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KZ Functor for Rational Cherednik Algebras

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1 Abstract

We study the Knizhnik-Zamolodchikov (KZ) functor, which maps rational Cherednik algebra modules to Iwahori-Hecke algebra modules. We define a rational Cherednik algebra (RCA) associated to a complex reflection group and prove an isomorphism result regarding a localisation of RCA. RCA modules and their horizontal sections are investigated, which are then pushed across the KZ functor via monodromy to produce Hecke algebra modules. From there, we compute examples of the KZ functor for cyclic groups and symmetric groups.

2 Introduction

Both rational Cherednik algebras and Iwahori-Hecke algebras are structures that appear extensively in the study of representation theory. In [Gin+03], the KZ functor is introduced, which maps RCA modules to Hecke algebra modules, allowing our knowledge about one structure to help with the understanding of the other.

A complex reflection group W is generated by reflections and acts on a complex vector space $\mathfrak{a}^*$. Associated to W, we define a rational Cherednik algebra $\tilde{\mathbb{H}}$ (see [EM10]) in terms of generators and relations. To allow certain denominators in the equations for later results, we consider a localisation $\tilde{\mathbb{H}}^0$ of the RCA. We reprove first a result from [Gri10] regarding relations of generators, then an isomorphism between $\tilde{\mathbb{H}}^0$ and $\mathcal{D}(\mathfrak{a}^0) \rtimes W$ from [Gin+03], where $\mathfrak{a}^0$ is the configuration space (see equation (43)). We then consider group representations of W, from which we induce RCA modules and investigate conditions for their horizontal sections, leading to systems of partial differential equations [Ram21]. RCA modules are mapped across the KZ functor by calculating monodromy matrices, which produce corresponding Hecke modules. Next, we explore explicit examples when W is a cyclic group or a symmetric group. In the case of a cyclic group, we are able to fully solve the partial differential equations to obtain horizontal sections and thus compute explicit parameters for the corresponding Hecke algebras (see [Ram08]).

Statement of Authorship

This project topic was first proposed by Prof. Arun Ram and later conducted under the supervision of Dr. Ting Xue and Prof. Ram. I also worked on this research in collaboration with

another AMSI vacation scholar Haris Rao and his supervisor Dr. Yaping Yang. The theory explored in this project was primarily developed in [Gin+03] and was presented by Prof. Ram in a series of five lectures [Ram21]. Although this report does not include any new results, I gained a better understanding of rational Cherednik algebras, KZ functor and related topics. I also computed examples of the KZ functor for the cyclic and symmetric groups, first individually and then in discussion with Haris. I wrote this report, which was reviewed by Dr. Xue and Prof. Ram.

3 Rational Cherednik algebras

3.1 Rational Cherednik algebras $\tilde{\mathbb{H}}$

Let W be a complex reflection group (see Appendix A), acting on a complex vector space $\mathfrak{a}^*$. Let $\kappa \in \mathbb{C}\backslash\{0\}$ and $c_s \in \mathbb{C}$ be complex parameters, such that $c_{wsw^{-1}} = c_s$ for all reflections $s \in R$ and $w \in W$.

The *rational Cherednik algebra* associated to W (see [EM10, Ch. 3 Prop. 3.2]) is the algebra $\tilde{\mathbb{H}}$ generated by x_μ, $y_{\lambda^\vee}$, t_w for $\mu \in \mathfrak{a}^*$, $\lambda^\vee \in \mathfrak{a}$ and $w \in W$ with relations (46) and (47) from symmetric algebras (see Appendix B), and

$$t_{wv} = t_v t_w \tag{1}$$

$$t_w x_\mu = x_{w\mu} t_w, \quad t_w y_{\lambda^\vee} = y_{w\lambda^\vee} t_w \tag{2}$$

$$y_{\lambda^\vee} x_\mu = x_\mu y_{\lambda^\vee} + \kappa\langle\mu, \lambda^\vee\rangle - \sum_{s\in R} c_s \langle\mu, \alpha_s^\vee\rangle\langle\alpha_s, \lambda^\vee\rangle t_s. \tag{3}$$

3.2 Localisation of $\tilde{\mathbb{H}}$

We may wish to rearrange (45′) as

$$\langle\mu, \alpha_s^\vee\rangle = \frac{x_\mu - x_{s\mu}}{x_{\alpha_s}}. \tag{45''}$$

To do so, we first need to allow the presence of x_{α_s} in the denominator. This motivates us to consider the following localisation of the rational Cherednik algebra.

Let $\Delta = \prod_{s\in R} x_{\alpha_s}$.

Let $\tilde{\mathbb{H}}^0$ be the algebra generated by x_μ, $y_{\lambda^\vee}$, t_w and $\frac{1}{\Delta}$ for $\mu \in \mathfrak{a}^*$, $\lambda^\vee \in \mathfrak{a}$ and $w \in W$ with relations (46), (47) and (1)-(3) from $\tilde{\mathbb{H}}$ and

$$\Delta\frac{1}{\Delta} = \frac{1}{\Delta}\Delta = 1. \tag{4}$$

Relation (3) describes how to interchange the order between $y_{\lambda^\vee}$ and x_μ. It may also be useful to explore the relation between $y_{\lambda^\vee}$ and a polynomial in $S(\mathfrak{a}^*)$.

Propositon 1. [Gri10, Ch. 2 Prop. 2.3]
Let $\lambda^\vee \in \mathfrak{a}$ and $f \in S(\mathfrak{a}^*)$. Then

$$y_{\lambda^\vee} f = f y_{\lambda^\vee} + \kappa\frac{\partial f}{\partial\lambda^\vee} - \sum_{s\in R} c_s\langle\alpha_s, \lambda^\vee\rangle\frac{f - sf}{x_{\alpha_s}} t_s \tag{5}$$

where $\frac{\partial f}{\partial\lambda^\vee}$ is a derivation of $S(\mathfrak{a}^*)$ determined by $\frac{\partial x_\mu}{\partial\lambda^\vee} := \langle\mu, \lambda^\vee\rangle$ for $\mu \in \mathfrak{a}^*$ and satisfies the Leibniz rule $\frac{\partial(fg)}{\partial\lambda^\vee} = \frac{\partial f}{\partial\lambda^\vee}g + f\frac{\partial g}{\partial\lambda^\vee}$.

Proof. We will prove this using an induction on the degree of f.

Base case: Substitute (45″) in (3),

$$\begin{aligned} y_{\lambda^\vee}x_\mu &= x_\mu y_{\lambda^\vee} + \kappa\langle\mu, \lambda^\vee\rangle - \sum_{s\in R} c_s\langle\mu, \alpha_s^\vee\rangle\langle\alpha_s, \lambda^\vee\rangle t_s \\ &= x_\mu y_{\lambda^\vee} + \kappa\frac{\partial x_\mu}{\partial\lambda^\vee} - \sum_{s\in R} c_s\langle\alpha_s, \lambda^\vee\rangle\frac{x_\mu - x_{s\mu}}{x_{\alpha_s}} t_s. \end{aligned}$$

Inductive step:

$$\begin{aligned} y_{\lambda^\vee}(fg) - (fg)y_{\lambda^\vee} &= (y_{\lambda^\vee}f - fy_{\lambda^\vee})g + f(y_{\lambda^\vee}g - gy_{\lambda^\vee}) \\ &= \left[\kappa\frac{\partial f}{\partial\lambda^\vee} - \sum_{s\in R} c_s\langle\alpha_s, \lambda^\vee\rangle\frac{f - sf}{x_{\alpha_s}}t_s\right]g + f\left[\kappa\frac{\partial g}{\partial\lambda^\vee} - \sum_{s\in R} c_s\langle\alpha_s, \lambda^\vee\rangle\frac{g - sg}{x_{\alpha_s}}t_s\right] \\ &= \kappa\left(\frac{\partial f}{\partial\lambda^\vee}g + f\frac{\partial g}{\partial\lambda^\vee}\right) - \sum_{s\in R} c_s\langle\alpha_s, \lambda^\vee\rangle\left(\frac{f - sf}{x_{\alpha_s}}sg + f\frac{g - sg}{x_{\alpha_s}}\right)t_s \\ &= \kappa\frac{\partial(fg)}{\partial\lambda^\vee} - \sum_{s\in R} c_s\langle\alpha_s, \lambda^\vee\rangle\left(\frac{fsg - sfsg + fg - fsg}{x_{\alpha_s}}\right)t_s \\ &= \kappa\frac{\partial(fg)}{\partial\lambda^\vee} - \sum_{s\in R} c_s\langle\alpha_s, \lambda^\vee\rangle\left(\frac{fg - s(fg)}{x_{\alpha_s}}\right)t_s. \end{aligned}$$

□

3.3 Another algebra $\mathcal{D}^0$

In the definition of $\tilde{\mathbb{H}}^0$, relation (3) is relatively complicated. We will now define a very similar algebra $\mathcal{D}^0$, where we will introduce an analogous but simpler relation (8). To differentiate between the two algebras, we will use the symbol $\partial_{\lambda^\vee}$ as a generator, in place of $y_{\lambda^\vee}$.

Let $\mathfrak{a}^0$ be the configuration space (see equation (43)). Let $\mathcal{D}^0 := \mathcal{D}(\mathfrak{a}^0) \rtimes W$ be the algebra generated by x_μ, $\partial_{\lambda^\vee}$, t_w, and $\frac{1}{\Delta}$ for $\mu \in \mathfrak{a}^*$, $\lambda^\vee \in \mathfrak{a}$ and $w \in W$ with relations (46), (1), (4) and

$$\partial_{\lambda^\vee+\gamma^\vee} = \partial_{\lambda^\vee} + \partial_{\gamma^\vee}, \quad \partial_{c\lambda^\vee} = c\partial_{\lambda^\vee}, \quad \partial_{\lambda^\vee}\partial_{\gamma^\vee} = \partial_{\gamma^\vee}\partial_{\lambda^\vee} \tag{6}$$

and

$$t_w x_\mu = x_{w\mu} t_w, \quad t_w \partial_{\lambda^\vee} = \partial_{w\lambda^\vee} t_w \tag{7}$$

$$\partial_{\lambda^\vee} x_\mu = x_\mu \partial_{\lambda^\vee} + \langle \mu, \lambda^\vee \rangle. \tag{8}$$

Observe that the definitions of the algebras $\tilde{\mathbb{H}}^0$ and $\mathcal{D}^0$ are very similar.

Propositon 2. [Gin+03, Ch. 5 Thm. 5.6] *There exists an isomorphism ϕ between $\tilde{\mathbb{H}}^0$ and $\mathcal{D}^0$, where*

$$\phi : \tilde{\mathbb{H}}^0 \to \mathcal{D}^0 \tag{9}$$

$$x_\mu \mapsto x_\mu \tag{10}$$

$$t_w \mapsto t_w \tag{11}$$

$$y_{\lambda^\vee} \mapsto \kappa\partial_{\lambda^\vee} - \sum_{s\in R} c_s \langle \alpha_s, \lambda^\vee \rangle \frac{1}{x_{\alpha_s}}(1 - t_s). \tag{12}$$

Remark. *Note that this isomorphism is not unique. In fact, one can easily replace the '1' in (12) by any arbitrary complex number $A \in \mathbb{C}$*

$$y_{\lambda^\vee} \mapsto \kappa\partial_{\lambda^\vee} - \sum_{s\in R} c_s \langle \alpha_s, \lambda^\vee \rangle \frac{1}{x_{\alpha_s}}(A - t_s) \tag{12'}$$

and the following proof would still hold.

Proof. Let the term $\sum_{s\in R} c_s \langle \alpha_s, \lambda^\vee \rangle \frac{1}{x_{\alpha_s}}(1-t_s)$ in (12) be denoted by T. It is clear that an inverse ϕ^{-1} exists with

$$\phi^{-1}(\partial_{\lambda^\vee}) = \kappa^{-1}\left[y_{\lambda^\vee} + T\right]. \tag{13}$$

Since an inverse map exists, ϕ is bijective. It remains to check the relations still hold in each algebra. It is clear to see that relations (46), (1) and (4) regarding x_μ and t_w hold in both algebras. Relations (47) and (6) also hold since ϕ as an algebra homomorphism is linear.

Moreover, T is commutative with t_w:

$$\begin{aligned}
t_w T &= t_w \left[\sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \frac{1}{x_{\alpha_s}} (1 - t_s) \right] \\
&= \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \frac{1}{x_{w\alpha_s}} t_w (1 - t_s) \\
&= \sum_{s \in R} c_s \langle w\alpha_s, w\lambda^\vee \rangle \frac{1}{x_{w\alpha_s}} (1 - t_{wsw^{-1}}) t_w \\
&= \sum_{wsw^{-1} \in R} c_s \langle \alpha_{wsw^{-1}}, \lambda^\vee \rangle \frac{1}{x_{w\alpha_s}} (1 - t_{wsw^{-1}}) t_w \\
&= T t_w.
\end{aligned}$$

Then, in $\tilde{\mathbb{H}}^0$, relation (2) is preserved by ϕ:

$$\begin{aligned}
t_w y_{\lambda^\vee} &\mapsto t_w \left[\kappa \partial_{\lambda^\vee} - T \right] \\
&= \left[\kappa \partial_{w\lambda^\vee} - T \right] t_w \\
&\mapsto y_{w\lambda^\vee} t_w.
\end{aligned}$$

Similarly, relation (7) in $\mathcal{D}^0$ is preserved by ϕ^{-1}.

Now we check relation (3) in $\tilde{\mathbb{H}}^0$ still holds:

$$\begin{aligned}
y_{\lambda^\vee} x_\mu &\mapsto \left[\kappa \partial_{\lambda^\vee} - T \right] x_\mu \\
&= \kappa \partial_{\lambda^\vee} x_\mu - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \frac{1}{x_{\alpha_s}} (1 - t_s) x_\mu \\
&= \kappa x_\mu \partial_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \left[\frac{x_\mu}{x_{\alpha_s}} - \frac{x_{s\mu}}{x_{\alpha_s}} t_s \right] \\
&= \kappa x_\mu \partial_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \left[\frac{x_\mu}{x_{\alpha_s}} - \frac{x_\mu}{x_{\alpha_s}} t_s + \frac{x_\mu - x_{s\mu}}{x_{\alpha_s}} t_s \right] \\
&= \kappa x_\mu \partial_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \left[\frac{x_\mu}{x_{\alpha_s}} (1 - t_s) + \langle \mu, \alpha_s^\vee \rangle t_s \right] \\
&= x_\mu \left[\kappa \partial_{\lambda^\vee} - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \frac{1}{x_{\alpha_s}} (1 - t_s) \right] + \kappa \langle \mu, \lambda^\vee \rangle - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee \rangle \langle \mu, \alpha_s^\vee \rangle t_s \\
&\mapsto x_\mu y_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle - \sum_{s \in R} c_s \langle \mu, \alpha_s^\vee \rangle \langle \alpha_s, \lambda^\vee \rangle t_s.
\end{aligned}$$

Similarly, relation (8) in $\mathcal{D}^0$ also holds. $\square$

4 RCA Modules

4.1 Inducing RCA modules from group representations

Let (E,π) be a representation of the complex reflection group W, where vector space $E = span\{e_1,\dots,e_d\}$ and $\pi : W \to GL(E)$. The group representation can then induce a $\tilde{\mathbb{H}}^0$-module, $\Delta(E)^0$, with the action of W described by π

$$t_w e_j = \pi(w) e_j \tag{14}$$

and the action of $S(\mathfrak{a})$ be given by

$$y_{\lambda^\vee} e_j = 0 \tag{15}$$

for $w \in W$, $\lambda^\vee \in \mathfrak{a}$ and $j \in \{1,\dots,d\}$.

Since no action of x_μ has been defined, any multiplication by x_μ will simply be added on. As a result, an element of $\Delta(E)^0$ has the form

$$p = p_1 e_1 + \cdots + p_d e_d$$

where $p_1,\ \cdots,\ p_d \in \mathbb{C}[x_1,\dots,x_n,\Delta^{-1}]$.

Note that with respect to $(e_1,\dots,e_d)$, each t_w acts as a $d\times d$ matrix, with

$$t_w e_j = \sum_{i=1}^{d} (t_w)_{ij} e_i. \tag{16}$$

Recall that relations (2) and (3) in $\tilde{\mathbb{H}}^0$ describes how to interchange the order among x_μ, $y_{\lambda^\vee}$ and t_w. Combined with (14) and (15), we have fully described the action of each generator of $\tilde{\mathbb{H}}^0$ on the module $\Delta(E)^0$.

4.2 Horizontal sections

Prop. 2 describes a mapping from $\partial_{\lambda^\vee}$ to an element of $\tilde{\mathbb{H}}^0$. Via (12), we obtain the corresponding action of $\partial_{\lambda^\vee}$ on $\Delta(E)^0$.

The *space of horizontal sections* of $\Delta(E)^0$ is

$$HS(\Delta(E)^0) = \{p \in \Delta(E)^0 \mid \partial_{\lambda^\vee} p = 0 \quad \forall \lambda^\vee \in \mathfrak{a}\}. \tag{17}$$

Propositon 3. [Ram21, Lec. 4 p. 4]
An element $p = p_1e_1 + \cdots + p_de_d \in \Delta(E)^0$ is a horizontal section if and only if

$$\frac{\partial p_i}{\partial x_k} = \kappa^{-1}\sum_{s\in R} c_s\langle \alpha_s, \epsilon_k^\vee\rangle \frac{1}{x_{\alpha_s}}\left(-p_i + \sum_{j=1}^{d}(t_s)_{ij}p_j\right) \tag{18}$$

for $i \in \{1,\dots,d\}$ and $k \in \{1,\dots,n\}$.

Remark. *If we were to use an alternative isomorphism described in (12′) to compute the corresponding action of $\partial_{\lambda^\vee}$, we will again need to replace '1' by the complex number A in the partial differential equations above*

$$\frac{\partial p_i}{\partial x_k} = \kappa^{-1}\sum_{s\in R} c_s\langle \alpha_s, \epsilon_k^\vee\rangle \frac{1}{x_{\alpha_s}}\left(-Ap_i + \sum_{j=1}^{d}(t_s)_{ij}p_j\right). \tag{18′}$$

Proof. Let $p = p_1e_1 + \cdots + p_de_d \in \Delta(E)^0$. p is a horizontal section if and only if

$$\begin{aligned}
&\partial_{\lambda^\vee}p = 0 \quad \forall \lambda^\vee \in \mathfrak{a}\\
&\iff \kappa^{-1}\left[y_{\lambda^\vee} + \sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{1}{x_{\alpha_s}}(1-t_s)\right]p = 0 \quad \forall\lambda^\vee\in\mathfrak{a} \quad \text{(substituting (12) in Prop. 2)}\\
&\iff y_{\lambda^\vee}p + \sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{1}{x_{\alpha_s}}(1-t_s)p = 0 \quad \forall\lambda^\vee\in\mathfrak{a}\\
&\iff \sum_{j=1}^{d} y_{\lambda^\vee}p_je_j + \sum_{s\in R}\sum_{j=1}^{d} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{1}{x_{\alpha_s}}(1-t_s)p_je_j = 0 \quad \forall\lambda^\vee\in\mathfrak{a}\\
&\iff \sum_{j=1}^{d} p_jy_{\lambda^\vee}e_j + \sum_{j=1}^{d}\kappa\frac{\partial p_j}{\partial\lambda^\vee}e_j - \sum_{j=1}^{d}\sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{p_j - sp_j}{x_{\alpha_s}}t_se_j\\
&+\sum_{s\in R}\sum_{j=1}^{d} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{1}{x_{\alpha_s}}(1-t_s)p_je_j = 0 \quad \forall\lambda^\vee\in\mathfrak{a} \quad \text{(using Prop. (1))}\\
&\iff \kappa\sum_{j=1}^{d}\frac{\partial p_j}{\partial\lambda^\vee}e_j = \sum_{j=1}^{d}\sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle\left(\frac{p_j - sp_j}{x_{\alpha_s}}t_s - \frac{1}{x_{\alpha_s}}(1-t_s)p_j\right)e_j \quad \forall\lambda^\vee\in\mathfrak{a}\\
&\iff \kappa\sum_{j=1}^{d}\frac{\partial p_j}{\partial\lambda^\vee}e_j = \sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{1}{x_{\alpha_s}}\sum_{j=1}^{d}(p_jt_se_j - p_je_j) \quad \forall\lambda^\vee\in\mathfrak{a}\\
&\iff \kappa\sum_{j=1}^{d}\frac{\partial p_j}{\partial\lambda^\vee}e_j = \sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{1}{x_{\alpha_s}}\sum_{j=1}^{d}\left(p_j\sum_{i=1}^{d}(t_s)_{ij}e_i - p_je_j\right) \quad \forall\lambda^\vee\in\mathfrak{a}.
\end{aligned}$$

Take the coefficient of e_i on both sides:

$$\begin{aligned}
&\iff \kappa\frac{\partial p_i}{\partial\lambda^\vee} = \sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle\frac{1}{x_{\alpha_s}}\left(\sum_{j=1}^{d}p_j(t_s)_{ij} - p_i\right) \quad \forall i\in\{1,\dots,d\},\lambda^\vee\in\mathfrak{a}\\
&\iff \frac{\partial p_i}{\partial x_k^\vee} = \kappa^{-1}\sum_{s\in R} c_s\langle\alpha_s,\epsilon_k^\vee\rangle\frac{1}{x_{\alpha_s}}\left(\sum_{j=1}^{d}p_j(t_s)_{ij} - p_i\right) \quad \forall i\in\{1,\dots,d\},k\in\{1,\dots,n\}.
\end{aligned}$$

□

5 Hecke modules via monodromy

Equation (18) provides a system of partial differential equations. To fully specify the horizontal sections of the RCA module we desire, initial conditions need to be given.

Let $a_0 \in \mathfrak{a}^0$ be a basepoint in the configuration space. Let $f_1, f_2, \dots, f_d \in HS(\Delta(E)^0)$ be horizontal sections with

$$f_j(a_0) = e_j. \tag{19}$$

These are the initial conditions for the partial differential equations in (18).

For each reflection $s \in R$, the *monodromy matrix* $T_s \in End(E)$ is given by

$$T_s^{-1} e_j = t_s^{-1} f_j(t_s a_0), \tag{20}$$

i.e.,

$$T_s = \begin{pmatrix} | & & | \\ f_1(t_s a_0) & \dots & f_d(t_s a_0) \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} & & \\ & t_s & \\ & & \end{pmatrix}. \tag{21}$$

Recall the vector space $E = span\{e_1, \dots, e_d\}$ is from the group representation of W, which induces the RCA module $\Delta(E)^0$. [Gin+03, Ch. 5 Th. 5.13] shows that the matrices T_s for $s \in R$ satisfy the Hecke relations of a Hecke algebra of W. Hence, the vector space E can be viewed as a Hecke module with generators T_s of the Hecke algebra acting by the matrices in (21) .

This process of mapping rational Cherednik algebra modules to Hecke algebra modules is the *KZ functor*.

6 Type $G(r,1,1)$

Let $W = \{1, t, \dots, t^{r-1} \mid t^r = 1\}$, which is a cyclic group of order r. W is a complex reflection group acting on a one-dimensional complex vector space $\mathfrak{a}^* = span\{\epsilon_1\} \cong \mathbb{C}$. (see Appendix A.5 for a detailed discussion on W.)

6.1 Rational Cherednik algebra

Let $\kappa \in \mathbb{C}\backslash\{0\}$ and $c_1,\ c_2,\ \dots,\ c_r \in \mathbb{C}$. Since each element of the cyclic group is its own conjugacy class, no pair of c_i and c_j has to be equal.

Let $\Delta = \prod_{l=1}^{r-1} x_{\alpha_i}$. A localisation $\tilde{\mathbb{H}}^0$ of the rational Cherednik algebra associated to the cyclic group W is the algebra generated by $x_1, y_1, t_1, \frac{1}{\Delta}$ with relations

$$t_1^r = 1 \tag{22}$$

$$\Delta\frac{1}{\Delta} = \frac{1}{\Delta}\Delta = 1 \tag{23}$$

$$t_1 x_1 = \zeta x_1 t_1, \quad t_1 y_1 = \zeta^{-1} y_1 t_1 \tag{24}$$

and

$$\begin{aligned} y_1 x_1 &= x_1 y_1 + \kappa - \sum_{l=1}^{r-1} c_l \langle \epsilon_1, \alpha_l^\vee \rangle \langle \alpha_l, \epsilon_1^\vee \rangle t_1^l && (25)\\ &= x_1 y_1 + \kappa - \sum_{l=1}^{r-1} c_l b_l \frac{(1-\zeta^l)}{b_l} t_1^l && (26)\\ &= x_1 y_1 + \kappa - \sum_{l=1}^{r-1} c_l (1-\zeta^l) t_1^l. && (27) \end{aligned}$$

Let $\mathcal{D}^0$ be the algebra generated by $x_1, \partial_1, t_1, \frac{1}{\Delta}$ with relations

$$t_1 x_1 = \zeta x_1 t_1, \quad t_1 \partial_1 = \zeta^{-1} \partial_1 t_1 \tag{28}$$

$$\partial_1 x_1 = x_1 \partial_1 + 1. \tag{29}$$

Using Prop. (2), $\tilde{\mathbb{H}}^0 \cong \mathcal{D}^0$ via the mapping

$$y_1 \mapsto \kappa \partial_1 - \sum_{l=1}^{r-1} c_l \frac{1}{x_1}(1 - t_1^l). \tag{30}$$

6.2 RCA modules

Let $E^{(j)} = span\{e_j\}$ with $t_1 e_j = \zeta^j e_j$ for $j \in \{0, 1, \dots, r-1\}$. These are the irreducible representations of $G(r,1,1)$. We can also take the direct sum of these irreducible representations to form the regular representation $E = span\{e_0, \dots, e_{r-1}\}$.

Let $\Delta(E)^0$ be the $\tilde{\mathbb{H}}^0$-module induced by E with action of W described above and $y_1 e_j = 0$ for $j \in \{0, \dots, r-1\}$.

In other words, t_1^l acts on $E = span\{e_0, e_1, \dots, e_{r-1}\}$ by

$$t_1^l = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta^l & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{(r-1)l} \end{bmatrix}.$$

We can similarly define $\Delta(E^{(j)})^0$ for the irreducible representations, with action given by the 1×1 matrix $t_1^l = [\zeta^{jl}]$.

Hence, an element of $\Delta(E)^0$ has the form

$$p = p_0 e_0 + \cdots + p_{r-1} e_{r-1}$$

and an element of $\Delta(E^{(j)})^0$ has the form

$$p = p_j e_j$$

where $p_0,\ \cdots,\ p_{r-1} \in \mathbb{C}[x_1, \Delta^{-1}]$.

Using the mapping (30) between y_1 and ∂_1, we can also compute the action of ∂_1 on $\Delta(E)^0$. This leads to the following conditions for $p = p_1 e_1 + \cdots + p_d e_d \in \Delta(E)^0$ to be a horizontal section using Prop. 3:

$$\frac{\partial p_i}{\partial x_1} = \kappa^{-1} \frac{1}{x_1} \sum_{l=1}^{r-1} c_l (\zeta^{il} - 1) p_i = k_i \frac{1}{x_1} p_i \tag{31}$$

for $i \in \{0, \ldots, r-1\}$ and $k_i = \kappa^{-1} \sum_{l=1}^{r-1} c_l(\zeta^{il} - 1)$.

The general solutions to the above differential equation are

$$p_i = C_i x_1^{k_i} \tag{32}$$

where $C_i \in \mathbb{C}$.

6.3 Monodromy in $\mathfrak{a}^0$

Fix a basepoint $a_0 = \mu_0 \in \mathfrak{a}^0 = \mathbb{C}\setminus\{0\}$.

Let $f_0, f_1, \ldots, f_{r-1} \in HS(\Delta(E)^0)$ with $f_j(a_0) = e_j$. To find such f_j, let $f_j = p_0 e_0 + \cdots + p_{r-1} e_{r-1}$, then

$$\begin{aligned} f_j(a_0) &= e_j \\ \sum_{i=0}^{r-1} p_i(a_0) e_i &= e_j \\ \sum_{i=0}^{r-1} C_i a_0^{k_i} e_i &= e_j \\ \implies C_j = a_0^{-k_i} \quad &\text{and} \quad C_i = 0\ \forall i \neq j \\ \implies \quad f_j &= a_0^{-k_j} x_1^{k_j} e_j. \end{aligned}$$

For $i = 1, \dots, r-1$, the monodromy matrix $T_i \in End(E)$ is given by

$$\begin{aligned} T_i^{-1} e_j &= t^{-i} f_j(t^i a_0) \\ &= t^{-i} f_j(\zeta^i a_0) \\ &= t^{-i} a_0^{-k_j} (\zeta^i a_0)^{k_j} e_j \\ &= t^{-i} \zeta^{ik_j} e_j \\ &= \zeta^{-ij} \zeta^{ik_j} e_j \\ &= \zeta^{i(k_j - j)} e_j \\ \implies \quad T_i e_j &= \zeta^{i(j-k_j)} e_j. \end{aligned} \tag{33}$$

Hence, the monodromy matrix of E can be written as

$$T_i = \begin{bmatrix} \zeta^{-k_0 i} & & \\ & \ddots & \\ & & \zeta^{(r-1-k_{r-1})i} \end{bmatrix}. \tag{34}$$

6.4 Hecke algebra and its modules

The *Hecke algebra* $H_{r,1,1}$ associated with a group of type $G(r,1,1)$ is generated by T_1 with

$$(T_1 - q_0)(T_1 - q_1)\cdots(T_1 - q_{r-1}) = 0 \tag{35}$$

for parameters $q_0, \ q_1, \ \dots, \ q_{r-1} \in \mathbb{C}$.

Assume q_i are all distinct, the simple $H_{r,1,1}$-modules are

$$E^{(j)} = span\{e_j\} \quad \text{with} \quad T_1 e_1^{(j)} = q_j e_j$$

for $j \in \{0, 1, \dots, r-1\}$ (see [Ram08, §1 Thm. 1.3]).

The direct sum of all simple modules also forms a module $E = span\{e_0, \dots, e_{r-1}\}$.

Consider the monodromy matrix T_i in (34) calculated in the previous section. Observe that T_1 with action on E given by $T_1 e_j = \zeta^{(j-k_j)} e_j$ satisfies

$$\prod_{j=0}^{r-1} (T_1 - \zeta^{(j-k_j)}) = 0 \tag{36}$$

which is the Hecke relation when parameters $q_j = \zeta^{(j-k_j)}$.

As a result, from the rational Cherednik algebra module $\Delta(E)^0$, we have indeed produced a corresponding Hecke module E with the generator T_1 of the Hecke algebra acting by $T_1 e_j = \zeta^{(j-k_j)} e_j$ for $j = 0, 1, \ldots, r-1$.

If we map the RCA module $\Delta(E^{(j)})^0$ across the KZ functor instead, then T_1 also has action on $E^{(j)}$ given by $T_1 e_j = \zeta^{(j-k_j)} e_j$. This produces a simple module $E^{(j)}$ of $H_{r,1,1}$ with parameter $q_j = \zeta^{(j-k_j)}$.

7 Type $G(1,1,n)$

Let $W = \langle s_i \mid i = 1, 2 \ldots, n-1\rangle$, where $s_i = (i, i+1)$, be a symmetric group of order $n!$. W acts on a complex n-dimensional vector space $\mathfrak{a}^* = span\{\epsilon_1, \ldots, \epsilon_n\} \cong \mathbb{C}^n$ via permutations of coordinates. (see Appendix A.6 for a detailed discussion on W.)

7.1 Rational Cherednik algebra

Let $\kappa \in \mathbb{C}\backslash\{0\}$ and $c \in \mathbb{C}$. Let $\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)$.

We will use x_i to denote x_{ϵ_i}, y_i to denote $y_{\epsilon_i^\vee}$ and t_i to denote t_{s_i}.

A localisation of the rational Cherednik algebra $\tilde{\mathbb{H}}^0$ is the algebra generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $t_1, \ldots, t_{n-1}$, and $\frac{1}{\Delta}$ with relations (46), (47), (1), (2), (4) and

$$y_i x_i = x_i y_i + \kappa - c \sum_{j \neq i} t_{(i,j)} \tag{37}$$

$$y_i x_j = x_j y_i + c t_{(i,j)}. \tag{38}$$

7.2 RCA modules and horizontal sections

The irreducible representations of the symmetric group W are indexed by partitions of n. We can also obtain other representations by taking the direct sum of the irreducible ones. From each group representation E, we can then induce a RCA module $\Delta(E)^0$.

For each RCA module induced from group representation $E = \{e_1, \ldots, e_d\}$, apply (3) to obtain the following conditions for $p = p_1 e_1 + \cdots + p_d e_d \in \Delta(E)^0$ to be a horizontal section:

$$\frac{\partial p_i}{\partial x_k} = \kappa^{-1} c \sum_{\ell \neq k} \frac{1}{x_k - x_\ell} \left(-p_i + \sum_{j=1}^{d} (t_{(\ell,k)})_{ij} p_j \right) \tag{39}$$

for $i \in \{1, \ldots, d\}$ and $k \in \{1, \ldots, n\}$.

For instance, when $n = 3$, an irreducible representation of W is indexed by the partition $(2,1)$. This partition has two standard Young tableaux $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$. Hence, this irreducible representation is of dimension two.

In this case, W acts by:

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then, using Proposition 3, an element $p = p_1 e_1 + p_2 e_2$ in the induced RCA module is a horizontal section if and only if the following partial differential equations are satisfied:

$$\begin{aligned}
\frac{\partial p_1}{\partial x_1} &= \frac{c}{\kappa}\left[\frac{1}{x_1 - x_3}(-\frac{3}{2}p_1 - \frac{3}{2}p_2)\right] \\
\frac{\partial p_1}{\partial x_2} &= \frac{c}{\kappa}\left[\frac{1}{x_2 - x_3}(-\frac{3}{2}p_1 + \frac{3}{2}p_2)\right] \\
\frac{\partial p_1}{\partial x_3} &= \frac{c}{\kappa}\left[\frac{1}{x_3 - x_1}(-\frac{3}{2}p_1 - \frac{3}{2}p_2) + \frac{1}{x_3 - x_2}(-\frac{3}{2}p_1 + \frac{3}{2}p_2)\right] \\
\frac{\partial p_2}{\partial x_1} &= \frac{c}{\kappa}\left[\frac{1}{x_1 - x_2}(-2p_2) + \frac{1}{x_1 - x_3}(-\frac{1}{2}p_1 - \frac{1}{2}p_2)\right] \\
\frac{\partial p_2}{\partial x_2} &= \frac{c}{\kappa}\left[\frac{1}{x_2 - x_1}(-2p_2) + \frac{1}{x_2 - x_3}(\frac{1}{2}p_1 - \frac{1}{2}p_2)\right] \\
\frac{\partial p_2}{\partial x_3} &= \frac{c}{\kappa}\left[\frac{1}{x_3 - x_1}(-\frac{1}{2}p_1 - \frac{1}{2}p_2) + \frac{1}{x_3 - x_2}(\frac{1}{2}p_1 - \frac{1}{2}p_2)\right].
\end{aligned}$$

A Hecke algebra associated with a symmetric group of order $n!$ is generated by $T_1, \ldots, T_{n-1}$ with relations

$$T_i T_j = T_j T_i \quad \text{if } |i - j| \geq 2 \tag{40}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{41}$$

$$(T_i - q)(T_i + q^{-1}) = 0 \tag{42}$$

for $i, j \in \{1, \ldots, n-1\}$ (see [Ram21, Lec. 5 p. 5-7]).

Unfortunately, we did not manage to solve these systems of partial differential equations to obtain the parameters q_i for the Hecke relations of symmetric groups.

8 Discussion and conclusion

Associated to a complex reflection group, a rational Cherednik algebra is defined and related results are explored. Infinite dimensional RCA modules are then pushed across the KZ functor by solving partial differential equations for horizontal sections and computing monodromy matrices.

Overall, three pieces of information are passed into the definition of a rational Cherednik algebra — a complex reflection group W, a vector space $\mathfrak{a}^*$ on which W acts and some complex parameters κ and c_s. After mapping across the KZ functor, this data is manifested through the relations of the corresponding Hecke algebra associated with W and its complex parameters q_j.

In the example for cyclic groups, the resulting monodromy matrices are computed explicitly and are verified to indeed produce Hecke modules. Although the example for symmetric groups is also calculated, the systems of partial differential equations have not been solved, which could be investigated further. I am also interested in exploring further properties of the rational Cherednik algebras and Hecke algebras, and investigate the KZ functor for other types of complex reflection groups in the future.

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Appendices

A Complex reflection groups

A.1 Complex reflections

Let $\mathfrak{a}^*$ be a $\mathbb{C}$-vector space of dimension n.

$$\mathfrak{a}^* := span\{\epsilon_1, \dots, \epsilon_n\}.$$

A linear transformation $s \in GL(\mathfrak{a}^*)$ is a *complex reflection* if it fixes a hyperplane in $\mathfrak{a}^*$, i.e.,

$$dim(\mathfrak{a}^s) = n - 1, \qquad \text{where } \mathfrak{a}^s = \{\mu \in \mathfrak{a}^* \mid s\mu = \mu\}.$$

We call such hyperplane $\mathfrak{a}^s$ the *reflecting hyperplane* of s. If $(e_2, e_3, \dots, e_n)$ is a basis of $\mathfrak{a}^s$ and $\alpha_s \in \mathfrak{a}^*$ is an nonzero element orthogonal to $\mathfrak{a}^s$, then with respect to the basis $(\alpha_s, e_2, e_3, \dots, e_n)$, the matrix of s is

$$\begin{bmatrix} m_s & 0 & \cdots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

for some $m_s \in \mathbb{C}$. Since $s\alpha_s = m_s\alpha_s$, α_s is an eigenvector of s with non-trivial eigenvalue m_s.

Figure 1 offers a visual intuition for the definition of a complex reflection in a two-dimensional complex vector space.

A.2 Complex reflection groups

A *complex reflection group* W acting on a complex vector space $\mathfrak{a}^*$ is a subgroup of $GL(\mathfrak{a}^*)$ generated by the set of complex reflections it contains. Let R be the set of complex reflections in W. Then, the set of reflecting hyperplanes of W is $\mathcal{A} = \{\mathfrak{a}^s \mid s \in R\}$.

The *configuration space* is

$$\mathfrak{a}^0 = \mathfrak{a}^* - \bigcup_{\mathfrak{a}^s \in \mathcal{A}} \mathfrak{a}^s. \tag{43}$$

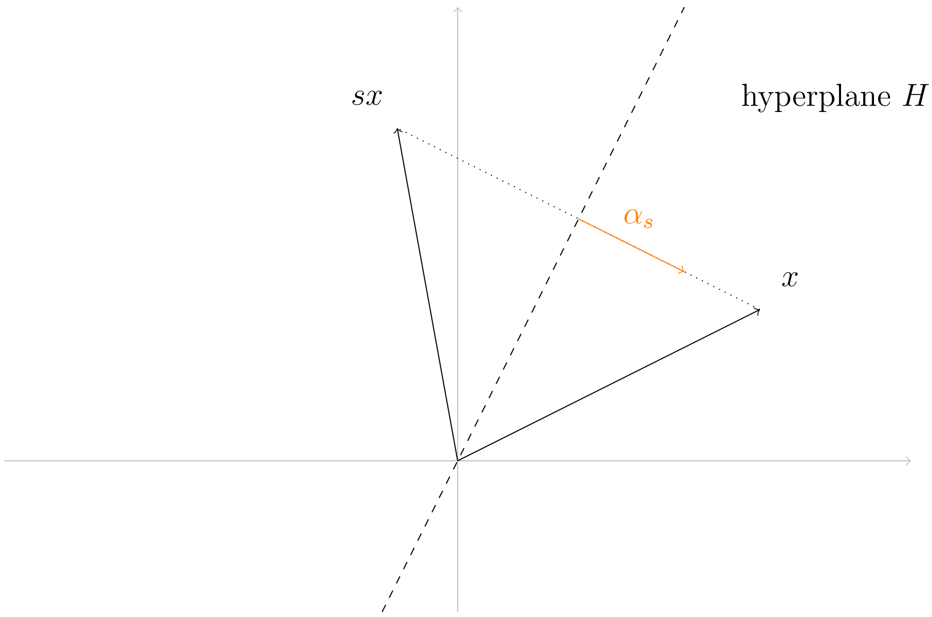

Figure 1: Visualisation of a complex reflection s.

A.3 Dual vector space

Let the dual vector space of $\mathfrak{a}^*$ be $\mathfrak{a}$, consisting of linear maps $\lambda^\vee : \mathfrak{a}^* \to \mathbb{C}$.

We define a pairing $\langle\, , \rangle : \mathfrak{a}^* \times \mathfrak{a} \to \mathbb{C}$ in the usual way with $\langle \mu, \lambda^\vee \rangle = \lambda^\vee(\mu)$. We choose a basis $\{\epsilon_1^\vee, \dots, \epsilon_n^\vee\}$ of the dual vector space $\mathfrak{a}$ so that $\langle \epsilon_i, \epsilon_j^\vee \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$.

Given the action of W on $\mathfrak{a}^*$, let the action on the dual vector space $\mathfrak{a}$ be given by $w\lambda^\vee(\mu) = \lambda^\vee(w^{-1}\mu)$ for $w \in W$, $\mu \in \mathfrak{a}^*$, $\lambda^\vee \in \mathfrak{a}$. This can also be written in the pairing notation as $\langle \mu, w\lambda^\vee \rangle = \langle w^{-1}\mu, \lambda^\vee \rangle$.

This is a valid group action since

$$\begin{aligned} v[w\lambda^\vee(\mu)] &= v\lambda^\vee(w^{-1}\mu) \\ &= \lambda^\vee(w^{-1}v^{-1}\mu) \\ &= \lambda^\vee((vw)^{-1}\mu) \\ &= (vw)\lambda^\vee(\mu) \quad \forall \mu \in \mathfrak{a}^* \\ &\implies v(w\lambda^\vee) = (vw)\lambda^\vee. \end{aligned}$$

Note that $\langle w\mu, w\lambda^\vee \rangle = \langle \mu, \lambda^\vee \rangle$.

A.4 More on eigenvectors of reflections

For each reflection $s \in R$, choose an element $\alpha_s^\vee \in \mathfrak{a}$ such that the kernel of $\alpha_s^\vee$ is the hyperplane fixed by s, ie,

$$\mathfrak{a}^s = \{\mu \in \mathfrak{a}^* \mid \langle \mu, \alpha_s^\vee \rangle = 0\}. \tag{44}$$

Recall that we let $\alpha_s \in \mathfrak{a}^*$ be an eigenvector of s with non-trivial eigenvalue m_s. For both α_s and $\alpha_s^\vee$, we are also free to choose up to any scalar multiple. Let us normalise α_s and $\alpha_s^\vee$ such that $\langle \alpha_s, \alpha_s^\vee \rangle = 1 - m_s$.

Lemma 1. [Gri10, Ch. 2 p. 8] *If $\alpha_s \in \mathfrak{a}^*$ is an eigenvector of s with non-trivial eigenvalue m_s such that $\langle \alpha_s, \alpha_s^\vee \rangle = 1 - m_s$, then*

$$s\mu = \mu - \langle \mu, \alpha_s^\vee \rangle \alpha_s \quad \forall \mu \in \mathfrak{a}^*. \tag{45}$$

Proof. For the eigenvector α_s, $s\alpha_s = m_s\alpha_s = (1 - \langle \alpha_s, \alpha_s^\vee \rangle)\alpha_s$.

Let $(e_2, e_3, \ldots, e_n)$ be a basis of the hyperplane $\mathfrak{a}^s$ fixed by s. Then, $se_i = e_i = e_i - \langle e_i, \alpha_s^\vee \rangle \alpha_s$ since e_i is in the kernel of $\alpha_s^\vee$, which means $\alpha_s^\vee(e_i) = \langle e_i, \alpha_s^\vee \rangle = 0$, for $i = 2, 3, \ldots, n$

We have now shown (45) is true for all elements of the basis $(\alpha_s, e_2, e_3, \ldots, e_n)$, and hence it holds for all $\mu \in \mathfrak{a}^*$. □

Note that the converse is also true: If (45) holds, then substituting $\mu = \alpha_s$ gives $s\alpha_s = (1 - \langle \alpha_s, \alpha_s^\vee \rangle)\alpha_s$. Since s only fixes a hyperplane, there exists α_s such that $\langle \alpha_s, \alpha_s^\vee \rangle \neq 0$. So α_s is indeed an eigenvector of s with non-trivial eigenvalue $m_s = 1 - \langle \alpha_s, \alpha_s^\vee \rangle$. Hence, we can used (45) to find an eigenvector α_s when computing examples.

A.5 Complex reflection groups of type $G(r, 1, 1)$

Let $r \in \mathbb{Z}_{>0}$. A complex reflection group of type $G(r, 1, 1)$ refers to a cyclic group of order r

$$W = \{1, t, \ldots, t^{r-1} \mid t^r = 1\}$$

acting on a complex vector space of dimension 1

$$\mathfrak{a}^* = span\{\epsilon_1\} \cong \mathbb{C}.$$

Let $\zeta = e^{\frac{2\pi i}{r}}$ be a r^{th} root of unity. Let the action of W on $\mathfrak{a}^*$ be given by $t\epsilon_1 = \zeta\epsilon_1$. Then for its dual vector space $\mathfrak{a} = span\{\epsilon_1^\vee\} \cong \mathbb{C}$, the action is $t\epsilon_1^\vee = \zeta^{-1}\epsilon_1^\vee$.

For the cyclic group W, the set of reflections is $R = \{t, t^2, \ldots, t^{r-1}\}$ since each t^i with non-zero exponent fixes the origin, which is a hyperplane of dimension 0. Hence, the set of reflecting hyperplanes is $\mathcal{A} = \{\{0\}\}$ and the configuration space is $\mathfrak{a}^0 = \mathfrak{a}^* - \{0\} = \mathbb{C}\setminus\{0\}$. Clearly, the set of reflections in W can generate the cyclic group, and thus W is indeed a complex reflection group.

For each reflection $s = t^i \in R$, let us denote α_s as α_i and $\alpha_s^\vee$ as $\alpha_i^\vee$ for the sake of clearer notations. Since $\mathfrak{a}^s = \{0\} = \{\mu \in \mathfrak{a}^* \mid \langle \mu, \alpha_s^\vee\rangle = 0\}$, $\alpha_i^\vee$ can be any non-zero element of $\mathbb{C}$. We can see that the choice here does not affect the RCA defined in section 6.1. For now, let us choose $\alpha_i^\vee = b_i\epsilon_1^\vee$ for some $b_i \in \mathbb{C}$. Then, choose $\alpha_i \in \mathfrak{a}^*$ to ensure $s\mu = \mu - \langle \mu, \alpha_s^\vee\rangle\alpha_s$:

$$\begin{aligned}
t^i\mu &= \mu - \langle \mu, \alpha_i^\vee\rangle\alpha_i \\
\zeta^i\mu_1\epsilon_1 &= \mu_1\epsilon_1 - (\mu_1 b_i)\alpha_i \\
\alpha_i &= \frac{(1-\zeta^i)\epsilon_1}{b_i}.
\end{aligned}$$

A.6 Complex reflection groups of type $G(1,1,n)$

Let $n \in \mathbb{Z}_{>0}$. A complex reflection group of type $G(1,1,n)$ is a symmetric group of order $n!$

$$W = \langle s_i \mid i = 1, 2 \ldots, n-1\rangle \text{ where } s_i = (i, i+1)$$

acting on a complex vector space of dimension n

$$\mathfrak{a}^* = span\{\epsilon_1, \ldots, \epsilon_n\} \cong \mathbb{C}^n$$

via permutations of coordinates.

Let the dual vector space be $\mathfrak{a} = span\{\epsilon_1^\vee, \ldots, \epsilon_n^\vee\}$.

The set of reflections in W is $R = \{(i,j) \mid 1 \le i < j \le n\}$, since each (i,j) fixes a hyperplane in which the i^{th} component equals to the j^{th} component.

$$\mathfrak{a}^{(i,j)} = \{\mu = (\mu_1, \ldots, \mu_n) \in \mathfrak{a}^* \mid \mu_i = \mu_j\}.$$

Hence, the set of reflecting hyperplanes of W is $\mathcal{A} = \{\mathfrak{a}^{(i,j)} \mid (i,j) \in R\}$. All reflections in W belong to the same conjugacy class since $(i,j) = s_{j-1}\cdots s_{i+1}s_is_{i+1}\cdots s_{j-1}$.

Using (44) and (45), we find that $\alpha^{\vee}_{(i,j)} = \epsilon^{\vee}_i - \epsilon^{\vee}_j$ and $\alpha_{(i,j)} = \epsilon_i - \epsilon_j$.

The configuration space is

$$\mathfrak{a}^0 = \{\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{a}^* \mid \mu_i \neq \mu_j \ \forall i \neq j\}\,.$$

B Symmetric algebras

Let $S(\mathfrak{a}^*)$ be the symmetric algebra of $\mathfrak{a}^*$, which is an algebra generated by x_μ for $\mu \in \mathfrak{a}^*$ with relations

$$x_{\mu+\nu} = x_\mu + x_\nu, \quad x_{c\mu} = cx_\mu, \quad x_\mu x_\nu = x_\nu x_\mu \tag{46}$$

for $\mu, \nu \in \mathfrak{a}^*,\ c \in \mathbb{C}$. For the simplicity of notation, let us write $x_i := x_{\epsilon_i}$ for $i = 1, \dots, n$.

Similarly, let $S(\mathfrak{a})$ be the symmetric algebra of the dual $\mathfrak{a}$, which is an algebra generated by $y_{\lambda^\vee}$ for $\lambda^\vee \in \mathfrak{a}$ with relations

$$y_{\lambda^\vee+\gamma^\vee} = y_{\lambda^\vee} + y_{\gamma^\vee}, \quad y_{c\lambda^\vee} = cy_{\lambda^\vee}, \quad y_{\lambda^\vee} y_{\gamma^\vee} = y_{\gamma^\vee} y_{\lambda^\vee} \tag{47}$$

for $\lambda^\vee, \gamma^\vee \in \mathfrak{a},\ c \in \mathbb{C}$. We write $y_i = y_{\epsilon_i^\vee}$ for $i = 1, \dots, n$.

Note that we can now write (45) as

$$x_{s\mu} = x_\mu - \langle \mu, \alpha_s^\vee \rangle x_{\alpha_s}. \tag{45$'$}$$