid. There is a similar construction for the case of group C^*-algebras.

Theorem 7.2 *Let $u \in G^{(0)}$ be a unit in the unit space of a locally compact Hausdorff étale groupoid G. Let $a \in C_c(G)$ and $\xi \in l^2(s^{-1}(u))$. Then the formula*

$$(\pi_\lambda^u(a)\xi)(g) = \sum_{r(h)=r(g)} a(h)\xi(h^{-1}g)$$

defines a bounded linear operator $\pi_\lambda^u : C_c(G) \to B(l^2(s^{-1}(u))$. It is a representation of $C_c(G)$ as a bounded operator on the Hilbert space $l^2(s^{-1}(u)))$.

For a detailed proof of this theorem, see Theorem 3.3.11 in Putnam (2019).

Definition 7.3 *The left regular representation is defined by the direct sum of the representations of $C_c(G)$ over the units in the unit space of G:*

$$\pi_\lambda = \bigoplus_{u \in G^{(0)}} \pi_\lambda^u$$

Definition 7.4 *The full C^*-algebra and the reduced C^*-algebra. For a locally compact Hausdorff étale groupoid, we write $C^*(G)$ to denote the C^*-algebra of G. $C^*(G)$ is the completion of $C_c(G)$ in the norm*

$$\|a\| = \sup\{\|\pi(a)\| | \pi : C_c(G) \to B(H) \text{ a representation of } C_c(G)\}$$

The reduced C^*-algebra is the completion of $C_c(G)$ in the norm

$$\|a\|_r = \|\pi_\lambda(a)\|, \text{ with } \pi_\lambda \text{ the left regular representation of } C_c(G)$$

where the norm of the representation is the operator norm in the associated Hilbert space.

The preceding definitions require numerous technical results which, for the sake of brevity, have been left out in this report. In particular, we have neglected the definitions of G-sets (also called bisections, as in Sims (2019)), which are core to the theory, as the continuous functions supported on G-sets form a dense set of $C_c(G)$. The main technicalities are in proving that the supremum over arbitrary representations of $C_c(G)$ is finite and well-defined. As it turns out, (see Lemma 3.3.16 Putnam (2019)), this is indeed true!

7.2 The C^*-algebra of a directed graph

This section of the appendix will introduce the construction of a C^*-algebra of a directed graph. The main resource for this section is Aidan Sims' notes on Operators and Graph Algebras, and also includes a handy introduction to Hilbert spaces and orthogonal projections in Hilbert spaces. The construction of a k-graph C^*-algebra mirrors that of directed graphs, albeit with technicalities.

Definition 7.5 *Let E be a row-finite directed graph with no sources. Let A be a C^*-algebra. A Cuntz-Krieger E-family in a C^*-algebra is a pair of sets (p, s), where $p = \{p_v | v \in E^0\}$ is a set of projections in A associated to each vertex of E and $s = \{s_e | e \in E^1\}$ is a set of partial isometries associated to each edge in E. The family (p, s) must satisfy the following conditions:*

(CK 1) $s_e^* s_e = p_{s(e)}$ *for any* $e \in E^1$, *and* $s_e \in s, p_v \in p$

(CK 2) $p_v = \sum_{r(e)=v} s_e s_e^*$

Let (p, s) be a Cuntz-Krieger E-family in a C^*-algebra A. The C^*-algebra generated by (p, s) is denoted $C^*(p, s) := C^*(p \cup s)$ and is defined as follows:

$C^*(p, s) := \{\cap_{i \in I} B_i | B_i$ C^*-subalgebras of A containing all elements p_v, s_e with $v \in E^0, e \in E^1\}$

This is the smallest C^*-subalgebra of A that contains the elements of p and s, and it is a theorem that $C^*(p, s)$ is indeed a C^*-algebra - see Lemma 4.10 [GraphAlgebra] In the next theorem, we write $E^* * E^*$ to denote the set $\{(u, v) \in E^* \times E^* | s(u) = s(v)\}$.

Theorem 7.3 *Let E be a row-finite directed graph with no sources. We write $V(E)$ to denote the complex vector space of finitely supported functions $f : E^* * E^* \to \mathbb{C}$. $V(E)$ is equipped with pointwise addition and scalar multiplication. Following [GraphAlgebra], let $M_{u,v}$ represent the point mass function that is 1 on $(u, v) \in E^* * E^*$ and 0 elsewhere. Then $M_{(u,v)}$ is a basis for $V(E)$. Define $* : V(E) \to V(E)$ by $M_{u,v}^* = M(v, u)$. Then $V(E)$ forms a $*$-algebra with involution $*$ and the multiplication of functions given by*

$$M_{u,v} M_{a,b} = \begin{cases} M_{u,bv'} & \text{if } v = av' \\ M_{ua',b} & \text{if } a = ua' \\ 0 & \text{otherwise} \end{cases}$$

Theorem 7.4 *Let E be a row-finite directed graph with no sources. For any Cuntz-Krieger E-family (q, t), there exists a $*$-homomorphism $\pi^0_{q,t} : V(E) \to C^*(q, t)$ such that*

$$\sum_{(u,v) \in E^* * E^*} a_{(u,v)} M_{u,v} \mapsto \sum_{(u,v) \in E^* * E^*} a_{(u,v)} t_u t_v^*$$

Define the function

$$N(a) := \sup\left\{\|\pi^0_{q,t}\| | (q,t) \text{ a Cuntz-Krieger family}\right\}$$

Then N defines a finite semi-norm on $V(E)$. The quotient $A_E = V(E)/\ker(N)$ is well-defined. The completion of A_E with the norm $\|a + \ker(N)\| := N(a)$ is denoted by $C^*(E)$ and is a C^*-algebra.

See Theorem 4.16 in Aidan Sims' notes for the proof. The quotient process removes the offending elements which are non-zero yet their semi-norms are zero - this allows the semi-norm to become a norm in the quotient space.

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