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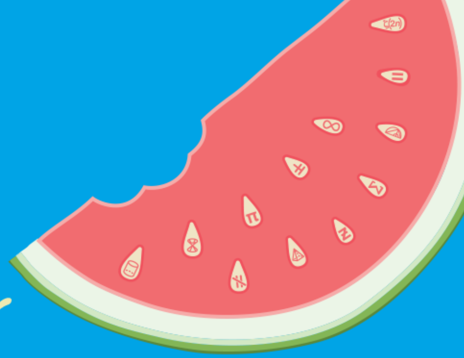

On the Generalised Möbius Function of a Finite Poset

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Abstract

This report aims to provide a new mathematical framework for the Generalised Möbius function of a finite partially ordered set, and investigate upper bounds on the sum of its values. The Möbius function has been studied previously in the literature and upper bounds for the absolute sum of its values are known. Using a theorem by Phillip Hall, we count the total sum of entries of the Möbius function for a poset as the difference between the number of odd and even chains in the poset. We extend upon this idea by introducing the Generalised Möbius function, which counts any linear combination of the number of odd and even chains in a poset. Under this generalised framework, we preset new results of the sums of Generalised Möbius function values, with a focus on a special class of posets - hierarchical posets. This is the class of posets for which the upper bound on the absolute sum of Generalised Möbius function values is achieved. Finding that Günter Matthias Ziegler proved a theorem on the upper bound for the absolute sum of Möbius function values using a "compression algorithm", we similarly apply a "generalised compression algorithm" to prove an upper bound for the absolute sum of Generalised Möbius function values under certain conditions. Finally, we present a conjecture on an upper bound for the absolute sum of Generalised Möbius function values when these conditions are removed.

Contents

1 Introduction

1.1 Overview

Subsection 1.2 recounts a brief history of the Möbius function of a finite poset.

Section 2 presents basic definitions, as well as the notation that is used throughout the report.

In Section 3, we present preliminary results on the Möbius function; notably, Theorem 3.3 by Phillip Hall forms the premise of many of the arguments used throughout the report.

In Section 4, we introduce the mathematical framework for the Generalised Möbius function, presenting basic properties in Subsection 4.1 and properties of the sums of Generalised Möbius function values and their upper bounds in Subsection 4.2. In Subsection 4.2, a key theorem, Theorem 4.21, is presented, providing an upper bound for the absolute sum of Generalised Möbius function values under certain conditions. Then, we extend on this Theorem in Conjecture 4.22, where we conjecture an upper bound for the absolute sum of Generalised Möbius function values when these conditions are removed. In Subsection 4.3, we present new results on the Generalised Chain-Poset Theorem, which states the relationship between the sums of Generalised Möbius function values for a poset, and the corresponding poset of chains. In this section, Theorem 4.24 states a concise relationship between these two values under certain conditions, whereas Theorem 4.25 generalises the relationship to all conditions. In Subsection 4.4, we provide examples of the use of the compression algorithm, which allows us to transform a poset into one with a greater absolute sum of its Generalised Möbius function values, while maintaining the same number of elements in the poset.

Finally, in Section 5, we conclude the report with suggestions for future work.

1.2 History of the Möbius Function

In this Section, we recount a brief history of the Möbius function.

Following August Ferdinand Möbius' introduction of the Möbius function, several other mathematicians including Edmond Laguerre, Julius Dedekind, Eric Temple Bell, Gian-Carlo Rota, Hassler Whitney, Louis Weisner and Phillip Hall used the Möbius function to solve various problems.

August Ferdinand Möbius introduced the Möbius function in 1831 in his paper [12], where he investigates the properties of the coefficients of a function. In particular, starting with a function $f = a_1x + a_2x^2 + \cdots$, he sought to determine coefficients $b_1, b_2, \ldots$ such that $x = b_1f(x) + b_2f(x^2) + \cdots$. Thereafter, the Möbius function became an important tool used in number theory. In 1837, Edmond Laguerre was the first to present the theory of Möbius inversion in the format used today in number theory [10]. It was Julius Dedekind in 1857, studying mathematics under Carl Friedrich Gauss, who first proved and stated Möbius inversion in his paper [4]. Then, taking advantage of prime number factorisation, Franz Mertens [11], in 1874, provided a more succinct definition of the Möbius function, and also introduced its notation, μ. With the help of modern abstract algebra, Eric Temple Bell, in 1915, treated the arithmetic functions seen in Möbius inversion from a ring structure perspective [2], providing an even more succinct definition of Möbius inversion.

However, it was Gian-Carlo Rota's fundamental paper on Möbius functions in 1964 [13] which marked the beginning of the "modern era" [7] of the Möbius function. Since then, the Möbius function has been used extensively to solve problems in combinatorics. Rota's paper expands on earlier work done by mathematicians such as Hassler Whitney [18] in 1932, Louis Weisner [17] in 1935 and Phillip Hall [9] in 1936, who used the theory of Möbius inversion to independently solve group theory problems. In his 1932 paper [18], Hassley Whitney demonstrates the use of logical expansion, which parallels the principle of inclusion-exclusion we know today, to problems in prime number theory, probability and the coloring of graphs. In 1936, Phillip Hall [8] stated an equivalent enumeration principle for any p-group (a group, whose order is a power of p, a prime number [6]). Weisner, in 1935, found similar results for Möbius inversion on prime-power groups [17].

However, neither Louis Weisner nor Phillip Hall seemed to be aware of the implications of their group theory work to combinatorics. It was Gian-Carlo Rota's paper that unified the structural and enumerative aspects of posets [7] and related topics such as inclusion-exclusion, Möbius inversion in number theory, colouring theory and flows in networks together [3].

2 Definitions

Definition 2.1. [15] A finite partially ordered set, or *poset*, is a set P equipped with a relation $\preceq$ that satisfies, for all $x, y, z \in P$, the following three properties:

(i) reflexivity: $x \preceq x$ for all $x \in P$;

(ii) anti-symmetry: if $x \preceq y$ and $y \preceq x$, then $x = y$;

(iii) transitivity: if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Definition 2.2. [15] x is a *minimal element* of a poset P if $x \preceq y$ for all $y \in P$.

Definition 2.3. [15] x is a *maximal element* of a poset P if $y \preceq x$ for all $y \in P$.

Definition 2.4. [15] A poset is *bounded* if it contains a unique minimal and maximal element.

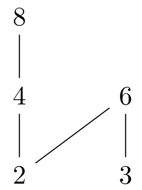
Definition 2.5. [14] $x \prec y$ indicates that an element y of a poset *covers* another element x. That is, $x \prec y$ and that no element $c \in P$ satisfies $x \prec c \prec y$.

Definition 2.6. [16] A *Hasse diagram* is a graphical representation of a poset drawn according to the following rules:

- If $x \prec y$, then the vertex corresponding to x is drawn lower than the vertex corresponding to y;
- A line segment is drawn between two vertices corresponding to x and y if and only if x covers y or y covers x.

Example 2.7. A simple poset on the integers $\{1, 2, \ldots, 10\}$, ordered by the "less than" relation $\leq$, is bounded since it has minimal element 1 and maximal element 10.

Example 2.8. The poset P on the subset of integers $\{2, 3, 4, 6, 8\}$ equipped with the relation of divisibility can be represented by the following Hasse diagram:



Furthermore, we can see that P is not bounded since it does not contain a unique minimal or maximal element.

Definition 2.9. [15] The *cardinality* of a set is the number of elements in the set.

Definition 2.10. [15] A set of $k+1$ elements $x_0, \ldots, x_k$ of a poset P satisfying

$$x_0 \prec \cdots \prec x_k$$

is a *chain* with cardinality $k+1$.

Definition 2.11. [15] $\mathcal{C}(x, y)$ denotes the set of non-empty chains that have x as the least element and y as the greatest element in the chain. That is, for any $z \in C$, where $C \in \mathcal{C}(x, y)$, then $x \preceq z$ and $z \preceq y$.

Definition 2.12. $\mathcal{C}_o^P(x, y)$ denotes the subset of chains of $\mathcal{C}(x, y)$ with odd cardinality. $\mathcal{C}_e^P(x, y)$ denotes the subset of chains of $\mathcal{C}(x, y)$ with even cardinality. It follows that $\mathcal{C}(x, y) = \mathcal{C}_o^P(x, y) \cup \mathcal{C}_e^P(x, y)$.

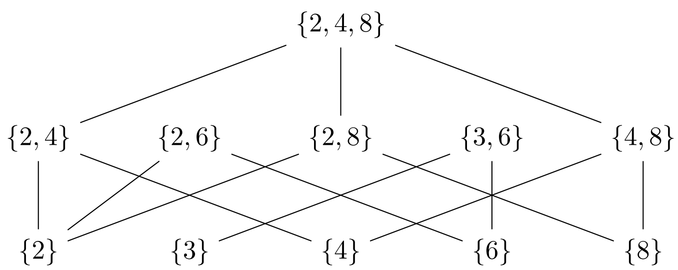
Definition 2.13. $\mathrm{Ch}(P)$ denotes the set of all chains of a poset P. Also, define $C_k(P)$ to be the subset of $\mathrm{Ch}(P)$ containing all chains with cardinality k. That is,

$$C_k(P) = \{C \in \mathrm{Ch}(P) : |C| = k\}.$$

Definition 2.14. $\mathcal{C}_o^P$ denotes the subset of chains of $\mathrm{Ch}(P)$ with odd cardinality. That is, $\mathcal{C}_o^P = \bigcup_{\text{k odd}} C_k(P)$. Similarly, $\mathcal{C}_e^P$ denotes the subset of chains of $\mathrm{Ch}(P)$ with even cardinality. That is, $\mathcal{C}_e^P = \bigcup_{\text{k even}} C_k(P)$. It follows that $\mathrm{Ch}(P) = \mathcal{C}_o^P \cup \mathcal{C}_e^P$.

Remark 2.15. $\mathrm{Ch}(P)$ forms a poset itself, under subset inclusion.

Example 2.16. For the poset P from Example 2.8, the poset $\mathrm{Ch}(P)$, ordered by subset inclusion, is represented by the following Hasse diagram:



Definition 2.17. [15] Two elements, x and y of a set, are *comparable* with respect to the binary relation $\prec$ if $x \prec y$ or $y \prec x$. They are *incomparable* if they are not comparable. That is, if $x \not\prec y$ and $y \not\prec x$.

Definition 2.18. [15] An *antichain* of a poset P is a subset of P of pairwise incomparable elements.

Example 2.19. For the poset from Example 2.8, the subsets $\{2,3\}$, $\{6,8\}$ and $\{4,6\}$ are antichains of P.

Definition 2.20. [14] The *zeta function* $\zeta : P \times P \mapsto \mathbb{Z}$ for all $x, y \in P$ is defined as follows:

$$\zeta(x,y) := \begin{cases} 1\,, & x \preceq y\,; \\ 0\,, & \text{otherwise.} \end{cases}$$

Remark 2.21. By the anti-symmetry of P, we can permute the row and column indices such that ζ is upper-triangular. Since $\zeta(x,x) = 1$ for each element x, it follows that $\det(\zeta) = 1$ and that ζ is invertible.

Definition 2.22. [14] The inverse matrix of ζ is the *Möbius function* $\mu : P \times P \mapsto \mathbb{Z}$ of P:

$$\zeta\mu = \mu\zeta = I\,.$$

Remark 2.23. Since μ is a matrix consisting of only 1 and 0 entries, and the determinant $\det \mu = 1$, it follows from the co-factor expression for the inverse

$$B^{-1} = \frac{1}{\det B}(\operatorname{cof}(B))^T$$

that cof(B) will have integer entries only and hence that the inverse ζ will also have integer entries only.

Example 2.24. For the poset from Example 2.16, if we order the rows and columns of the adjacency matrix for P so that it is upper-triangular, then ζ and μ are given by the following matrices:

$$\zeta = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mu = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 2.25. Define $\mu_P^{a,b}(x,y) : P \times P \to \mathbb{R}$ by

$$\mu^{a,b}(x,y) = a|\mathcal{C}_o(x,y)| + b|\mathcal{C}_e(x,y)|.$$

where $a, b \in \mathbb{R}$. The corresponding matrix $\mu_P^{a,b} \in \mathbb{R}^{n\times n}$ is the *generalised Möbius function.*

Remark 2.26. The subscript/superscript P in $\mathcal{C}_o^P, \mathcal{C}_e^P, \mathcal{C}_o^P(x,y), \mathcal{C}_e^P(x,y), \mu_P^{a,b}(x,y)$ and $\mu_P^{a,b}$ can be omitted when there is no confusion about the poset in question.

3 Preliminary results on the Möbius function

Lemma 3.1. [1] *The (x, y)th entry of the matrix $(\zeta - I)^k$ is the number of chains C in $\mathcal{C}(x, y)$ of cardinality $|C| = k + 1$.*

Proof. For $k = 1$,

$$(\zeta - I)(x, y) = \begin{cases} 1 & x \prec y; \\ 0 & \textit{otherwise}. \end{cases}$$

Since $\{x, y\}$ is the only chain in $\mathcal{C}(x, y)$ with cardinality 2, the lemma is true for $k = 1$.
Assume that the lemma is true for $k = n$. The $(x, y)^{th}$ entry of the matrix $(\zeta - I)^{n+1}$ is

$$\sum_z (\zeta - I)^n(x, z)(\zeta - I)(z, y) = \sum_{z : z \prec y} (\zeta - I)^n(x, z),$$

which is the number of chains in $\mathcal{C}(x, y)$ with z as the $(n+1)^{th}$ term and y as the $(n+2)^{nd}$ term. Hence, it is equal to the number of chains with cardinality $|C| = n + 2$ by the induction hypothesis. □

The following corollary expresses the number of elements in the poset $\mathrm{Ch}(P)$.

Corollary 3.2. $(2I - \zeta)^{-1}$ *is the enumerator matrix for the total number of chains. In particular, entry (x, y) of $(2I - \zeta)^{-1}$ equals $\mathcal{C}(x, y)$. Hence, the number of chains in P is the sum of entries in this matrix.*

Proof. Begin by summing the geometric series

$$I + (\zeta - I)t + (\zeta - I)^2 t^2 + \cdots = (I - (\zeta - I)t)^{-1}.$$

Substituting $t = 1$, we obtain

$$I + (\zeta - I) + (\zeta - I)^2 + \cdots = (2I - \zeta)^{-1}.$$

Therefore,

$$|\mathrm{Ch}(P)| = \sum_{x, y \in P} (2I - \zeta)^{-1}(x, y).$$

□

Theorem 3.3. (Hall [9]) *For all $x, y \in P$,*

$$\mu(x, y) = - \sum_{C \in \mathcal{C}(x, y)} (-1)^{|C|}.$$

Proof. Begin by writing

$$\begin{aligned} \mu(x, y) &= \zeta^{-1}(x, y) \\ &= (I - (\zeta - I))^{-1}(x, y) \\ &= I(x, y) - (\zeta - I)(x, y) + (\zeta - I)^2(x, y) + \cdots \\ &= (I(x, y) + (\zeta - I)^2(x, y) + \cdots) - ((\zeta - I)(x, y) + (\zeta - I)^3(x, y) + \cdots), \end{aligned}$$

which is the difference between the number of chains in $\mathcal{C}(x,y)$ of odd cardinality and of even cardinality. Hence,

$$\begin{aligned}\mu(x,y) &= \sum_{C\in\mathcal{C}_o(x,y)} 1 - \sum_{C\in\mathcal{C}_e(x,y)} 1\\ &= -\sum_{C\in\mathcal{C}_o(x,y)} (-1)^{|C|} - \sum_{C\in\mathcal{C}_e(x,y)} (-1)^{|C|}\\ &= -\sum_{C\in\mathcal{C}(x,y)} (-1)^{|C|}\,.\end{aligned}$$

□

Corollary 3.4. [1] *The total sum of Möbius function values for a poset P is*

$$\sum_{x,y\in P} \mu(x,y) = -\sum_{C\in\mathrm{Ch}(P)} (-1)^{|C|}\,.$$

Proof. From Theorem 3.3, it follows that

$$\sum_{x,y\in P} \mu(x,y) = -\sum_{x,y\in P}\sum_{C\in\mathcal{C}(x,y)} (-1)^{|C|} = -\sum_{C\in\mathrm{Ch}(P)} (-1)^{|C|}\,.$$

□

Corollary 3.5. [1] *If P has a minimal element $\hat{0}$ or a maximal element $\hat{1}$, then*

$$\sum_{x,y\in P} \mu(x,y) = 1\,.$$

Proof. We will prove the case when P has a maximal element $\hat{1}$. By the definition of a maximal element, $x \preceq \hat{1}$ for all $x \in P$. This implies that every chain of cardinality ℓ with greatest element $z \neq \hat{1}$ can be extended to a chain of cardinality $\ell+1$ with greatest element $\hat{1}$. By Corollary 3.4, $\sum\limits_{x,y\in P} \mu(x,y)$ counts the difference between the number of chains with odd and even cardinality in P, so

$$\sum_{x,y\in P\setminus\{\hat{1}\}} \mu(x,y) = -\sum_{x\in P\setminus\{\hat{1}\}} \mu(x,\hat{1}).$$

Also, since $\mu(\hat{1},\hat{1}) = 1$, we can write

$$\sum_{x,y\in P} \mu(x,y) = \sum_{x,y\in P\setminus\{\hat{1}\}} \mu(x,y) + \sum_{x\in P\setminus\{\hat{1}\}} \mu(x,\hat{1}) + \mu(\hat{1},\hat{1}) = 1\,.$$

Similarly, if P has a minimal element, we can uniquely increase the cardinality of every chain C in P, where $\hat{0} \notin C$, by one, simply by including $\hat{0}$ at the start of the chain and the result follows. □

4 The Generalised Möbius Function

4.1 Basic properties of the Generalised Möbius Function

Lemma 4.1. $\mu^{a,b}$ *is upper triangular with diagonal entries all equal to a.*

Proof. If $y \prec x$, then $\mathcal{C}_o(x,y) = \mathcal{C}_e(x,y) = \emptyset$ so $\mu^{a,b}(x,y) = 0$. Hence, $\mu^{a,b}$ is upper triangular.

If $x = y$, then $\mathcal{C}(x,y) = \{\{x\}\}$ so $|\mathcal{C}_o(x,y)| = 1$ and $|\mathcal{C}_e(x,y)| = 0$ so $\mu^{a,b}(x,y) = a$. □

Lemma 4.2. $\mu^{a+c,b+d} = \mu^{a,b} + \mu^{c,d}$ *for all* $a,b,c,d \in \mathbb{R}$.

Proof. From Definition 2.25, for all $x, y \in P$,

$$\begin{aligned}\mu^{a+c,b+d}(x,y) &= (a+c)|\mathcal{C}_o(x,y)| + (b+d)|\mathcal{C}_e(x,y)| \\ &= (a|\mathcal{C}_o(x,y)| + b|\mathcal{C}_e(x,y)|) + (c|\mathcal{C}_o(x,y)| + d|\mathcal{C}_e(x,y)|) \\ &= \mu^{a,b}(x,y) + \mu^{c,d}(x,y).\end{aligned}$$

It follows that the matrix $\mu^{a+c,b+d} = \mu^{a,b} + \mu^{c,d}$ for all $a,b,c,d \in \mathbb{R}$. □

Lemma 4.3. $\mu^{ka,kb} = k\mu^{a,b}$ *for all* $a,b,k \in \mathbb{R}$.

Proof. From Definition 2.25, for all $x, y \in P$,

$$\begin{aligned}\mu^{ka,kb}(x,y) &= ka|\mathcal{C}_o(x,y)| + kb|\mathcal{C}_e(x,y)| \\ &= k(a|\mathcal{C}_o(x,y)| + b|\mathcal{C}_e(x,y)|) \\ &= k\mu^{a,b}(x,y).\end{aligned}$$

It follows that the matrix $\mu^{ka,kb} = k\mu^{a,b}$ for all $k,a,b \in \mathbb{Z}$. As a special case, when $k = -1$, then we can see that

$$\mu^{-a,-b} = -\mu^{a,b} \text{ for all } a,b \in \mathbb{Z}.$$

□

Lemma 4.4. *For all* $a, b \in \mathbb{R}$ *and* $x, y \in P$ *such that* $x \neq y$,

$$\mu^{a,b}(x,y) = \sum_{z \in A_x} \mu^{b,a}(z,y) + \sum_{z \notin A_x} \frac{|\mu^{a,b}(z,y)|}{|A_{x,z}|},$$

where $A_x = \{z : x \prec z\}$ *are the covering elements of* x *and* $A_{x,z} = \{a : x \prec a \prec z\} \subseteq A_x$.

Proof. Suppose that $x \neq y$ and $C = \{x, w, \ldots, y\} \in \mathcal{C}_o(x,y)$.

If $w \in A_x$, then the mapping of C to $\{w, \ldots, y\} \in \mathcal{C}_e(w,y)$ is one-to-one.

If $w \notin A_x$ (w does not cover x), then there exists $z \neq w$ such that $z \in A_x$ and $z \prec w$. That is, $z \in A_{x,z}$. If we "swap" x with z in C, we will obtain a new chain $\{z, w, \ldots, y\} \in \mathcal{C}_o(z,y)$ such that z covers x. Note that we can choose z to be any element of the set $A_{x,z}$, whose cardinality is $|A_{x,w}|$. So for every such w, we can map the chain $\{x, w, \ldots, y\}$ to exactly $|A_{x,w}|$ corresponding chains of the form $\{z, w, \ldots, y\} \in \mathcal{C}_o(z,y)$. Hence, counting the size of $|\mathcal{C}_o(z,y)|$ overcounts the size of $\mathcal{C}_o(x,y)$ by a multiple of $A_{x,z}$. Therefore,

$$|\mathcal{C}_o(x,y)| = \sum_{z \in A_x} |\mathcal{C}_e(z,y)| + \sum_{z \notin A_x} \frac{|\mathcal{C}_o(z,y)|}{|A_{x,z}|}.$$

Now, suppose that $x \neq y$ and $C = \{x, w, \dots, y\} \in \mathcal{C}_e(x,y)$. Note that C has at least 2 elements and has exactly 2 elements if and only if $y \in A_x$. In this case, $|\mathcal{C}_e(x,y)| = |\{\{x,y\}\}| = 1$. Furthermore for this case, $\sum_{z\in A_x} |\mathcal{C}_o(z,y)| = |\{y\}| = 1$ and $\sum_{z\notin A_x} \frac{|\mathcal{C}_o(z,y)|}{|A_{x,z}|} = 0$.

On the other hand, if $x \not\prec y$, then elements of $\mathcal{C}_e(x,y)$ contain at least 4 elements so by similar arguments as previously when $C \in \mathcal{C}_e(x,y)$, we can conclude that

$$|\mathcal{C}_e(x,y)| = \sum_{z\in A_x} |\mathcal{C}_o(z,y)| + \sum_{z\notin A_x} \frac{|\mathcal{C}_e(z,y)|}{|A_{x,z}|}$$

for all x, y such that $x \neq y$.

Hence,

$$\begin{aligned}
\mu^{a,b}(x,y) &= a|\mathcal{C}_o(x,y)| + b|\mathcal{C}_e(x,y)| \\
&= a\left(\sum_{z\in A_x} |\mathcal{C}_e(z,y)| + \sum_{z\notin A_x} \frac{|\mathcal{C}_o(z,y)|}{|A_{x,z}|}\right) + b\left(\sum_{z\in A_x} |\mathcal{C}_o(z,y)| + \sum_{z\notin A_x} \frac{|\mathcal{C}_e(z,y)|}{|A_{x,z}|}\right) \\
&= \sum_{z\in A_x} b|\mathcal{C}_o(z,y)| + a|\mathcal{C}_e(x,y)| + \frac{1}{|A_{x,z}|}\sum_{z\notin A_x} a|\mathcal{C}_o(z,y)| + b|\mathcal{C}_e(x,y)| \\
&= \sum_{z\in A_x} \mu^{b,a}(z,y) + \sum_{z\notin A_x} \frac{\mu^{a,b}(z,y)}{|A_{x,z}|}.
\end{aligned}$$

Note that using this counting techinque, the result cannot be generalised to matrix form due to the presence of the x-dependent variable $|A_{x,z}|$. □

Lemma 4.5.

$$\zeta^{-1} = \mu = \mu^{1,-1}; \tag{4.1}$$

$$(2I - \zeta)^{-1} = \mu^{1,1} \tag{4.2}$$

Proof. Equation 4.1 follows from Theorem 3.3. Equation 4.2 follows from Theorem 3.2.

Lemma 4.6.

$$\mu^{a,b} = \frac{a-b}{2}\mu^{1,-1} + \frac{a+b}{2}\mu^{1,1}$$

Proof. Using Lemma 4.5, for all $x, y \in P$, we can write

$$\mu^{1,-1}(x,y) = |\mathcal{C}_o(x,y)| - |\mathcal{C}_e(x,y)|$$

and

$$\mu^{1,1}(x,y) = |\mathcal{C}_o(x,y)| + |\mathcal{C}_e(x,y)|.$$

Hence,

$$\begin{aligned}
\frac{a-b}{2}\mu^{1,-1}(x,y) + \frac{a+b}{2}\mu^{1,1}(x,y) &= \frac{a-b}{2}|\mathcal{C}_o(x,y)| - \frac{a-b}{2}|\mathcal{C}_o(x,y)| + \frac{a+b}{2}|\mathcal{C}_o(x,y)| + \frac{a+b}{2}|\mathcal{C}_e(x,y)| \\
&= a|\mathcal{C}_o(x,y)| + b|\mathcal{C}_e(x,y)|
\end{aligned}$$

$$= \mu^{a,b}(x,y).$$

Since this holds for all $x, y \in P$, the result follows.

$\square$

While the Möbius function was initially defined as the inverse of the zeta function, we will reverse the direction of definition and instead, define the generalised zeta function as the inverse of the generalised Möbius function, as shown in the following lemma.

The *generalised zeta function* $\zeta^{a,b}$ is the inverse of the generalised Möbius function $\mu^{a,b}$. Both functions can be expressed in terms of ζ as shown below.

Lemma 4.7.

$$\mu^{a,b} = \zeta^{-1}((a-b)I + b\zeta)(2I-\zeta)^{-1}$$

Proof. From Lemma 4.6, we can write

$$\begin{aligned}
\mu^{a,b} &= \frac{a-b}{2}\zeta^{-1} + \frac{a+b}{2}(2I-\zeta)^{-1} \\
&= \frac{a-b}{2}\zeta^{-1} + \frac{a+b}{2}\zeta^{-1}\zeta(2I-\zeta)^{-1} \\
&= \zeta^{-1}\left(\frac{a-b}{2}I + \frac{a+b}{2}\zeta(2I-\zeta)^{-1}\right) \\
&= \zeta^{-1}\left(\frac{a-b}{2}(2I-\zeta)(2I-\zeta)^{-1} + \frac{a+b}{2}\zeta(2I-\zeta)^{-1}\right) \\
&= \zeta^{-1}\left(\frac{a-b}{2}(2I-\zeta) + \frac{a+b}{2}\zeta\right)(2I-\zeta)^{-1} \\
&= \zeta^{-1}((a-b)I + b\zeta)(2I-\zeta)^{-1}.
\end{aligned}$$

$\square$

Lemma 4.8.

$$\zeta^{a,b} = (2-\zeta)((a-b)I + b\zeta)^{-1}\zeta$$

Proof. Taking the inverse of $\mu^{a,b}$ from Lemma 4.7, the expression is obtained. To clarify,

$$\zeta^{a,b}\mu^{a,b} = (2-\zeta)((a-b)I + b\zeta)^{-1}\zeta\zeta^{-1}((a-b)I + b\zeta)(2I-\zeta)^{-1} = I$$

and

$$\mu^{a,b}\zeta^{a,b} = \zeta^{-1}((a-b)I + b\zeta)(2I-\zeta)^{-1}(2-\zeta)((a-b)I + b\zeta)^{-1}\zeta = I.$$

$\square$

We can generalise Lemmas 4.6, 4.7 and 4.8 by expressing $\mu^{a,b}$ in terms of $\mu^{c,d}$ and $\mu^{e,f}$ for any $c, d, e, f \in \mathbb{R}$ such that $cf - de \neq 0$.

Lemma 4.9. *For any $a, b, c, d, e, f \in \mathbb{R}$ such that $cf - de \neq 0$,*

$$\mu^{a,b} = \alpha\mu^{c,d} + \beta\mu^{e,f},$$

where $\alpha = \frac{af-be}{cf-de}$ and $\beta = \frac{cb-da}{cf-de}$.

Proof. For all x, y, we require that

$$a|\mathcal{C}_o(x,y)| + b|\mathcal{C}_e(x,y)| = \alpha(c|\mathcal{C}_o(x,y)| + d|\mathcal{C}_e(x,y)|) + \beta(c|\mathcal{C}_o(x,y)| + d|\mathcal{C}_e(x,y)|).$$

Equating coefficients of $|\mathcal{C}_o(x,y)|$ and $|\mathcal{C}_e(x,y)|$, we obtain the system of linear equations

$$a = \alpha c + \beta e$$
$$b = \alpha d + \beta f$$

which yields the required expression for α and β. The condition $cf - de \neq 0$ ensures that the matrix $\begin{pmatrix} c & e \\ d & f \end{pmatrix}$ is invertible. □

Lemma 4.10. *For any $a, b, c, d, e, f \in \mathbb{R}$ such that $cf - de \neq 0$,*

$$\mu^{a,b} = \mu^{c,d}(\alpha\zeta^{e,f} + \beta\zeta^{c,d})\mu^{e,f},$$

where $\alpha = \frac{af-be}{cf-de}$ and $\beta = \frac{cb-da}{cf-de}$

Proof. By Lemma 4.9,

$$\begin{aligned}
\mu^{a,b} &= \alpha\mu^{c,d} + \beta\mu^{e,f} \\
&= \alpha\mu^{c,d} + \beta\mu^{c,d}\zeta^{c,d}\mu^{e,f} \\
&= \mu^{c,d}(\alpha I + \beta\zeta^{c,d}\mu^{e,f}) \\
&= \mu^{c,d}(\alpha\zeta^{e,f}\mu^{e,f} + \beta\zeta^{c,d}\mu^{e,f}) \\
&= \mu^{c,d}(\alpha\zeta^{e,f} + \beta\zeta^{c,d})\mu^{e,f}.
\end{aligned}$$

□

Corollary 4.11. *For any $a, b, c, d, e, f \in \mathbb{R}$ such that $cf - de \neq 0$,*

$$\zeta^{a,b} = \zeta^{e,f}(\alpha\zeta^{e,f} + \beta\zeta^{c,d})^{-1}\zeta^{c,d},$$

where $\alpha = \frac{af-be}{cf-de}$ and $\beta = \frac{cb-da}{cf-de}$.

Proof. The expression follows by taking the inverse of $\mu^{a,b}$ given in Lemma 4.10. □

4.2 Sums of Generalised Möbius Function Values

Lemma 4.12. *If a poset P has a minimal element $\hat{0}$, then for all $a, b \in \mathbb{R}$,*

$$\sum_{x,y \in P} \mu^{a,b}(x,y) = \sum_{x,y \in P \setminus \hat{0}} \mu^{a+b,a+b}(x,y) + 1 = (a+b) \sum_{x,y \in P \setminus \hat{0}} \mu^{1,1}(x,y) + 1.$$

Proof. We count the number of chains in P which do not contain $\hat{0}$ and then uniquely extend each chain using $\hat{0}$, increasing its cardinality by 1, and hence resulting in a new chain with opposite parity. That is,

$$\begin{aligned}
\sum_{x,y \in P} \mu^{a,b}(x,y) &= \sum_{x,y \neq \hat{0}} \mu^{a,b}(x,y) + \sum_{x \neq \hat{0}} \mu^{a,b}(x,\hat{0}) + \mu^{a,b}(\hat{0},\hat{0}) \\
&= \sum_{x,y \in P \setminus \hat{0}} (a|\mathcal{C}_o(x,y)| + b|\mathcal{C}_e(x,y)|) + \sum_{x,y \in P \setminus \hat{0}} (a|\mathcal{C}_e(x,y)| + b|\mathcal{C}_o(x,y)|) + 1 \\
&= \sum_{x,y \in P \setminus \hat{0}} ((a+b)|\mathcal{C}_o(x,y)| + (a+b)|\mathcal{C}_e(x,y)|) + 1 \\
&= \sum_{x,y \in P \setminus \hat{0}} \mu^{a+b,a+b}(x,y) + 1 \\
&= (a+b) \sum_{x,y \in P \setminus \hat{0}} \mu^{1,1}(x,y) + 1.
\end{aligned}$$

The last line follows from Lemma 4.3. □

Lemma 4.13. *If a poset P has a maximal element $\hat{1}$, then for all $a, b \in \mathbb{R}$,*

$$\sum_{x,y \in P} \mu^{a,b}(x,y) = \sum_{x,y \in P \setminus \hat{1}} \mu^{a+b,a+b}(x,y) + 1 = (a+b) \sum_{x,y \in P \setminus \hat{1}} \mu^{1,1}(x,y) + 1.$$

Proof. The proof follows the same arguments as in Lemma 4.12 but instead, we count the number of chains in P which do not contain $\hat{1}$ and then uniquely extend each chain using $\hat{1}$, increasing its cardinality by 1, and hence resulting in a new chain with opposite parity. □

Lemma 4.14. *If a poset P is bounded (has a minimal element $\hat{0}$ and a maximal element $\hat{1}$), then for all $a, b \in \mathbb{R}$,*

$$\begin{aligned}
\sum_{x,y \in P} \mu^{a,b}(x,y) &= \sum_{x,y \in P \setminus (\hat{0} \cup \hat{1})} \mu^{(a+b)^2,(a+b)^2}(x,y) + (a+b+1) \\
&= (a+b)^2 \sum_{x,y \in P \setminus (\hat{0} \cup \hat{1})} \mu^{1,1}(x,y) + (a+b+1).
\end{aligned}$$

Proof. First we apply Lemma 4.13 to P to obtain

$$\sum_{x,y \in P} \mu^{a,b}(x,y) = (a+b) \sum_{x,y \in P \setminus \hat{1}} \mu^{1,1}(x,y) + 1.$$

Note that P is bounded, then $P \setminus \hat{1}$ has a minimal element $\hat{0}$ so applying Lemma 4.12 to $P \setminus \hat{1}$, we obtain

$$\sum_{x,y \in P} \mu^{a,b}(x,y) = (a+b) \left((a+b) \sum_{x,y \in (P \setminus \hat{1}) \setminus \hat{0}} \mu^{1,1}(x,y) + 1 \right) + 1$$

$$= (a+b)^2 \sum_{x,y \in P \setminus (\hat{1} \cup \hat{0})} \mu^{1,1}(x,y) + (a+b+1)$$
$$= \sum_{x,y \in P \setminus (\hat{0} \cup \hat{1})} \mu^{(a+b)^2,(a+b)^2}(x,y) + (a+b+1).$$

The last line follows from Lemma 4.3. □

Definition 4.15. [5] A *hierarchical poset* P is a set with m antichain layers $L_1, \dots, L_m$, where the elements of each layer are less than the elements in the layers above:

$$\text{if } x \in L_i \text{ and } y \in L_j, \text{ then } x \prec y \text{ if and only if } i < j\,.$$

For each $i = 1, \dots, m$, let $n_i = |L_i|$.

Theorem 4.16. *Let P be a hierarchical poset with m levels $L_1, \dots, L_m$, $n_k = |L_k|$ for each $k = 1, \dots, m$ and let $L = \{1, 2, \dots, m\}$. Then for all $a, b \in \mathbb{R}$,*

$$\sum_{x,y \in P} \mu^{a,b}(x,y) = a \sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i + b \sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i.$$

Proof. By definition of a hierarchical poset, for all partitions $A \subseteq L$, there are exactly $\prod\limits_{i \in A} n_i$ chains of cardinality $|A|$ in the form $\{x_{\ell_1}, \dots x_{\ell_{|A|}}\}$. Here, $x_i \in L_i$ and $\ell_k \in A$ for all $k = 1, \dots, |A|$ such that $\ell_i > \ell_j$ if and only if $i > j$. Hence for all $k \geq 1$

$$|C_k(P)| = \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i.$$

Then, the expression follows from the definition of the Generalised Möbius function. □

Corollary 4.17. *Let P be a hierarchical poset with m levels $L_1, \dots, L_m$ and let $n_k = |L_k|$ for each $k = 1, \dots, m$.*

$$\sum_{x,y \in P} \mu^{1,1}(x,y) = \prod_{i=1}^{m} (1 + n_i) - 1 \tag{4.3}$$

and

$$\sum_{x,y \in P} \mu^{1,-1}(x,y) = 1 - \prod_{i=1}^{m} (1 - n_i). \tag{4.4}$$

Proof. By Theorem 4.16, when $a = b = 1$, we have

$$\sum_{x,y \in P} \mu^{1,1}(x,y) = \sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i + \sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i$$
$$= \sum_{k} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i$$
$$= \prod_{i=1}^{m} (1 + n_i) - 1.$$

Next, by Theorem 4.16, when $a = 1$, and $b = -1$, we have

$$\begin{aligned}\sum_{x,y\in P}\mu^{1,-1}(x,y) &= \sum_{k\text{ odd}}\sum_{\substack{A\subseteq L\\|A|=k}}\prod_{i\in A}n_i - \sum_{k\text{ even}}\sum_{\substack{A\subseteq L\\|A|=k}}\prod_{i\in A}n_i\\ &= \sum_{\substack{A\subseteq L\\|A|=1}}\prod_{i\in A}n_i - \sum_{\substack{A\subseteq L\\|A|=2}}\prod_{i\in A}n_i + \cdots + (-1)^{m+1}\sum_{\substack{A\subseteq L\\|A|=m}}\prod_{i\in A}n_i\\ &= 1 - \prod_{i=1}^{m}(1-n_i).\end{aligned}$$

□

Theorem 4.18. *Let P be a hierarchical poset with m levels $L_1, \ldots, L_m$ and let $n_k = |L_k|$ for each $k = 1, \ldots, m$. Then for all $n \geq 1$,*

$$\sum_{x,y\in P}\mu^{1,1}(x,y) \leq 2^n - 1.$$

The poset P^* which achieves this upper bound has $m = n$ and $n_k = 1$ for each $k = 1, \ldots, m$. Note that P^* is a chain of length n and is isomorphic to the poset of n distinct integers ordered by $\leq$.

Proof. We seek to maximise $\prod_{i=1}^{m}(1+n_i) - 1$ subject to the constraint $\sum_{i=1}^{m} n_i = n$. Let $\{n_1^*, \ldots n_m^*\}$ be the integer partition of n which maximises the objective function.

Suppose for some $i = 1, \ldots, m$ that $n_i^* = 2k$, where $k \geq 1$. Then since

$$(1+k)(1+k) - 1 = 2k + k^2 > 2k = (1+2k) - 1,$$

we can always increase the product by replacing $2k$ with k and k, without changing the sum $\sum_{i=1}^{m} n_i$. Hence, $n_i^* \neq 2k$ for any $i = 1, \ldots, m$.

Next, suppose for some $i = 1, \ldots, m$ that $n_i = 2k + 1$, where $k \geq 1$. Then since

$$(1+k)(1+k+1) - 1 = k^2 + 3k + 1 > 2k + 1 = (1 + 2k + 1) - 1,$$

we can always increase the product by replacing $2k$ with k and $k+1$, without changing the sum $\sum_{i=1}^{m} n_i$. Hence, $n_i^* \neq 2k+1$ for all $i = 1, \ldots, m$ and $k \geq 0$.

Therefore, $n_i^* = 1$ for all $i = 1, \ldots, m$. It follows that $m = k$ and

$$\prod_{i=1}^{m}(1+n_i) - 1 \leq 2^n - 1.$$

□

Theorem 4.19. *For any poset P on $n \geq 1$ elements and any a, b,*

$$\sum_{x,y\in P}\mu^{a,b}(x,y) \leq \max_{\substack{\{n_i:1\leq i\leq m\}\\ n_1+\ldots+n_m=n}}\left(a\sum_{k\text{ odd}}\sum_{\substack{A\subseteq L\\|A|=k}}\prod_{i\in A}n_i + b\sum_{k\text{ even}}\sum_{\substack{A\subseteq L\\|A|=k}}\prod_{i\in A}n_i\right),$$

where $L = \{1, 2, \ldots, m\}$. Furthermore, this bound is achieved when P is a hierarchical poset with m levels $L_1, \ldots, L_m$ such that $n_k = |L_k|$ for each $k = 1, \ldots, m$.

Proof. The result is equivalent to the statement that for any poset on n elements, there exists a hierarchical poset on n elements with a greater sum of its Generalised Möbius function values.

We will prove the result by induction on the number of elements k. The result is trivially true for $k = 1$. Suppose the result is true for all $k < n$. Let P be any poset on n elements with c_0 maximal elements, the set of which is denoted by $C_0 = \max(P) = \{x_1, \dots, x_{c_0}\}$, i.e. $\mu^{a,b}(x_j, x) = 0$ for all $x \in P \setminus x_j$ where $j = 1, \dots, c_0$. Define the *best element*, $x^* \in C_0$, as the element of C_0 which maximally contributes to $\sum_{x,y \in P} \mu^{a,b}(x, y)$, i.e.

$$\sum_{x \in P} \mu^{a,b}(x, x^*) = \max_{1 \leq j \leq m} \sum_{x \in P} \mu^{a,b}(x, x_j)\,.$$

Construct P_1 from P as

$$P_1 = (P - C_0) \cup \{x_1, \dots, x_{c_0}\}$$

with partial ordering for $x, z \in P - C_0$ given by

$$x \prec_{P_1} x_j \iff x \prec_P x^*$$

and

$$x \prec_{P_1} z \iff x \prec_P z$$

so that the x_j's are incomparable maximal elements of P'. Consider the total sums involving chains with elements only from $P - C_0$, only from C_0 and those with maximal element x^* to obtain:

$$\begin{aligned}
\sum_{x,y \in P_1} \mu^{a,b}_{P_1}(x, y) &= \sum_{x,y \in P - C_0} \mu^{a,b}(x, y) + \sum_{x \in C_0} \mu^{a,b}(x, x) + c_0 \sum_{x,y \in P - C_0} \mu^{a,b}(x, x^*) \\
&\geq \sum_{x,y \in P - C_0} \mu^{a,b}(x, y) + \sum_{x \in C_0} \mu^{a,b}(x, x) + \sum_{i=1}^{c_0} \sum_{x,y \in P - C_0} \mu^{a,b}(x, x_i) \\
&= \sum_{x,y \in P} \mu^{a,b}_P(x, y).
\end{aligned}$$

We can repeat this algorithm on the poset P_1 by letting $C_1 = \max(P_1)$ and so on, resulting in an ordering of posets where

$$\sum_{x,y \in P} \mu^{a,b}(x, y) \leq \sum_{x,y \in P_1} \mu^{a,b}(x, y) \leq \sum_{x,y \in P_2} \mu^{a,b}(x, y) \leq \cdots\,.$$

Note that $C_0 \subseteq C_1 \subseteq \dots$ as all maximal elements of P_i remain maximal elements of P_{i+1}, but C_{i+1} may contain elements not in C_i. However, as the number of maximal elements cannot exceed the size n of the poset, this procedure has to stop after a finite number of steps, i.e., when $C_{i+1} = C_i$. At this point, C_i resembles one anti-chain layer of a hierarchical poset, since every element of C_i has the same partial ordering. Then, the algorithm is repeated on $P_i - C_i$ to produce the next layer of a hierarchical poset and so on.

It is sufficient to prove that at the last step, say constructing P_1 from P as above, where $C_1 = C_0$, that C_0 resembles one anti-chain layer of a hierarchical poset (that is, being comparable to every element below it).

So, we will prove that if $C_1 = C_0$, then $y \prec x$ for all $y \in P - C_0$ and all $x \in C_0$.

Suppose that for some $j^* = 1, \dots, c_0$ that there exists a $y \in P - C_0$ such that $y \not\prec x_{j^*}$. Since $y \notin C_0$, it is not a maximal element of P, so it must be comparable to at least one element of C_0. Hence, there exists a y'

such that $y \preceq y' \prec x$ for some $x \in C_0$. Let X the the set of all such x such that $y' \prec x$. If $x^* \notin X$, then y' will become a maximal element of P_1, i.e. $y' \in C_1$. However, as $C_1 = C_0$, this is a contradiction. If $x^* \in X$, then since $y' \prec x^*$, after applying the compression algorithm to P, we will have that $y' \prec x$ for all $x \in C_1$. However as $C_1 = C_0$, and this argument can be applied to all such y, this contradicts the initial assumption that every element of C_0 is not greater than every element not in $P - C_0$. Hence, being greater than every element below it, C_0 resembles the topmost layer of a hierarchical poset, and repeating the algorithm on $P - C_0$ and so on, will produce a hierarchical poset.

To summarise, on each repetition, we apply the compression algorithm to the poset P, iterating the algorithm until we've peeled off a subset of the poset and replaced it with one "hierarchical layer", say C, of the same size. We repeat the algorithm on $P - C$, so that after q repetitions, we will have produced a hierarchical poset with q levels.

Hence, it is left to prove that after each repetition, we have produced a poset whose sum of generalised Möbius function values is less than that of some hierarchical poset of the same size. Following the previous notation, it is required to prove that

$$\sum_{x,y\in P_1} \mu^{a,b}(x,y) \leq a \sum_{k \text{ odd}} \sum_{\substack{A\subseteq L\\ |A|=k}} \prod_{i\in A} n_i + b \sum_{k \text{ even}} \sum_{\substack{A\subseteq L\\ |A|=k}} \prod_{i\in A} n_i$$

for some partition $\sum_{i=1}^{m} n_i = n$.

Without loss of generality, let C_0 be the final maximal layer of P after a sufficient number of iterations of the compression algorithm, i.e. when $C_1 = C_0$. From before, we have that for all $x \in C_0$ and all $y \in P - C_0$, that $y \prec x$. Hence,

$$\begin{aligned}\sum_{x,y\in P_1} \mu^{a,b}(x,y) &= \sum_{x,y\in P-C_0} \mu^{a,b}(x,y) + \sum_{x\in C_0} \mu^{a,b}(x,x) + c_0 \sum_{x,y\in P-C_0} \mu^{a,b}(x,x^*)\\ &= \sum_{x,y\in P-C_0} \mu^{a,b}(x,y) + ac_0 + c_0 \sum_{x,y\in P-C_0} \mu^{b,a}(x,y).\end{aligned}$$

The last line follows from the fact that $\mu^{a,b}(x,x) = a$ and every chain (of length greater than one) with maximal element x^* is in one-to-one correspondence with a chain in $P - C_0$, but with different parity. Simplifying, we get

$$\begin{aligned}\sum_{x,y\in P_1} \mu^{a,b}(x,y) &= \sum_{x,y\in P-C_0} \left(\mu^{a,b}(x,y) + c_0\mu^{b,a}(x,y)\right) + ac_0\\ &= \sum_{x,y\in P-C_0} \mu^{a+bc_0,b+ac_0}(x,y) + ac_0\end{aligned}$$

by Lemma 4.2 and Lemma 4.3. By the induction hypothesis on $P - C_0$, there exists a partition $\sum_{i=1}^{m'} n'_i = n - c_0$ such that

$$\sum_{x,y\in P-C_0} \mu^{a+bc_0,b+ac_0}(x,y) \leq (a+bc_0) \sum_{k \text{ odd}} \sum_{\substack{A\subseteq L\\ |A|=k}} \prod_{i\in A} n'_i + (b+ac_0) \sum_{k \text{ even}} \sum_{\substack{A\subseteq L\\ |A|=k}} \prod_{i\in A} n'_i + ac_0,$$

where $L = \{1, \dots, m'\}$. Let $n_i = n'_i$ for all $i = 1, ..., m'$ and $n_m = c_0$ so that $m = m' + 1$ and $\sum_{i=1}^{m} n_i = n$. Note that

$$\sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i = \sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i - c_0 \left(1 + \sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i \right)$$

and

$$\sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i = \sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i - c_0 \left(\sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i \right).$$

Therefore,

$$\begin{aligned} \sum_{x,y \in P_1} \mu^{a,b}(x,y) \leq (a + bc_0) & \left[\sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i - c_0 \left(1 + \sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i \right) \right] \\ & + (b + ac_0) \left[\sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i - c_0 \sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i \right] + ac_0. \end{aligned}$$

Simplifying, we get

$$\begin{aligned} \sum_{x,y \in P_1} \mu^{a,b}(x,y) \leq (a + bc_0) & \sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i + (b + ac_0) \sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n_i \\ & - c_0 \left[(b + ac_0) \sum_{k \text{ odd}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i + (a + bc_0) \sum_{k \text{ even}} \sum_{\substack{A \subseteq L \\ |A|=k}} \prod_{i \in A} n'_i \right] - bc_0^2. \end{aligned}$$

Hence,

$$\sum_{x,y \in P_1} \mu^{a,b}(x,y) \leq \sum_{x,y \in HP} \mu_{HP}^{a+bc_0, b+ac_0}(x,y) - c_0 \sum_{x,y \in HP'} \mu_{HP'}^{b+ac_0, a+bc_0}(x,y) - bc_0^2,$$

where HP is the hierarchical poset with m levels of size $n_1, \dots n_m$ and HP' is the hierarchical poset with m' levels of size $n'_1, \dots n'_{m'}$. Simplifying, we get

$$\begin{aligned} \sum_{x,y \in P_1} \mu^{a,b}(x,y) \leq \sum_{x,y \in HP} \mu_{HP}^{a,b}(x,y) + c_0 & \left(\sum_{x,y \in HP} \mu_{HP}^{b,a}(x,y) - c_0 \sum_{x,y \in HP'} \mu_{HP'}^{a,b}(x,y) \right) \\ - c_0 & \sum_{x,y \in HP'} \mu_{HP'}^{b,a}(x,y) - bc_0^2. \end{aligned}$$

Noting that

$$\sum_{x,y \in HP} \mu_{HP}^{b,a}(x,y) = \sum_{x,y \in HP'} \mu_{HP'}^{b,a}(x,y) + bc_0 + c_0 \sum_{x,y \in HP} \mu_{HP'}^{a,b}(x,y),$$

we then have

$$\sum_{x,y \in P_1} \mu^{a,b}(x,y) \leq \sum_{x,y \in HP} \mu_{HP}^{a,b}(x,y) + c_0 \left(\sum_{x,y \in HP'} \mu_{HP'}^{b,a}(x,y) + bc_0 \right)$$

$$- c_0 \sum_{x,y \in HP'} \mu^{b,a}_{HP'}(x,y) - bc_0^2$$
$$= \sum_{x,y \in HP} \mu^{a,b}_{HP}(x,y),$$

and we are done. □

Theorem 4.20. *Let P be a poset on n elements. For any a, b such that $a, b \geq 0$,*

$$\sum_{x,y \in P} \mu^{a,b}(x,y) \leq (a+b)2^{n-1} - b.$$

Furthermore, the upper bound is achieved when P is a chain on n elements.

Proof. Suppose P is a hierarchical poset with n elements and m layers with respective sizes n_i for $i = 1, \ldots, m$ where $n_m = k$ and $n_{m-1} = 0$ for some $k \geq 2$. Furthermore, suppose that P' is a hierarchical poset with m layers with respective sizes n_i' for $i = 1, \ldots, m$ where $n_i' = n_i$ for $i = 1, \ldots, m-2$ and $n_{m-1} = 1$ and $n_m = k-1$.

Let $n = \sum_{i=1}^{n}$ so that P and P' both have n elements, with $m-2$ identical layers.

Note that setting a layer to size zero does not effect the total sum of its Generalised Möbius function values, hence setting $n_{m-1} = 0$ for poset P is chosen for convenience.

By Lemma 4.6 and Corollary 4.17, for poset P, we have

$$\begin{aligned}
\sum_{x,y \in P} \mu^{a,b}(x,y) &= \frac{a-b}{2}\left(1 - \prod_{i=1}^{m}(1-n_i)\right) + \frac{a+b}{2}\left(\prod_{i=1}^{m}(1+n_i) - 1\right) \\
&= -b - \frac{a-b}{2}(1-k)\prod_{i=1}^{m-2}(1-n_i) + \frac{a+b}{2}(1+k)\prod_{i=1}^{m-2}(1+n_i).
\end{aligned}$$

Furthermore, for poset P',

$$\begin{aligned}
\sum_{x,y \in P'} \mu^{a,b}(x,y) &= \frac{a-b}{2}\left(1 - 0(1-k)\prod_{i=1}^{m-2}(1-n_i)\right) + \frac{a+b}{2}\left((1+1)(1+k)\prod_{i=1}^{m}(1+n_i) - 1\right) \\
&= -b + \frac{a+b}{2}(2k)\prod_{i=1}^{m-2}(1+n_i) \\
&= -b + \frac{a+b}{2}(1+k)\prod_{i=1}^{m-2}(1+n_i) - \frac{a+b}{2}(1-k)\prod_{i=1}^{m-2}(1+n_i).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{x,y \in P'} \mu^{a,b}(x,y) - \sum_{x,y \in P} \mu^{a,b}(x,y) &= \frac{a+b}{2}(k-1)\prod_{i=1}^{m-2}(1+n_i) - \frac{a-b}{2}(k-1)\prod_{i=1}^{m-2}(1-n_i) \\
&= \frac{a-b}{2}(k-1)\prod_{i=1}^{m-2}(1+n_i) - \frac{a-b}{2}(k-1)\prod_{i=1}^{m-2}(1-n_i) + b(k-1)\prod_{i=1}^{m-2}(1+n_i) \\
&= \frac{a-b}{2}(k-1)\left(\prod_{i=1}^{m-2}(1+n_i) - \prod_{i=1}^{m-2}(1-n_i)\right) + b(k-1)\prod_{i=1}^{m-2}(1+n_i).
\end{aligned}$$

Note that

$$\prod_{i=1}^{m-2}(1+n_i)-\prod_{i=1}^{m-2}(1-n_i)=2\sum_{k\text{ odd}}\sum_{\substack{A\subseteq L\\|A|=k}}\prod_{i\in A}n_i\geq 0,$$

where $L=\{1,\ldots,m-2\}$. Hence, if $a\geq b\geq 0$, then

$$\sum_{x,y\in P'}\mu^{a,b}(x,y)-\sum_{x,y\in P}\mu^{a,b}(x,y)>0\Rightarrow\sum_{x,y\in P'}\mu^{a,b}(x,y)>\sum_{x,y\in P}\mu^{a,b}(x,y).$$

On the other hand, if we write $\sum_{x,y\in P'}\mu^{a,b}(x,y)-\sum_{x,y\in P}\mu^{a,b}(x,y)$ in a different way, we can see that

$$\begin{aligned}\sum_{x,y\in P'}\mu^{a,b}(x,y)-\sum_{x,y\in P}\mu^{a,b}(x,y)&=\frac{b+a}{2}(k-1)\prod_{i=1}^{m-2}(1+n_i)+\frac{b-a}{2}(k-1)\prod_{i=1}^{m-2}(1-n_i)\\&=\frac{b-a}{2}(k-1)\prod_{i=1}^{m-2}(1+n_i)+\frac{b-a}{2}(k-1)\prod_{i=1}^{m-2}(1-n_i)+a(k-1)\prod_{i=1}^{m-2}(1+n_i)\\&=\frac{b-a}{2}(k-1)\left(\prod_{i=1}^{m-2}(1+n_i)+\prod_{i=1}^{m-2}(1-n_i)\right)+a(k-1)\prod_{i=1}^{m-2}(1+n_i).\end{aligned}$$

Note that

$$\prod_{i=1}^{m-2}(1+n_i)+\prod_{i=1}^{m-2}(1-n_i)=2\left(1+\sum_{k\text{ even}}\sum_{\substack{A\subseteq L\\|A|=k}}\prod_{i\in A}n_i\right)\geq 0,$$

where $L=\{1,\ldots,m-2\}$. Hence, if $b\geq a\geq 0$, then

$$\sum_{x,y\in P'}\mu^{a,b}(x,y)-\sum_{x,y\in P}\mu^{a,b}(x,y)>0\Rightarrow\sum_{x,y\in P'}\mu^{a,b}(x,y)>\sum_{x,y\in P}\mu^{a,b}(x,y).$$

Therefore, for $a,b\geq 0$, then for any hierarchical poset, we can always replace a layer of length $k\geq 2$ with 2 layers of length 1 and $k-1$, without changing the number of elements but increasing the sum of the poset's Generalised Möbius function values. Since the ordering of layers in a hierarchical poset does not change this sum, we can repeatedly replace layers in this manner until we have maximised the sum. The hierarchical poset with such a maximal sum is one with all layers having length 1. Hence, by Theorem 4.19, for any poset P on n elements and any $a,b\geq 0$,

$$\begin{aligned}\sum_{x,y\in P}\mu^{a,b}(x,y)&\leq\frac{a-b}{2}\left(1-\prod_{i=1}^{m}(1-1)\right)+\frac{a+b}{2}\left(\prod_{i=1}^{m}(1+1)-1\right)\\&=(a+b)2^{n-1}-b.\end{aligned}$$

□

Theorem 4.21. *Let P be a poset on n elements. For any a,b such that $ab\geq 0$,*

$$\left|\sum_{x,y\in P}\mu^{a,b}(x,y)\right|\leq\left|(a+b)2^{n-1}-b\right|.$$

Proof. If $a, b \geq 0$, the result follows from Theorem 4.20.

Alternatively, if $a, b \leq 0$, then

$$\begin{aligned}\left|\sum_{x,y\in P} \mu^{a,b}(x,y)\right| &= \left|-\sum_{x,y\in P} \mu^{-a,-b}(x,y)\right| \\ &= \sum_{x,y\in P} \mu^{-a,-b}(x,y).\end{aligned}$$

Since $-a \geq -b \geq 0$, then by Theorem 4.20,

$$\begin{aligned}\sum_{x,y\in P} \mu^{-a,-b}(x,y) &\leq \left(\frac{-a-b}{2}\right) 2^n + b \\ &= \left|(a+b)2^{n-1} - b\right|.\end{aligned}$$

□

Conjecture 4.22. *Let P be a poset on n elements. If $a + b \neq 0$, then*

$$\left|\sum_{x,y\in P} \mu^{a,b}(x,y)\right| \leq \left|(a+b)2^{n-1} - b\right|.$$

The bound is achieved when P is a chain on n elements. If $a + b = 0$ and $n \geq 17$, then

$$\left|\sum_{x,y\in P} \mu^{a,b}(x,y)\right| \leq |a|(3^k 4^\ell - (-1)^{k+l}),$$

where $k = 5\lceil \frac{n}{5} \rceil - n$ and $\ell = n - 4\lceil \frac{n}{5} \rceil$. The bound is achieved when P is a hierarchical poset with $m = k + \ell$ levels whose sizes $n_1 = |L_1|, \ldots, n_k = |L_m|$ satisfy

$$\{n_1, \ldots, n_m\} = \{\underbrace{4, \ldots, 4}_{k}, \underbrace{5, \ldots, 5}_{\ell}\}.$$

Proof. All that is remaining to show is the case when $ab < 0$ and $a + b \neq 0$. □

4.3 Generalisation of the Chain Poset Theorem

Let P be a poset on n elements and $\mathrm{Ch}(P)$ be the corresponding poset whose elements are the chains of P ordered by subset inclusion.

Lemma 4.23. *Let $C \in Ch(P)$ and $a = -b$. The column sums of the generalised Möbius function is*

$$\sum_{C'\in Ch(P)} \mu^{a,b}(C',C) = a(-1)^{|C|+1}. \tag{4.5}$$

Proof. We will first prove by strong induction on $|C|$ for the case when $a = 1$ and $b = -1$. When $|C| = 1$, then

$$\sum_{C'\in \mathrm{Ch}(P)} \mu^{1,-1}(C',C) = \mu^{1,-1}(C,C) = 1 = 1(-1)^{1+1}.$$

Hence, the claim is true when $|C| = 1$. Now, assume that the claim is true for all integers $1 \le |C| \le k-1$ and suppose that $|C| = k$. Note that

$$\begin{aligned}\sum_{C' \in \mathrm{Ch}(P)} \mu^{1,-1}\mu(C', C) &= \sum_{C' \subseteq C} \mu^{1,-1}\mu(C', C) \\ &= \mu^{1,-1}\mu(C, C) + \sum_{\emptyset \neq C' \subset C} \mu^{1,-1}\mu(C', C) \\ &= 1 - \sum_{\emptyset \neq C' \subset C} \sum_{C'' \in \mathrm{Ch}(P)} \mu^{1,-1}\mu(C'', C')\end{aligned}$$

since every such even/odd chain in $\mathcal{C}(C', C)$ is in one-to-one correspondence with every such odd/even chain in $\mathcal{C}(C'', C')$. Since $C' \subseteq C \Rightarrow |C'| \le |C|$, then by the inductive hypothesis,

$$\begin{aligned}\sum_{C' \in \mathrm{Ch}(P)} \mu^{1,-1}(C', C) &= 1 - \sum_{\emptyset \neq C' \subset C} (-1)^{|C'|+1} \\ &= 1 - \sum_{k=1}^{|C|-1} \binom{|C|}{k} (-1)^k \\ &= 1 - (1-1)^{|C|} - (-1)^0 - (-1)^{|C|} \\ &= (-1)^{|C|+1}.\end{aligned}$$

Therefore, for all $|C| \ge 1$, $\sum\limits_{C' \in \mathrm{Ch}(P)} \mu^{1,-1}(C', C) = (-1)^{|C|+1}$.

Hence, for any a, b such that $a = -b$, it follows from Lemma 4.3 that

$$\sum_{C' \in \mathrm{Ch}(P)} \mu^{a,-a}(C', C) = a \sum_{C' \in \mathrm{Ch}(P)} \mu^{1,-1}(C', C) = a(-1)^{|C|+1}.$$

□

Theorem 4.24. *(Generalised Chain-Poset Theorem)*
For any poset P, and any a, b such that $a = -b$,

$$\sum_{C', C \in Ch(P)} \mu^{a,b}(C', C) = \sum_{x,y \in P} \mu^{a,b}(x, y).$$

Proof. By Lemma 4.23,

$$\begin{aligned}\sum_{C', C \in Ch(P)} \mu^{a,-a}(C', C) &= \sum_{C \in \mathrm{Ch}(P)} \sum_{C' \in \mathrm{Ch}(P)} \mu^{a,-a}(C', C) \\ &= \sum_{C \in \mathrm{Ch}(P)} a(-1)^{|C|+1} \\ &= \sum_{C', C \in P} \mu^{a,-a}(C', C).\end{aligned}$$

□

Theorem 4.25. *For any poset P and any a, b,*

$$\sum_{C', C \in Ch(P)} \mu^{a,b}(C', C) = \sum_{x,y \in P} \mu^{a,b}(x, y) + \frac{a+b}{2} \sum_{C \in Ch(P)} (T(C) - 1),$$

where

$$T(C) = \sum_{C' \in Ch(P)} \mu^{1,1}(C', C) = \sum_{1 \leq n_m < \cdots < n_1 \leq |C|} \binom{|C|}{n_1} \prod_{i=1}^{m} \binom{n_i}{n_{i+1}}.$$

Furthermore,

$$\sum_{C', C \in Ch(P)} \mu^{a,b}(C', C) = \sum_{x,y \in P} \mu^{a,b}(x, y)$$

if and only if $a = -b$ or P is an antichain on n elements.

Proof. By Lemma 4.6,

$$\begin{aligned}\sum_{C', C \in \mathrm{Ch}(P)} \mu^{a,b}(C', C) &= \frac{a-b}{2} \sum_{C', C \in \mathrm{Ch}(P)} \mu^{1,-1}(C', C) + \frac{a+b}{2} \sum_{C', C \in \mathrm{Ch}(P)} \mu^{1,1}(C', C) \\ &= \frac{a-b}{2} \sum_{C \in \mathrm{Ch}(P)} (-1)^{|C|} + \frac{a+b}{2} \sum_{C \in \mathrm{Ch}(P)} T(C),\end{aligned}$$

where $T(C) = \sum_{C' \in \mathrm{Ch}(P)} \mu^{1,1}(C', C)$. Note that since the elements of $\mathrm{Ch}(P)$ are ordered by inclusion, we can count the total number of chains $T(C)$ by finding subsets of C with cardinalities $n_1, \ldots n_m$ such that the n_i's are decreasing with i. That is,

$$T(C) = \sum_{1 \leq n_m < \ldots < n_1 \leq |C|} \binom{|C|}{n_1} \prod_{i=1}^{m} \binom{n_i}{n_{i+1}}.$$

On the other hand, for poset P, we have

$$\begin{aligned}\sum_{x,y \in P} \mu^{a,b}(x, y) &= \frac{a-b}{2} \sum_{x,y \in P} \mu^{1,-1}(x, y) + \frac{a+b}{2} \sum_{x,y \in P} \mu^{1,1}(x, y) \\ &= \frac{a-b}{2} \sum_{C \in \mathrm{Ch}(P)} (-1)^{|C|} + \frac{a+b}{2} \sum_{C \in \mathrm{Ch}(P)} 1.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{C', C \in \mathrm{Ch}(P)} \mu^{a,b}(C', C) &= \sum_{x,y \in P} \mu^{a,b}(x, y) - \frac{a+b}{2} \sum_{C \in \mathrm{Ch}(P)} 1 + \frac{a+b}{2} \sum_{C \in \mathrm{Ch}(P)} T(C) \\ &= \sum_{x,y \in P} \mu^{a,b}(x, y) + \frac{a+b}{2} \sum_{C \in \mathrm{Ch}(P)} (T(C) - 1).\end{aligned}$$

It follows that $\sum_{C', C \in \mathrm{Ch}(P)} \mu^{a,b}(C', C) = \sum_{x,y \in P} \mu^{a,b}(x, y)$ if and only if $a = -b$ or P is an antichain on n elements ($T(C) = 1$ for all $C \in \mathrm{Ch}(P)$). □

4.4 The Compression Algorithm

The "compression" algorithm used in the proof of Theorem 4.19 replaces a subset of a poset P with an antichain layer of the same length so that it resembles a layer of a hierarchical poset. The procedure is summarised below:

1. Let $C_0 = \max(P)$ be the maximal elements of P;

2. choose $x^* \in C_0$ such that x^* maximises $\sum_{x \in P} \mu^{a,b}(x, x_1)$ for all $x_1 \in C_0$;

3. construct a new poset P_1 from P as

$$P_1 = (P - C_0) \cup \{y_1, \dots, y_m\}$$

with partial ordering for $x \in P - M$ given by

$$x \prec_{P_1} y_j \iff x \prec_P x^*$$

for all $j = 1, \dots, m$.

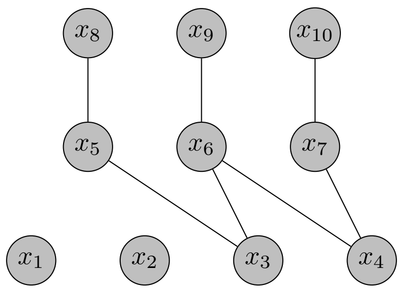
4. for $i = 1, \dots, k$, let $C_i = \max(P_i)$, where k is the minimal number of steps such that $C_{k+1} = C_k$. Repeat steps 2 and 3 k times; that is, until the maximal elements are unchanged.

5. let $P := P - C_k$. Repeat steps 1-4 until all the C_k's resemble the layers of a hierarchical poset.

We will now illustrate examples of using the "compression" algorithm to transform a poset into one with a greater total sum of its Generalised Möbius function values, while maintaining the same number of elements. Moreover, we will see that the final poset is in the class of hierarchical posets.

Example 4.26. In this example, we will maximise $\sum_{x,y \in P} \mu^{2,3}(x, y)$ where P is a poset on the set ordered as $\{x_1, x_2, \dots, x_{10}\}$ with the following Hasse diagram, Generalised Möbius function $\mu^{2,3}$ and Generalised Zeta matrix $\zeta^{2,3}$:

Hasse diagram for P



Generalised Möbius function $\mu^{2,3}$:

$$\mu^{2,3} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & 3 & 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 2 & 0 & 3 & 3 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Generalised Zeta matrix $\zeta^{2,3}$:

$$\zeta^{2,3} = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & -0.75 & -0.75 & 0 & -0.125 & -0.125 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & -0.75 & -0.75 & 0 & -0.125 & -0.125 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & -0.75 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}$$

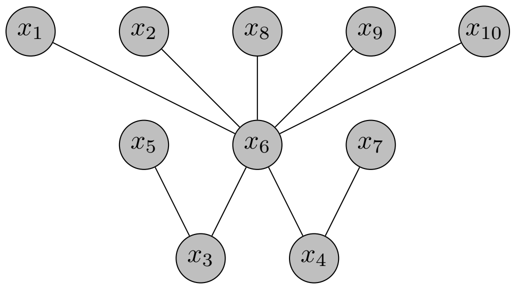
The sum of $\mu^{2,3}$ values is 61. The maximal elements of P are $C_0 = \{x_1, x_2, x_8, x_9, x_{10}\}$. From the Generalised

Möbius function $\mu^{2,3}$, we can see that the columns sums for elements in M are as follows:

$$\sum_{x \in P} \mu(x, x_1) = 2; \qquad \sum_{x \in P} \mu(x, x_2) = 2; \qquad \sum_{x \in P} \mu(x, x_8) = 10;$$
$$\sum_{x \in P} \mu(x, x_9) = 15; \qquad \sum_{x \in P} \mu(x, x_{10}) = 10.$$

Hence, the element of C_0 that maximally contributes to the sum is x_9. Hence, we let $x^* = x_9$ and construct P_1 so that it has the following Hasse diagram:

Hasse diagram for P_1

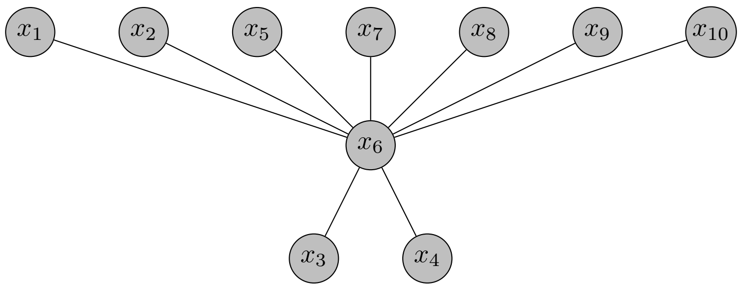


Now, P_1 has total Möbius sum 97 and maximal elements $C_1 = \{x_1, x_2, x_5, x_7, x_8, x_9, x_{10}\}$. We have the column sums

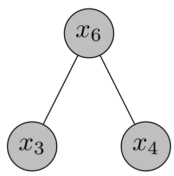
$$\sum_{x \in P} \mu(x, x_1) = 15; \quad \sum_{x \in P} \mu(x, x_2) = 15; \quad \sum_{x \in P} \mu(x, x_5) = 5; \quad \sum_{x \in P} \mu(x, x_7) = 5;$$
$$\sum_{x \in P} \mu(x, x_8) = 15; \quad \sum_{x \in P} \mu(x, x_9) = 15; \quad \sum_{x \in P} \mu(x, x_{10}) = 15.$$

So we can let $x^* = x_1$. Then the transformed poset P_2 has the following Hasse diagram:

Hasse diagram for P_2



The total sum of Generalised Möbius functions for P_2 is 117. The maximal elements of P_2 are $C_2 = \{x_1, x_2, x_5, x_7, x_8, x_9, x_{10}\} = C_1$. Since the maximal elements are unchanged, we have successfully created one layer of a hierarchical poset. Now, we will repeat the algorithm on $P - C_2$. Let $P := P - C_2$ for convenience of notation. P has the following Hasse diagram:



The total sum of Generalised Möbius functions for P is 12. The maximal elements of P are $C_0 = \{x_6\}$ so we have nothing to do. Repeating the algorithm on $P - C_0$, we will notice that the maximal elements of $P - C_0$ are $\{x_3, x_4\}$ and they have the same partial ordering. Hence, the compression algorithm will not change the poset P_2 any further. Therefore, we have successfully found a hierarchical poset P_2 (with layers of sizes 2,1 and 7), whose sum of Generalised Möbius function values is greater than that of the original poset P.

5 Future Work

Future work involves completing the proof of Conjecture 4.22 for the upper bound on $\left|\sum_{x,y\in P}\mu^{a,b}(x,y)\right|$ for all a, b. We may also group the cardinality of chains by modulus and hence, further generalise the Möbius function and investigate properties of the sums of its values. In this sense, the Generalised Möbius function studied in this report is the case when the modulus is chosen to be 2.

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