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Optimal Reinsurance Contract from A Principal-Agent Perspective

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1 Abstract

This report is an investigation into an Excess-Loss reinsurance relationship within a continuous-time Principal-Agent framework where it is assumed that the claims process can be described by the Cramer-Lundberg model and the Agent's (Insurer), and Principal's (Re-insurer) wealth equation's adhere to the Expected-Premium Principal. This is subject to conditions regarding how the wealth is allocated at any given time. Based upon the paper *Robust reinsurance contracts with uncertainty about jump risk (Hu, Chen, & Wang 2018)*[5]. Bellman's Principle of optimality is utilised, with the respective Hamilton-Jacobin-Bellman Equation for the wealth dynamics of the Principal and Agent being solved to find the value functions. During this procedure the optimal values for the Agent's Risk Retention (Insurer's Retention Rate) and the Principal's Safety Loading Factor (Reinsurance price) is expressed. A form of the claim distribution is assumed to produce explicit solutions. The nature of these solutions, namely the Safety Loading Factor and the Risk Retention rate is analysed using Sensitivity Analysis.

Contents

2 Introduction

This report is primarily based on the papers *Robust reinsurance contracts with uncertainty about jump risk by (Hu, Chen & Wang 2018)* and *Reinsurance contract design when the insurer is ambiguity-averse (Hu & Wang 2019)*. These two papers similarly elucidate an optimisation process of the Safety Loading Factor and Risk Retention factor of the Principal and Agent respectively for given wealth processes of an Insurer and a Re-Insurer. The model presented by *(Hu et al. 2018)* was constructed under the assumption of uncertainty in regards to the claim intensity rate of the Poisson Process. In this research report, it is assumed that there is no uncertainty is held by both parties and, it is to this extent that it must be stated that this report is the representation of a specific case of the generalised model outlined in the aforementioned papers. This is with the distinction that the process of solving for the parameters produced in this model requires numerical methods to approximate, which was able to be mitigated through the utilisation of a jump intensity adjuster function in the original paper.[5]

Re-insurance contracts are risk management tools used by Insurance companies to transfer risk to external entities. They are contracts between Insurance and Re-insurance firms where there is an agreement that the insurance company will transfer some level of risk to the re-insurer in the form of the re-insurer agreeing the indemnify the insurance company in the case of a loss. This has the effect of reducing the probability of the insurance company being liable to meet claim obligations in excess of what is predicted. Reinsurance serves many functions within the financial sector, it reduces the unearned premium reserves and increases the insurance companies underwriting capacity. The process of optimising the parameters such as the Re-insurer's Safety Loading Factor and Insurer's Risk Retention factor is an important part of ensuring that risk retained by the Insurer is effectively and efficiently pooled, or transferred. In this report it is assumed that the Reinsurance contract which is agreed upon in this Reinsurance relationship is an Excess-loss contract. This stipulates that if the obligations faced by an Insurance company in relation to claims to policyholders are over a pre-specified monetary value then the excess amount owing over that level is covered for by the Reinsurance company.

This report will consist of three main sections followed by Sensitivity Analysis. The first two sections will encompass the derivation of the optimised reinsurance proportion and Safety Loading Factor, and the corresponding value functions for the Insurer and the Re-insurer. The third section will outline an assumed claim distribution function as outlined in *Hu et al. (2019)* to produce semi-analytic representations of the optimised parameters. The Agent section will be positioned from the perspective of the insurance company with the goal of trying to determine expressions for the value function of the insurer and solve the optimal control problem for the Risk Retention parameter. The second section will focus on the Principal problem from the Re-insurer's perspective with the goal of determining an expression for the Re-insurer's value function and the optimal Safety Loading Factor. This report will utilise the Hamilton-Jacobin-Bellman (HJB) Equation along with assuming the dynamics of the claim process can be modelled using the Cramér–Lundberg (C-L) model, and that the Expected

Premium Principle is valid in the representation of the predicted wealth dynamics to determine an expression for the optimal Retention Rate and optimal Safety Loading Factor for the Principal. It is as a result of solving the HJB equation, that the value functions for both the Re-insurer's and insurance contracts will be derived.

The choice of using the Principal-Agent framework for optimising the optimal retention value and optimal re-insurance price parameters given the wealth dynamics of a two-party system is especially useful in this scenario as it allows us to take into consideration the perceptions of both the Insurer and the Re-Insurer. This means that the optimal solution combines the preferences of both parties which is important in this context as the perceptions held by the Principal or Agent can change the optimal dynamics of the parameters.

Following the derivation of the optimal Safety Loading Factor and retention rate functions, and both the Agent's and Principal's value functions, sensitivity analysis will be conducted to visually elucidate behaviours that can arise within the model whilst attempting to draw conclusions from the optimal directions taken by the Principal and Agent within the model derived.

Statement of Authorship

The workload was divided as follows:

1. Patrick Gillen. Completed mathematic formulation, derived theoretical results, wrote the report, wrote R code.
2. Ning Wang. Proofread document, provided sources, taught concepts, assisted in the technical aspects.

3 Model

This model is constructed within in a filtered probability space $(\Omega, F, (F_{t\in[0,T]}), P)$. In which the filtration is produced by a Compound Poisson Process, where the dynamics of the claim process can be modelled by the C-L model.

$$dC_t = cdt - d\sum_{i=1}^{N(t)} Z_i$$

Where $\sum_{i=1}^{N(t)} Z_i$ is a Compound Poisson Process that acts to display the net number of claims to a time t, $c \in \mathbf{R}^+$ is a positive constant that represents the claim premium return, $\{N(t)\}_{t\in[0,T]}$ is an homogeneous Poisson process with a corresponding intensity $\eta \in \mathbf{R}^+$. For the Compound Poisson process it is assumed that the claim sizes $\{Z_i, i \geq 0\}$ are identically distributed and independent with a probability distribution function $f(Z)$, a cumulative distribution function $F(Z)$ and a corresponding inverse cumulative distribution function/survival function $\bar{F}(Z)$ such that $Z_i > 0$ and that the process is also independent of the homogeneous Poisson process. It is also assumed that $E[Z_i] = \mu$.

4 Agent Problem

4.1 Wealth Equation

The model for the Agent's wealth process is created under the assumption that in one period the Agent would receive income from insurance premiums and income from the returns generated from a deposit account with a fixed rate of return. In the same period the insurance company would be subject to the obligations arising from claim's that are still held by the insurer, and pay for make payments to the reinsurer for the contracts reinsured. It is assumed that changes in wealth will adhere to the Expected Premium Principle. The control variables are defined such that $p, \beta \in C([0,T] \mapsto [0,1])$. The control variable $p = \{p_t, t \in [0,T]\}$ will be defined to be the Agent's Risk Retention and, $\beta = \{\beta_t, t \in [0,T]\}$ be the principal's Safety Loading Factor, $\theta \in \mathbf{R}^+$ be the Safety Loading Factor for the insurer. Since the Re-Insurance contract is of the form of an Excess-Loss contract, the amount reinsured is defined to be the losses which exceed a certain agreed upon level this boundary is determined by the suggested Risk Retention which can be expressed as $Z_i - Min[Z_i, p_t]$. $w : [0,T] \mapsto \mathbf{R}$ is defined to be the state variable of the system that denotes the wealth process. Then w_t satisfies the following equation:

$$\begin{aligned} dw_t &= [(1+\theta)\eta\mu - (1+\beta_t)\eta\mathbf{E}_t\left[Z_i - Min[Z_i, p_t]\right]]\,dt + rw_t dt - d\sum_{i=1}^{N(t)} Min[Z_i, p_t] \\ &= \left((\theta - \beta_t)\eta\mu + \eta(1+\beta_t)\mathbf{E}_t[Min[Z_i, p_t]]\right)dt + rw_t dt - d\sum_{i=1}^{N(t)} Min[Z_i, p_t] \\ &= \left[(\theta - \beta_t)\eta\mu + \eta(1+\beta_t)\int_0^{p_t} \bar{F}(Z)dZ + rw_t\right]dt - d\sum_{i=1}^{N(t)} Min[Z_i, p_t] \end{aligned} \quad (1)$$

In this case it can be seen that $c = (1+\theta)\eta\mu - (1+\beta_t)\eta\mathbf{E}_t\left[Z_i - Min[Z_i, p_t]\right] + rw_t$ and the Expected Reinsurance cost and insurance income are both a result of the expected premium principle. In this scenario HJB equation will be constructed with the goal being trying to maximise the expected utility of the terminal value. As outlined inn *Hu et al. (2018)* a function $v \in C^{(1,2)}([0,T] \times \mathbf{R} \mapsto \mathbf{R})$ is defined such that:

$$v_t(t,w) + Sup_{p\in[0,1]}\{L(t,w;p)\} = 0$$

Where $L : [0,T) \times \mathbf{R} \times [0,1] \mapsto \mathbf{R}$ is comprised of the wealth dynamics of the the system and the Expected change in the value following a jump, or change in the claims process.

$$L(t,w;p) = A(t,w;p)v_w(t,w;p) + \mathbf{E}[v(t,w-y(Z_i)) - v(t,w)]$$

$$dw_t = A(t,w;p)dt - d\sum_{i=1}^{N(t)} y(Z_i)$$

.

Comparing the coefficient functions of v_w with Equation (1) yields:

$$\begin{aligned} A(t,w;p) =& (\theta-\beta_t)\eta\mu + \eta(1+\beta_t)\int_0^{p_t}\bar{F}(Z)dZ + rw \\ y(Z_i) =& Min[Z_i, p] \end{aligned}$$

Substituting this into the HJB Equation results in the following:

$$\begin{aligned} v_t(t,w) + Sup_{p\in[0,1]}\Bigg\{ \left[(\theta-\beta_t)\eta\mu + \eta(1+\beta_T)\int_0^{p_t}\bar{F}(Z)dZ + rw\right] v_w(t,w;p) \\ +\eta\mathbf{E}_t[v(t,w-Min[Z_i,p]) - v(t,w)]\Bigg\} = 0 \end{aligned} \tag{2}$$

Letting utility function be the exponential function, results in the objective and value function being of the form:

$$J(t,w) = v(t,w;p) = Sup_{p\in[0,1]}E[U(w(T))] = Sup_{p\in[0,1]}E\left[-\frac{e^{-\lambda w(T)}}{\lambda}\right]$$

With the terminal value being of the form:

$$U(w(T)) = v(T,w;p) = -\frac{e^{-\lambda w}}{\lambda}$$

Where $\lambda \in (0,1]0$ is a measure of risk aversion. The higher the risk aversion parameter the more risk-averse the Agent is. For $v(t,w;p)$ such that $t \in [0,T)$. I will use the assumed value function as presented by *Hu, Chen, and Wang (2018)*$_{[4]}$. This value function will be of the form:

$$v(t,w;p) = -\frac{exp\left(-\lambda(we^{r(T-t)} + h(t))\right)}{\lambda} \tag{3}$$

Where $h \in C^2([0,T] \mapsto \mathbf{R})$ is a function to be determined with the boundary condition that $h(T) = 0$

4.2 Optimising Risk Retention Factor

In this section, the supremum of the Risk Retention factor will be derived. Taking the first partial derivatives of the assumed value function produces:

$$v_t(t,w;p) = \lambda(we^{r(T-t)} - h'(t))v$$
$$v_w(t,w;p) = -\lambda e^{r(T-t)}v$$

As displayed in Equation (22) in Appendix (9.1) the Expectation of the change in value following the jump in claims process can be expressed as:

$$\mathbf{E}_t[v(t,w-Min[Z_i,p]) - v(t,w)] = v(t,w)\lambda e^{r(T-t)}\int_0^p exp(\lambda Ze^{r(T-t)})\bar{F}(Z)dZ \tag{4}$$

Substituting the result from Equation (4) into Equation (2) produces:

$$\begin{aligned}&\lambda(we^{r(T-t)} - h'(t))v + Sup_{p\in[0,1]}\Bigg\{\left[(\theta-\beta_t)\eta\mu + \eta(1+\beta_t)\int_0^p \bar{F}(Z)dZ + rw\right](-\lambda e^{r(T-t)}v)\\ &+\eta v\lambda e^{r(T-t)}\int_0^p exp(\lambda Ze^{r(T-t)})\bar{F}(Z)dZ\Bigg\} = 0\end{aligned} \tag{5}$$

The process of solving for the Agent's optimal Risk Retention factor is displayed in the Appendix (9.2). It is of the form:

$$p_t^* = \frac{ln(1+\beta_t)}{\lambda}e^{-r(T-t)} \tag{6}$$

Analysing this equation we can make inferences about the nature of a optimised reinsurance relationship where the Principal's Safety Loading Factor is not optimal. If the Agent's risk aversion parameter increases this would result in optimal stop-loss value increases. This is in line with what is predicted as a higher risk aversion parameter would imply that the Agent is less comfortable with higher risk. In this equation we can also see that as time tends to maturity then optimal stop-loss point will decrease.

Assuming that Equation (5) consists of the Agent's optimized Risk Retention factor and solving for the Insurer's auxiliary value function we get the following:

$$\begin{aligned}h'(t) &= \eta e^{r(T-t)}\int_0^{p_t^*} exp(\lambda Ze^{r(T-t)})\bar{F}(Z)dZ - (\theta-\beta_t)\eta\mu e^{r(T-t)} - \eta(1+\beta_t)e^{r(T-t)}\int_0^{p_t^*}\bar{F}(Z)dZ\\ \Longrightarrow h(t) &= -\int_t^T \eta e^{r(T-\alpha)}\int_0^{p_\alpha^*} exp(\lambda Ze^{r(T-\alpha)})\bar{F}(Z)dZd\alpha + \int_t^T(\theta-\beta_\alpha)\eta\mu e^{r(T-\alpha)}d\alpha\\ &+\int_t^T \eta(1+\beta_\alpha)e^{r(T-\alpha)}\int_0^{p_\alpha^*}\bar{F}(Z)dZd\alpha\end{aligned} \tag{7}$$

Since the Safety Loading Factor is dependent on time solving the Insurer's auxiliary function requires an explicit solution to the Principal's optimal Safety Loading Factor. Since it will be shown that an explicit solution for the principal's optimised Safety Loading Factor cannot be found the Insurer's auxiliary value function will be of a semi-analytical nature requiring numerical analysis to determine approximations for the function.

5 Principal Problem

5.1 Wealth Equation

In the same manner as what was presented in the Agent section it is assumed that Principal's claim process can be modelled using the C-L model. It is also assumed that in one period the reinsurance company will receive returns from a deposit account compounding at a fixed rate $r \in \mathbf{R}^+$. The reinsurance firm will also receive the premiums from the reinsurance contacts and pay the claims which they have re-insured. It is also assumed the the parameters p, β, μ are defined as in the same way as in the Agent problem section. The state variable $\hat{w} : [0, T] \mapsto \mathbf{R}$ that represents will then satisfy the following equation:

$$
\begin{aligned}
d\hat{w}_t &= r\hat{w}_t dt + (1+\beta_t)\eta \mathbf{E}[Z_i - Min(Z_i, p_t)]dt - d\sum_{i=1}^{N(t)}[Z_i - Min[Z_i, p_t]] \\
&= r\hat{w}_t dt + (1+\beta_t)\eta\mu dt - (1+\beta_t)\eta \int_0^{p_t} \bar{F}(Z)dZdt - d\sum_{i=1}^{N(t)}[Z_i - Min[Z_i, p_t]]
\end{aligned}
\tag{8}
$$

In the same manner as what occurred in the Agent section the Equation (8) is substituted into the HJB equation. If it is assumed that the equation includes the optimised Risk Retention parameter it will be of the form:

$$
\begin{aligned}
\hat{v}_t(t, \hat{w}) + Sup_{\beta \in [0,1]} \Bigg\{ & \left[r\hat{w}_t + (1+\beta_t)\eta\mu - (1+\beta_t)\eta \int_0^{p_t^*} \bar{F}(Z)dZ \right] \hat{v}_{\hat{w}}(t, \hat{w}; \beta) \\
& + \eta \mathbf{E}_t[\hat{v}(t, \hat{w} - Z_i + Min[Z_i, p_t^*]) - \hat{v}(t, \hat{w})] \Bigg\} = 0
\end{aligned}
\tag{9}
$$

The Principal's utility function will be assumed to be of the same form as the Agent's being represented in the form of an exponential function. This results in the objective and value function being of the form:

$$\hat{J}(t, \hat{w}) = \hat{v}(t, \hat{w}; \beta) = Sup_{\beta \in [0,1]} E[U(\hat{w}(T))] = Sup_{\beta \in [0,1]} E\left[-\frac{e^{-\gamma \hat{w}(T)}}{\gamma} \right]$$

With the terminal value being of the form:

$$U(\hat{w}(T)) = \hat{v}(T, \hat{w}; \beta) = -\frac{e^{-\gamma \hat{w}}}{\gamma}$$

In this case $\gamma \in (0, 1]$ is a measure of the Principal's risk aversion. The assumed value function as presented by *Hu, Chen, and Wang (2018)*[4] will again be used, this produces a value function of the form:

$$\hat{v}(t, \hat{w}; \beta) = -\frac{exp\left(-\gamma(\hat{w}e^{r(T-t)} + g(t))\right)}{\gamma} \tag{10}$$

Where $g \in C^2([0, T] \mapsto \mathbf{R})$ is a function to be determined with the terminal condition that $g(T) = 0$

5.2 Optimising Safety Loading Factor

As displayed in the Appendix (9.3) the expected change in value of the re-insurer following a jump in the claims process can be expressed as:

$$\mathbf{E}_t[\hat{v}(t, \hat{w} - Z_i + Min[Z_i, p]) - \hat{v}(t, \hat{w})] = \hat{v}(t, \hat{w}) exp(-\gamma p e^{r(T-t)}) \int_p^{\infty} exp(\gamma Z e^{r(T-t)}) f(Z)dZ - \bar{F}(p)$$

Assuming that the HJB equation consists of the optimised Safety Loading Factor, optimised agent Risk Retention factor, and substituting the above expression and the value function and the derivatives of the value function produces a HJB equation of the form:

$$\begin{aligned}&\gamma(\hat{w}e^{r(T-t)} - g'(t))\hat{v} - \gamma e^{r(T-t)}\hat{v}\left[r\hat{w}_t + (1+\beta_t^*)\eta\mu - (1+\beta_t^*)\eta\int_0^{p_t^*}\bar{F}(Z)dZ\right]\\&+\eta\hat{v}exp(-\gamma p_t^* e^{r(T-t)})\int_{p_t^*}^{\infty}exp(\gamma Z e^{r(T-t)})f(Z)dZ - \eta\hat{v}\bar{F}(p_t^*) = 0\end{aligned} \tag{11}$$

As displayed in Equation (29) in Appendix (9.4), the Principal's Optimised Safety Loading Factor without making assumptions in regard to the nature of the claim distribution is a solution to the following equation:

$$\begin{aligned}&-\gamma e^{r(T-t)}\mu + \gamma e^{r(T-t)}\int_0^{p_t^*}\bar{F}(Z)dZ + (1+\beta_t^*)\gamma e^{r(T-t)}\bar{F}(p_t^*)\frac{\partial p^*}{\partial \beta^*}\\&-\gamma e^{r(T-t)}exp(-\gamma p_t^* e^{r(T-t)})\frac{\partial p^*}{\partial \beta^*}\int_{p_t^*}^{\infty}exp(\gamma Z e^{r(T-t)})f(Z)dZ\\&+exp(-\gamma p_t^* e^{r(T-t)})\frac{\partial}{\partial \beta_t^*}\left[\int_{p_t^*}^{\infty}exp(\gamma Z e^{r(T-t)})f(Z)dZ\right] - \bar{F}'(p_t^*)\frac{\partial p_t^*}{\partial \beta_t^*} = 0\end{aligned} \tag{12}$$

An explicit solution for the Principal's Safety Loading Factor requires information in regard to the to claim size distribution. Solving for the Re-insurer's Auxiliary function produces an equation of the form:

$$\begin{aligned}&g'(t) = e^{r(T-t)}(1+\beta_t)\eta\int_0^{p_t^*}\bar{F}(Z)dZ - e^{r(T-t)}\eta(1+\beta_t)\mu\\&+\frac{\eta}{\gamma}exp(-\gamma p_t^* e^{r(T-t)})\int_{p_t^*}^{\infty}exp(\gamma Z e^{r(T-t)})f(Z)dZ - \frac{\eta}{\gamma}\bar{F}(p_t^*)\\\Longrightarrow\ &g(t) = \int_t^T e^{r(T-t)}\eta(1+\beta_\alpha)\mu d\alpha - \int_t^T e^{r(T-\alpha)}(1+\beta_\alpha)\eta\int_0^{p_\alpha^*}\bar{F}(Z)dZd\alpha\\&-\int_t^T\frac{\eta}{\gamma}exp(-\gamma p_\alpha^* e^{r(T-\alpha)})\int_{p_\alpha^*}^{\infty}exp(\gamma Z e^{r(T-\alpha)})f(Z)dZd\alpha + \int_t^T\frac{\eta}{\gamma}\bar{F}(p_\alpha^*)d\alpha\end{aligned} \tag{13}$$

A semi-analytical solution for the Principal's auxiliary value function also requires knowledge about the claim size distribution. These solutions will be found in the following section.

6 Claim distribution assumption

In this section an explicit expression for the Insurer's and Re-insurer's auxiliary function and the Re-insurer's Safety Loading Factor will be found. To find an explicit solution for the value function of the Agent, I will follow *D. Hu, et al.(2018)*[5] by will assuming that the claim sizes will follow an exponential distribution.

$$\begin{aligned} f(Z) &= ke^{-kZ} \\ F(Z) &= \int_0^Z ke^{-ka}da = 1 - e^{-kZ} \\ \bar{F}(Z) &= e^{-kZ} \end{aligned} \tag{14}$$

For some $k \in \mathbf{R}^+$ which represents the mean claim size.

6.1 Optimal Safety Loading Factor

As displayed in Equation (29) in Appendix (9.3), substituting the derivative of the Agent's Risk Retention parameter with respect to the Safety Loading Factor into Principal's HJB Equation, and substituting the optimised Risk Retention factor into all terms not requiring a claim assumption produces:

$$\begin{aligned} &-\gamma e^{r(T-t)}\mu + \gamma e^{r(T-t)}\int_0^{p_t^*}\bar{F}(Z)dZ + \frac{\gamma}{\lambda}\bar{F}(p_t^*) - \frac{\gamma}{\lambda}(1+\beta)^{-\frac{\gamma}{\lambda}-1}\int_p^\infty exp(\gamma Z e^{r(T-t)})f(Z)dZ \\ &+ (1+\beta)^{-\frac{\gamma}{\lambda}}\frac{d}{d\beta}\left[\int_p^\infty exp(\gamma Z e^{r(T-t)})f(Z)dZ\right] - \frac{e^{-r(T-t)}\bar{F}'(p_t^*)}{(1+\beta)\lambda} = 0 \end{aligned} \tag{15}$$

Substituting the claim distribution assumption into Equation (15) produces

$$\begin{aligned} &-\gamma e^{r(T-t)}\mu + \gamma e^{r(T-t)}\int_0^{p_t^*}e^{-kZ}dZ + \frac{\gamma}{\lambda}e^{-kp_t^*} - \frac{\gamma k}{\lambda}(1+\beta)^{-\frac{\gamma}{\lambda}-1}\int_p^\infty exp(\gamma Z e^{r(T-t)})e^{-kZ}dZ \\ &+ k(1+\beta)^{-\frac{\gamma}{\lambda}}\frac{d}{d\beta}\left[\int_p^\infty exp(\gamma Z e^{r(T-t)})e^{-kZ}dZ\right] + \frac{ke^{-r(T-t)}e^{-kp_t^*}}{(1+\beta)\lambda} = 0 \\ &-\gamma e^{r(T-t)}\mu + \frac{\gamma}{k}e^{r(T-t)}[1-e^{-kp_t^*}] + \frac{\gamma}{\lambda}e^{-kp_t^*} + \frac{\gamma k}{(\gamma e^{r(T-t)}-k)\lambda}(1+\beta)^{-\frac{\gamma}{\lambda}-1}exp(p_t^*(\gamma e^{r(T-t)}-k)) \\ &-\frac{k(1+\beta)^{-\frac{\gamma}{\lambda}}}{\gamma e^{r(T-t)}-k}\frac{d}{d\beta}\left[exp(p_t^*(\gamma e^{r(T-t)}-k))\right] + \frac{ke^{-r(T-t)}e^{-kp_t^*}}{(1+\beta)\lambda} = 0 \end{aligned} \tag{16}$$

In the process of integrating the third and forth terms in Equation (16) there is an implicit assumption that $\gamma e^{r(T-t)} - k < 0$ bounding the integral. Substituting the optimised Agent's Risk Retention factor into Equation (16) produces:

$$\begin{aligned} &-\gamma e^{r(T-t)}\mu + \frac{\gamma e^{r(T-t)}}{k}[1-(1+\beta)^{-\frac{k}{\lambda}e^{-r(T-t)}}] + \frac{\gamma(1+\beta)^{-\frac{k}{\lambda}e^{-r(T-t)}}}{\lambda} \\ &+ \frac{\gamma k(1+\beta)^{\frac{-ke^{-r(T-t)}}{\lambda}-1}}{(\gamma e^{r(T-t)}-k)\lambda} - \frac{ke^{-r(T-t)}(1+\beta)^{\frac{-ke^{-r(T-t)}}{\lambda}-1}}{\lambda} + \frac{e^{-r(T-t)}k(1+\beta)^{-\frac{k}{\lambda}e^{-r(T-t)}-1}}{\lambda} = 0 \end{aligned}$$

This results in the optimal Safety Loading Factor satisfying the following equation:

$$(1+\beta^*)^{-\frac{k}{\lambda}e^{-r(T-t)}}\left[\frac{1}{\lambda} - \frac{e^{r(T-t)}}{k}\right] + \frac{k(1+\beta^*)^{-\frac{k}{\lambda}e^{-r(T-t)}-1}}{(\gamma e^{r(T-t)}-k)\lambda} - e^{r(T-t)}\mu + \frac{e^{r(T-t)}}{k} = 0$$

An expression for the optimised parameter is to be calculated using numerical methods. This will be calculated in the following sensitivity analysis section. Analysing the equation above we see that if the Agent's risk aversion parameter is equal to 0 the function is not defined. This is also true if the Principal's risk aversion parameter is equal to 0.

Substituting the claim distribution assumption and the optimal proportion of retained claims in the second derivative of the Principal's HJB Equation with respect to the Safety Loading Factor produces as the following:

$$\begin{aligned}\frac{dL^2}{d\beta^2}(t,\hat{w}) &= \eta\hat{v}\Big(-\frac{\gamma}{\lambda^2}(1+\beta)^{-\frac{k}{\lambda}e^{-r(T-t)}-1}(ke^{-r(T-t)}-\lambda) - \frac{\gamma k(1+\beta^*)^{-\frac{k}{\lambda}e^{-r(T-t)}-2}}{(\gamma e^{r(T-t)}-k)\lambda^2}\Big(ke^{-r(T-t)}-\lambda\Big)\Big)\\ &= -\frac{\gamma\eta\hat{v}}{\lambda^2}(1+\beta)^{-\frac{k}{\lambda}e^{-r(T-t)}-2}(ke^{-r(T-t)}-\lambda)\Big(1+\beta+\frac{k}{\gamma e^{r(T-t)}-k}\Big)\end{aligned}$$

To ensure that the parameter is a supremum we must place the condition that

$$ke^{-r(T-t)} - \lambda > 0$$

$$1+\beta+\frac{k}{\gamma e^{-r(T-t)}-k} > 0$$

In the process of modelling the optimal parameters it must be verified that the optimal safety loading parameter adheres to this assumption in the sensitivity analysis section.

6.2 Agent's Optimal Retention rate

Since the Agent's Optimal Retention rate is dependent on the Principal's Safety Loading Factor, and an explicit solution to the Safety Loading Factor is dependent on assuming a form of the claims distribution. If the Optimised Safety Loading Factor is included into the Optimal Retention rate, it will be of the form:

$$p_t^* = \frac{ln(1+\beta_t^*)}{\lambda}e^{-r(T-t)} \tag{17}$$

6.3 Insurer's Auxiliary Value function

In this section a semi-analytical solution to the Insurer's and Re-insurer's value function. Substituting the claim distribution assumption into the expression for the Auxiliary function produces:

$$\begin{aligned}h(t) &= -\int_t^T \eta e^{r(T-\alpha)}\int_0^{p_\alpha^*} exp(\lambda Z e^{r(T-\alpha)})e^{-kZ}dZd\alpha + \int_t^T(\theta-\beta_\alpha)\eta\mu e^{r(T-\alpha)}d\alpha\\ &+ \int_t^T \eta(1+\beta)e^{r(T-\alpha)}\int_0^{p_\alpha^*} e^{-kZ}dZd\alpha\\ &= -\int_t^T \frac{\eta e^{r(T-\alpha)}exp(p_\alpha^*(\lambda e^{r(T-\alpha)}-k))}{\lambda e^{r(T-\alpha)}-k}d\alpha + \int_t^T(\theta-\beta_\alpha)\eta\mu e^{r(T-\alpha)}d\alpha\\ &+ \int_t^T \frac{\eta(1+\beta)e^{r(T-\alpha)}[1-e^{-kp_\alpha^*}]}{k}d\alpha\end{aligned} \tag{18}$$

Since the Agent's Risk Retention parameter is dependent on the Principal's Safety Loading Factor which does not have an explicit solution. Following the substitution the Agent's optimal retention rate produces we get the following:

$$h(t) = -\int_t^T \frac{\eta e^{r(T-\alpha)}(1+\beta_\alpha^*)^{1-\frac{k}{\lambda}e^{r(T-t)}}}{\lambda e^{r(T-\alpha)} - k} d\alpha + \int_t^T (\theta - \beta_\alpha)\eta\mu e^{r(T-\alpha)} d\alpha + \int_t^T \frac{\eta(1+\beta_\alpha^*)e^{r(T-\alpha)}[1-(1+\beta_\alpha^*)^{\frac{-k}{\lambda}e^{-r(T-\alpha)}}]}{k} d\alpha$$

To determine the nature of the Insurer's value function the auxiliary function must be numerically estimated.

6.4 Re-insurer's Auxiliary Value function

In Section (5) the Re-insurer's Auxiliary function was derived. Substituting the claim distribution assumption into Equation (13) produces:

$$\begin{aligned} g(t) &= \int_t^T e^{r(T-t)}(1+\beta_\alpha)\mu d\alpha - \int_t^T e^{r(T-\alpha)}(1+\beta_\alpha)\int_0^{p_\alpha^*} e^{-kZ} dZ d\alpha \\ &- \int_t^T \frac{\eta}{\gamma} exp(-\gamma p e^{r(T-\alpha)}) \int_{p_\alpha^*}^\infty exp(\gamma Z e^{r(T-\alpha)}) k e^{-kZ} dZ d\alpha + \int_t^T \frac{\eta}{\gamma} e^{-kp_\alpha^*} d\alpha \\ &= \int_t^T e^{r(T-t)}(1+\beta_\alpha)\mu d\alpha + \int_t^T \frac{e^{r(T-\alpha)}}{k}(1+\beta_\alpha)(e^{-kp_\alpha^*} - 1) d\alpha \\ &+ \int_t^T \frac{k\eta}{\gamma(\gamma e^{r(T-\alpha)} - k)} e^{-kp_\alpha^*} d\alpha + \int_t^T \frac{\eta}{\gamma} e^{-kp_\alpha^*} d\alpha \end{aligned} \tag{19}$$

Substituting the Agent's Optimal Retention rate given that the Principal's Safety Loading Factor is optimal into Equation (19) produces the following form of the Re-insurer's Auxiliary Function:

$$\begin{aligned} g(t) &= \int_t^T e^{r(T-\alpha)}(1+\beta_\alpha)\mu d\alpha + \int_t^T \frac{e^{r(T-\alpha)}}{k}(1+\beta_\alpha)((1+\beta_\alpha^*)^{-\frac{k}{\lambda}e^{-r(T-\alpha)}} - 1) d\alpha \\ &+ \int_t^T \frac{k\eta}{\gamma(\gamma e^{r(T-\alpha)} - k)} (1+\beta_\alpha^*)^{-\frac{k}{\lambda}e^{-r(T-\alpha)}} d\alpha + \int_t^T \frac{\eta}{\gamma}(1+\beta_\alpha^*)^{-\frac{k}{\lambda}e^{-r(T-\alpha)}} d\alpha \end{aligned} \tag{20}$$

Since this function is also dependent on the Principal's Safety Loading Factor numerical methods will have to be used to estimate the nature of the function.

7 Sensitivity Analysis

In the previous section the claim distribution assumption led to semi-analytical solutions to the value functions and control parameters. In this section the behaviour of the Principal and the Agent given changes in the parameters will be observed. This will be achieved by utilisation of sensitivity analysis. This type of analysis will highlight the effects on the Retention rate, the Reinsurance price given changes in the risk aversion parameters. I assume in this case the the Insurer is risk prone and the Re-insurer high a moderate level of risk aversion.

Parameters	r	μ	γ	λ	t	T
Values	3%	0.2	0.03	0.6	0.5	1

7.1 Optimal Reinsurance Price and Re-Insurer Safety Loading Factor functions

In the derivation of the optimal retention rate and Safety Loading Factor assumptions were made about the nature of the the claim distribution parameter k. If is assumed that:

$$k = 4e^{r(T-t)}$$

Then the conditions specified on k will be satisfied since it is stipulated that $\lambda, \gamma \leq 1$

7.1.1 Re-insurer's Safety Loading Factor

The figure below represents the optimal Safety Loading Factor given changes in the risk aversion parameter's of the Principal and Agent.

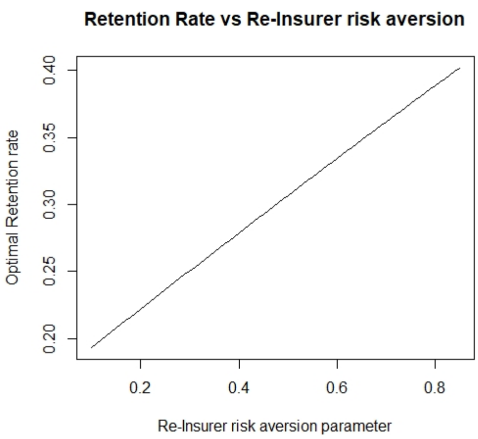

Figure 1: The effects of the Re-Insurer risk aversion on the retention rate

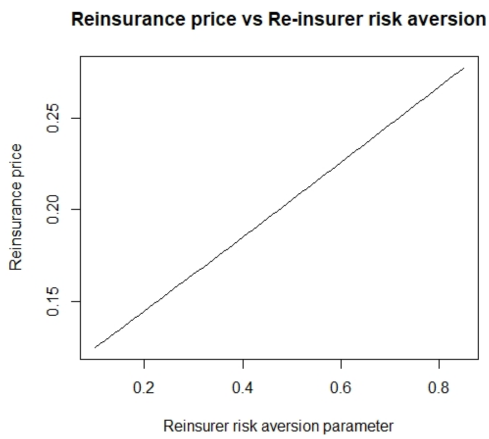

Figure 2: The effects of the Re-Insurer risk aversion on the retention rate

In Figure 1 there is displayed a positive relationship between the Re-insurer's risk aversion and the reinsurance price. If the re-insurer is more risk-averse then they will demand a higher premium for the extra risk that they will be incurring by re-insuring the Insurer's insurance contracts. As a result of the higher premiums charged by the re-insurer, the perceived value of the reinsurance contract diminishes resulting in a higher proportion of retained claims. This complement's the relationship displayed in Figure 2 which elucidates a positive relationship between the Retention rate and the Re-insurer's risk aversion.

7.1.2 Insurer's Risk Retention

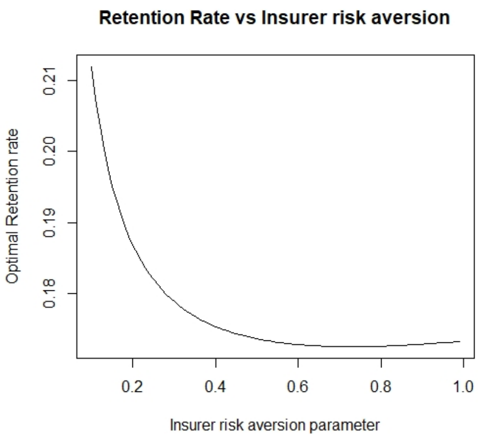

Figure 3: The effects of the Insurer risk aversion on the retention rate

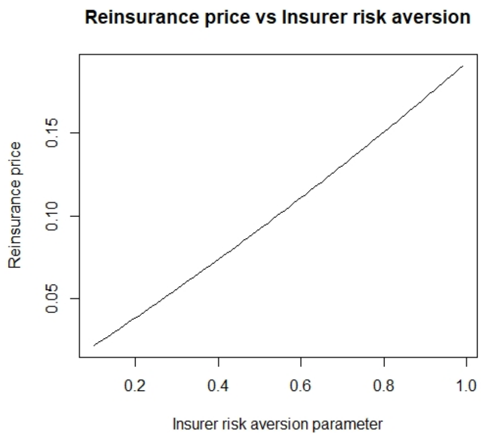

Figure 4: The effects of the Insurer risk aversion on the Reinsurance price

Figure 3 depicts a negative relationship between the retention rate and the Insurer's risk aversion parameter. This intuitively makes sense since the more risk averse the insurer is, the lower number of claims they are willing to retain due to their perception of the benefits received from the risk incurred is lower. This will result in an increase in the demand for reinsurance ceteris paribus. The figure adjacent represents a positive relationship between the reinsurance price and the Insurer's risk aversion. As previously stated since the Insurer's risk aversion parameter is increasing the demand for reinsurance will increase. This increase in demand will lead to a higher reinsurance premium charged by the insurer.

8 Conclusion

In this report, I have conducted analysis within the Principal-Agent framework where the claim process is assumed to be modelled using Cramer-Lundburg Model. Through the utilisation of the HJB equation in conjunction with the assumption regarding the form of the Principal's and Agent's utility function and the claim distribution, expressions for the optimal reinsurance price and retention rate were derived. The HJB equation also allowed for the derivation of the value function for both the insurer and re-insurer. In the sensitivity analysis section is was displayed that an Insurer's risk aversion parameter will result in a decrease in the optimal Retention Rate and an increase in the Safety Loading Factor. It was also displayed that an increase in the Re-Insurer's Risk Aversion parameter resulted in an increase in the Retention Rate and Safety Loading Factor.

9 Appendix

9.1 Expectation of change in Agent's Value jump

Simplifying the Expectation as displayed in Equation (2) produces:

$$\begin{aligned}
&\mathbf{E}_t[v(t, w - Min[Z_i, p]) - v(t, w)] \\
&= \mathbf{E}_t[v(t, w - Min[Z_i, p])] - \mathbf{E}[v(t, w)] \\
&= \mathbf{E}_t\left[- \frac{exp\left(-\lambda(w - Min[Z, p])e^{r(T-t)} + h(t))\right)}{\lambda} \right] - \mathbf{E}[v(t, w)] \\
&= - \int_0^\infty \frac{exp\left(-\lambda(w - Min[Z, p])e^{r(T-t)} + h(t))\right)}{\lambda} f(Z)dZ - \int_0^\infty v(t, w) f(Z)dZ \\
&= v(t, w; p)\left(\int_0^{p_t^*} exp\left(\lambda Z e^{r(T-t)}\right) f(Z)dZ + \int_{p_t^*}^\infty exp\left(\lambda p_t^* e^{r(T-t)}\right) f(Z)dZ \right. \\
&\left. - \int_0^\infty f(Z)dZ \right)
\end{aligned}$$

Focusing the first term of the above Equation. I define a function:

$$D(Z) = \int_0^{p_t^*} exp\left(\lambda Z e^{r(T-t)}\right) f(Z)dZ$$

Integration by parts is used to simplify to the expression. By letting:

$$\begin{aligned}
&u(Z) = exp\left(\lambda Z e^{r(T-t)}\right) \\
&v_z(Z) = f(Z) \\
\implies &\int_0^{p_t^*} u(Z) v_z(Z)dZ = \Big[u(Z)v(Z)\Big]_0^{p_t^*} - \int_0^{p_t^*} u_z(Z)v(Z)dZ
\end{aligned}$$

We can express the function in the form:

$$\begin{aligned}
&\int_0^{p_t^*} exp\left(\lambda Z e^{r(T-t)}\right) f(Z)dZ \\
&= \left[exp\left(\lambda Z e^{r(T-t)}\right) F(Z)\right]_0^{p_t^*} - \int_0^{p_t^*} \lambda e^{r(T-t)} exp\left(\lambda Z e^{r(T-t)}\right) F(Z)dZ \\
&= exp\left(\lambda p_t^* e^{r(T-t)}\right) F(p_t^*) - \int_0^{p_t^*} \lambda e^{r(T-t)} exp\left(\lambda Z e^{r(T-t)}\right) F(Z)dZ \\
&= exp\left(\lambda p_t^* e^{r(T-t)}\right) - \bar{F}(p_t^*) exp\left(\lambda p_t^* e^{r(T-t)}\right) - \int_0^{p_t^*} \lambda e^{r(T-t)} exp\left(\lambda Z e^{r(T-t)}\right) dZ \\
&+ \int_0^{p_t^*} \lambda e^{r(T-t)} exp\left(\lambda Z e^{r(T-t)}\right) \bar{F}(Z)dZ \\
&= exp\left(\lambda p_t^* e^{r(T-t)}\right) - \bar{F}(p_t^*) exp\left(\lambda p_t^* e^{r(T-t)}\right) - \left[exp\left(\lambda Z e^{r(T-t)}\right)\right]_0^{p*} \\
&+ \int_0^{p_t^*} \lambda e^{r(T-t)} exp\left(\lambda Z e^{r(T-t)}\right) \bar{F}(Z)dZ \\
&= -\bar{F}(p_t^*) exp\left(\lambda p_t^* e^{r(T-t)}\right) + \int_0^{p_t^*} \lambda e^{r(T-t)} exp\left(\lambda Z e^{r(T-t)}\right) \bar{F}(Z)dZ + 1
\end{aligned} \tag{21}$$

Substituting the result from Equation (21) back into the expectation term produces:

$$\begin{aligned}&\mathbf{E}_t[v(t,w-Min[Z_i,p])-v(t,w)]\\&=v(t,w;p)\Bigg(-\bar{F}(p_t^*)exp\left(\lambda p_t^* e^{r(T-t)}\right)+\int_0^{p_t^*}\lambda e^{r(T-t)}exp\left(\lambda Z e^{r(T-t)}\right)\bar{F}(Z)dZ+1\\&+exp\left(\lambda p_t^* e^{r(T-t)}\right)\bar{F}(p_t^*)-1\Bigg)\\&=\int_0^{p_t^*}\lambda e^{r(T-t)}exp\left(\lambda Z e^{r(T-t)}\right)\bar{F}(Z)dZ\end{aligned}\tag{22}$$

9.2 Agent's optimal Risk Retention factor

Setting first derivative of Equation (5) with respect to the Risk Retention parameter equal to 0 produces:

$$\begin{aligned}&\eta(1+\beta)\bar{F}(p^*)v_w+\eta\lambda\ e^{r(T-t)}exp(\lambda p^* e^{r(T-t)})\bar{F}(p^*)v=0\\&(1+\beta)v_w+\lambda\ e^{r(T-t)}exp(\lambda p^* e^{r(T-t)})v=0\end{aligned}\tag{23}$$

Substituting the derivatives of the value function into Equation (23) yields:

$$\begin{aligned}&(1+\beta)\lambda e^{r(T-t)}v=\lambda\ e^{r(T-t)}exp(\lambda p^* e^{r(T-t)})v\\&(1+\beta)=exp(\lambda p^* e^{r(T-t)})\\&ln\big(1+\beta\big)=\lambda p^* e^{r(T-t)}\\&p_t^*=\frac{ln\big(1+\beta\big)}{\lambda}e^{-r(T-t)}\end{aligned}\tag{24}$$

Taking the second derivative of Equation (5) with respect to the Agent's Risk Retention factor produces:

$$\begin{aligned}&-\eta(1+\beta)\bar{F}'(p)\lambda e^{r(T-t)}v+\eta\lambda\ e^{r(T-t)}exp(\lambda p e^{r(T-t)})\bar{F}'(p)v+\eta\lambda^2 e^{2r(T-t)}exp(\lambda p e^{r(T-t)})\bar{F}(p)v\\&=\eta v\lambda e^{r(T-t)}(exp(\lambda p e^{r(T-t)})\bar{F}'(p)+\lambda e^{r(T-t)}exp(\lambda p e^{r(T-t)})\bar{F}(p)-(1+\beta)\bar{F}'(p))\\&=k\eta v\lambda e^{r(T-t)}(-exp(p(\lambda e^{r(T-t)}-k))+\lambda e^{r(T-t)}exp(p(\lambda e^{r(T-t)}-k))+k(1+\beta)e^{-kp})\end{aligned}\tag{25}$$

The final line of Equation (25) is produced by substituting the claim distribution assumption into the preceding expression. Since the value function is less than 0 for all values of time and wealth. For this parameter to be a supremum of the HJB Equation

$$-exp(p(\lambda e^{r(T-t)}-k))+\lambda e^{r(T-t)}exp(p(\lambda e^{r(T-t)}-k))+k(1+\beta)e^{-kp}>0$$

9.3 Expectation of change in Principal's Value jump

$$
\begin{aligned}
&\mathbf{E}_t[v(t, w - Z_i + Min[Z_i, p]) - v(t, w)] \\
&= \mathbf{E}_t[v(t, w - Z_i + Min[Z_i, p])] - \mathbf{E}[v(t, w)] \\
&= \mathbf{E}_t[-\frac{exp\left(-\gamma(w - Z + Min[Z, p])e^{r(T-t)} + h(t))\right)}{\gamma}] - \mathbf{E}[v(t, w)] \\
&= -\int_0^\infty \frac{exp\left(-\gamma(w - Z + Min[Z, p])e^{r(T-t)} + h(t))\right)}{\gamma} f(Z)dZ - \int_0^\infty v(t, w) f(Z)dZ \\
&= v(t, w; p)]\left[\int_0^{p_t^*} exp\left(-\gamma(-Z + Z)e^{r(T-t)}\right) f(Z)dZ + \int_{p_t^*}^\infty exp\left(-\gamma(-Z + p_t^*)e^{r(T-t)}\right) f(Z)dZ \right. \\
&\left. - \int_0^\infty f(Z)dZ\right] \\
&= v(t, w; p)]\left[\left[F(Z)\right]_0^p + exp\left(-\gamma p_t e^{r(T-t)}\right) \int_{p_t^*}^\infty exp\left(\gamma Z e^{r(T-t)}\right) f(Z)dZ - 1\right] \\
&= v(t, w; p)]\left[1 - \bar{F}(p) - F(0) + exp\left(-\gamma p_t e^{r(T-t)}\right) \int_{p_t^*}^\infty exp\left(\gamma Z e^{r(T-t)}\right) f(Z)dZ - 1\right]
\end{aligned}
$$

Applying the initial condition that F(0)=0, as a result of Z being defined for positive real numbers we have the expression

$$
\mathbf{E}_t[v(t, w - Z_i + Min[Z_i, p]) - v(t, w)] = v(t, w)\left[exp(-\gamma p e^{r(T-t)}) \int_p^\infty exp(\gamma Z e^{r(T-t)}) f(Z)dZ - \bar{F}(p)\right]
$$

9.4 Principal's Optimised Safety Loading Factor

Taking the first derivative of Equation (11) with respect to the Safety Loading Factor and setting it equal to 0 produces:

$$
\begin{aligned}
&-\gamma e^{r(T-t)}\hat{v}\left[\eta\mu - \frac{\partial}{\partial\beta_t}\left((1 + \beta_t)\eta\int_0^{p_t^*} \bar{F}(Z)dZ\right)\right] \\
&+ \frac{\partial}{\partial\beta_t}\left(\eta\hat{v}exp(-\gamma p_t^* e^{r(T-t)}) \int_{p_t^*}^\infty exp(\gamma Z e^{r(T-t)}) f(Z)dZ - \eta\hat{v}\bar{F}(p_t^*)\right) = 0 \\
&-\gamma e^{r(T-t)}\mu + \gamma e^{r(T-t)}(1 + \beta_t)\frac{\partial}{\partial\beta_t}\left(\int_0^{p_t^*} \bar{F}(Z)dZ\right) + \gamma e^{r(T-t)}\mu\int_0^{p_t^*} \bar{F}(Z)dZ \\
&+ \frac{\partial}{\partial\beta_t}\left(exp(-\gamma p e^{r(T-t)}) \int_{p_t^*}^\infty exp(\gamma Z e^{r(T-t)}) f(Z)dZ - \bar{F}(p_t^*)\right) = 0
\end{aligned}
$$

Since the Agent's Risk Retention factor is a function of the Safety Loading Factor, the Safety Loading Factor must satisfy the following Equation:

$$
\begin{aligned}
&-\gamma e^{r(T-t)}\mu + \gamma e^{r(T-t)}\int_0^{p_t^*} \bar{F}(Z)dZ + (1 + \beta_t)\gamma e^{r(T-t)}\bar{F}(p_t^*)\frac{\partial p_t^*}{\partial\beta_t} \\
&-\gamma e^{r(T-t)} exp(-\gamma p_t^* e^{r(T-t)})\frac{\partial p_t^*}{\partial\beta_t}\int_{p_t^*}^\infty exp(\gamma Z e^{r(T-t)}) f(Z)dZ \\
&+ exp(-\gamma p_t^* e^{r(T-t)})\frac{\partial}{\partial\beta_t}\left[\int_{p_t^*}^\infty exp(\gamma Z e^{r(T-t)}) f(Z)dZ\right] - \bar{F}'(p_t^*)\frac{\partial p_t^*}{\partial\beta_t} = 0
\end{aligned}
\tag{26}
$$

The derivative of the Risk Retention factor with respect to the Safety Loading Factor is of the form:

$$\frac{\partial p_t^*}{\partial \beta_t} = \frac{e^{-r(T-t)}}{(1+\beta_t)\lambda} \tag{27}$$

Substituting Equation (27) into Equation (26), and the optimised retention rate into all terms other than the claim function produces:

$$\begin{aligned} &-\gamma e^{r(T-t)}\mu + \gamma e^{r(T-t)}\int_0^{p_t^*}\bar{F}(Z)dZ + \frac{\gamma}{\lambda}\bar{F}(p_t^*) - \frac{\gamma}{\lambda}(1+\beta_t)^{-\frac{\gamma}{\lambda}-1}\int_{p_t^*}^{\infty} exp(\gamma Z e^{r(T-t)})f(Z)dZ \\ &+(1+\beta)^{-\frac{\gamma}{\lambda}}\frac{\partial}{\partial \beta_t}\left[\int_{p_t^*}^{\infty} exp(\gamma Z e^{r(T-t)})f(Z)dZ\right] - \frac{e^{-r(T-t)}\bar{F}'(p_t^*)}{(1+\beta_t)\lambda} = 0 \end{aligned} \tag{28}$$

To determine whether the parameter is the supremium of the function the second derivative of Equation (11) with respect to the Principal's Safety Loading Factor is taken, the general form of the second derivative without taking into account the assumed claim distribution is of the form:

$$\begin{aligned} \frac{d^2}{dp^2}L(t,w) =& v\eta\left(\gamma e^{r(T-t)}\bar{F}(p_t^*)\frac{dp}{d\beta} + \gamma e^{r(T-t)}\bar{F}(p_t^*)\frac{dp}{d\beta} + (1+\beta_t)\gamma e^{r(T-t)}\bar{F}'(p_t^*)\left(\frac{dp}{d\beta}\right)^2\right. \\ &\gamma^2 e^{2r(T-t)}exp(-\gamma p e^{r(T-t)})\left(\frac{dp}{d\beta}\right)^2\int_p^{\infty} exp(\gamma Z e^{r(T-t)})f(Z)dZ \\ &-2\gamma e^{r(T-t)}exp(-\gamma p e^{r(T-t)})\frac{dp}{d\beta}\frac{d}{d\beta}\left[\int_p^{\infty} exp(\gamma Z e^{r(T-t)})f(Z)dZ\right] \\ &\left.+ exp(-\gamma p e^{r(T-t)})\frac{d^2}{d\beta^2}\left[\int_p^{\infty} exp(\gamma Z e^{r(T-t)})f(Z)dZ\right] - \bar{F}''(p_t^*)\left(\frac{dp}{d\beta}\right)^2\right) \end{aligned} \tag{29}$$

9.5 Numerical Examples code

```
library(plotly)
lambda<-seq(from=0.01,to=0.99, by=0.01)
gamma<-seq(from=0.1,to=0.80, by=0.01)
mu<-0.2
r=0.03
t2= 1
t= 0.5
b<-seq(from=0,to=length(lambda))
k<-function(gamma,lambda){exp(r*(t2-t))}
#Functions
beta<-function(b,mu,lambda,gamma,t2,t){(1+b)^((-k(gamma,lambda)/lambda)*exp(-r*(t2-t)))*
    (1/lambda-(exp(r*(t2-t))/k(gamma,lambda)))+(k(gamma,lambda)*(1+b)^(-k(gamma,lambda)/lambda
    *exp(-r*(t2-t))-1))/((gamma*exp(r*(t2-t))-k(gamma,lambda))*lambda)-exp(r*(t2-t))*mu+
    exp(r*(t2-t))/k(gamma,lambda)}
p<-function(lambda,gamma,Optimal_beta,r, t,t2){log(1+Optimal_beta)*exp(-r*(t2-t))/lambda}
#Optimised Beta graph
rm(v)
interval<-lambda
v<-rep(0,1)
for(i in interval){
        v<-append(v,c(uniroot(beta, interval=c(0,100),tol=
        0.000000000001,mu=mu,lambda=i,gamma=gamma,t2=t2,t=t)$root))}
indices <- c(1)
Optimal_beta<- v[-indices]
plot(lambda,Optimal_beta,type='l',xlab ='Insurer risk aversion parameter',ylab="Reinsurance price",
     main="Reinsurance price vs Insurer risk aversion" )
#Optimised Beta graph_Gamma
interval<-gamma
rm(v)
v<-rep(0,1)
for(i in interval){
        v<-append(v,c(uniroot(beta, interval=c(0,100),tol=
        0.000000000001,mu=mu,lambda=lambda,gamma=i,t2=t2,t=t)$root))}
indices <- c(1)
Optimal_beta<- v[-indices]
```

```
plot(gamma,Optimal_beta, type='l',xlab ='Reinsurer risk aversion parameter',ylab="Reinsurance price",
     main="Reinsurance price vs Re-insurer risk aversion" )
#Optimal Retention Rate vs Gamma or Lambda
Op_Retention_r<-p(lambda,gamma,Optimal_beta,r,t,t2)
plot(lambda,Op_Retention_r,type='l',xlab ='Insurer risk aversion parameter'
    ,ylab="Optimal Retention rate", main="Retention Rate vs Insurer risk aversion")
plot(gamma,Op_Retention_r,type='l',xlab ='Re-Insurer risk aversion parameter'
,ylab="Optimal Retention rate", main="Retention Rate vs Re-Insurer risk aversion")

Condition_1<-function(b,k,gamma,r,t,t2){
        1+b+k(gamma,lambda)/(gamma*exp(r*(t2-t))-k(gamma,lambda))
}
Condition_2<-function(p,b,r,t,t2){
 exp(p*(lambda*exp(r*(t2-t))-k(gamma,lambda)))*(lambda*exp(r*(t2-t))-1)
 +k(gamma,lambda)*(1+b)*exp(-k(gamma,lambda)*p)
}
for( i  in 1:length(Optimal_beta)){
if(Condition_1(Optimal_beta[i],k,gamma[i],r,t,t2)<0){
        stop("Not Optimal")
}}

for( i  in 1:length(Optimal_beta)){
        if(Condition_2(Op_Retention_r[i],Optimal_beta[i],r,t,t2)<0){
                stop("Not Optimal")
        }}
        help('stop')
Condition_2(Op_Retention_r,Optimal_beta,r,t,t2)
Condition_1(Optimal_beta,k,gamma,r,t,t2)
```

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