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Geodesic Growth in a Virtually Heisenberg Group

Aden Northcote

Supervised by Murray Elder

University of Technology Sydney

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Abstract

The growth function of a group G measures how quickly the ball of radius n grows with respect to some generating set $S = \{s_1, s_2, \ldots, s_k\}$, where each group element can be expressed as a word constructed of generators and their inverses in S with maximum length of n. The *geodesic growth* function for a finitely generated group with respect to a generating set S measures the number of words with the minimum length required to represent the corresponding group element. Using previous results proving a polynomial upper bound on the geodesic growth function of a virtually 2-step nilpotent group, we are able to prove a lower bound for the same group.

1 Prelude

1.1 Acknowledgements

I would like to thank Murray Elder for the ideas and guidance he provided over the course of this research project, and for the interest in mathematics he fostered before it.

2 Introduction

The growth function of a group G measures how quickly the ball of radius n grows with respect to some generating set $S = \{s_1, s_2, \ldots, s_k\}$, where each group element can be expressed as a word constructed of generators and their inverses in S with maximum length of n. The *geodesic growth* function for a finitely generated group with respect to a generating set S measures the number of words with the minimum length required to represent the corresponding group element.

Gromov's 1981 theorem on groups of polynomial growth proved that any finitely generated group of polynomial growth has a finite index nilpotent subgroup. A direct consequence of this is that if a group has polynomial geodesic growth with respect to some finite generating set then that group is virtually nilpotent. Previous work (Bridson et al. 2012) has shown that any strictly nilpotent group that is not virtually cyclic has exponential geodesic growth. These results were extended by Bishop who proved that for virtually abelian groups have either polynomial or exponential geodesic growth with regards to any generating set (Bishop 2021). Building on these findings (Bishop and Elder 2020) constructed an example of a virtually 2-step nilpotent group with polynomial geodesic growth with respect to a certain finite generating set .

By using the work of (Blachère 2003), Bishop and Elder were able to prove a polynomial upper bound on the geodesic growth rate of the constructed group. Here, these results are extended to provide a lower bound on the group's geodesic growth function.

2.1 Statement of Authorship

The main goals of the project were suggested and supervised by Elder. Experimental code capable of producing explicit geodesic words was written by Northcote, and was extended from previous work by Bishop (Bishop

2020). The content of this report was written by Northcote with support from Elder, and is built conceptually upon many different sources and texts, with a full list provided in the bibliography. This project was funded by AMSI.

3 Group Theory

Many of these concepts and definitions are adapted from the work of (Lauritzen 2003), with further ideas regarding lower central series and nilpotent groups adapted from (Coxeter and Moser 2013; Lang 2012).

3.1 Definition of a Group

A *group* is a pair $(G, \circ)$ consisting of a set G and a composition $\circ : G \times G \to G$, where *associative*,

$$\forall s \in G \mid s_1 \circ (s_2 \circ s_3) = (s_1 \circ s_2) \circ s_3,$$

where there exists an *identity* element $e \in G$,

$$\forall s \in G \; \exists e \in G \mid e \circ s = s \circ e = s,,$$

and where for every element $s \in G$ there exists an *inverse* element s^{-1},

$$\forall s \in G \; \exists s^{-1} \in G \mid s \circ s^{-1} = s^{-1} \circ s = e.$$

Note that if

$$\forall x, y \in G \mid x \circ y = y \circ x,$$

then the group G is called *abelian*.

3.2 Generating Sets

A *generating set* X for a group G is a subset $X \subseteq G$ such that every element $g \in G$ can be finitely expressed as a combination of elements $w_i \in X^*$ and their inverses. If G is generated by X, we write $G = \langle X \rangle$.

3.3 Words Over a Generating Set and Geodesic Representatives

Each product of elements $w = w_1 w_2 \ldots w_k$ where w_i or $w_i^{-1} \in X^*$ is called a *word* over the generating set X. The group element corresponding to the word w is written $\overline{w} \in G$ and we denote the *word length* of w by $|w|_X = k$. For each element $g \in G$ we write $\ell_X(g) = \min\{|w|_X \mid \overline{w} = g\}$ for the *length* of an element with respect to the generating set X. A word $w \in X$ is a *geodesic* if $\ell_x(\overline{w}) = |w|_X$.

3.4 Geodesics and the Geodesic Growth Function

A word $w \in X$ is *geodesic* if

$$\ell_x(\overline{w}) = |w|_X.$$

Further, the *geodesic growth function*, $\gamma_X : \mathbb{N} \to \mathbb{N}$ of a group G with respect to the generating set X is defined by

$$\gamma_X(n) = \{w \in X^* \mid \ell_X(\overline{w}) = |w|_X \leq n\}.$$

3.5 Subgroups

A *subgroup* of a group G is a non-empty subset $H \subseteq G$ where the composition of G forms a group with the subset H, i.e., where the pair $(H, \circ)$ form a group by the definition given in section 3.1.

3.6 Normal Subgroups

A subgroup N of a group G is called *normal* if

$$\forall g \in G \mid gNg^{-1} = \left\{gng^{-1} \mid n \in N\right\} = N.$$

We denote the normal subgroup N of a group G by

$$N \trianglelefteq G.$$

3.7 Normal Closure

The *normal closure* of a subset S of a group G is the subgroup generated by the set of all elements of form $g^{-1}sg$, where $g \in G$ and $s \in S$.

3.8 Group Presentations

A *presentation* is a set of group generators S and a set of relations $R \subseteq S \cup S^{-1}$ among the generators that completely describe a group G, denoted by

$$G = \langle S \mid R\rangle$$

More formally, a group isomorphic to G can be construct from (S, R) as follows

1. Let F_s be the free group with free basis S.
2. Let $\langle\langle R\rangle\rangle$ denote the normal closure of the set R.
3. Then $G \cong F_s/\langle\langle R\rangle\rangle$

3.9 Lower Central Series and Nilpotent Groups

The *lower central series* of a group G is constructed by defining $G_1 = G$, and letting $G_{i+1} = [G_i, G]$. A *nilpotent group* is a group G whose lower central series terminates in the trivial group $\{1\}$ after finitely many steps.

$$G = G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_s \trianglelefteq G_{s+1} = \{1\}.$$

The *nilpotency class* of G is the number of steps s taken for the lower central series of G to resolve to the trivial group $G_{s+1} = \{1\}$. G is also said to be *s-step nilpotent*.

4 Discrete Heisenberg Group $\mathscr{H}$

4.1 Definition

The *discrete Heisenberg group* $\mathscr{H}$ is the group of upper-triangular 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

where $x, y, z \in \mathbb{Z}$. Further, define

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can verify that $[a, b] = c$, where $[a, b]$ denotes the *commutator* $aba^{-1}b^{-1}$, thus showing that

$$\mathscr{H}(\mathbb{Z}) = \langle a, b, c \rangle = \langle a, b \rangle .$$

Further, the discrete Heisenberg group has the standard presentation

$$\mathscr{H} = \langle a, b \mid [a, [a, b]] = 1, [b, [a, b]] = 1 \rangle .$$

This presentation can also be represented visually through a Cayley graph as shown in figure 4.1 Observe that every element $g \in \mathscr{H}$ corresponds to a unique normal form word $[a, b]^z b^y a^x$, which following the convention of Blachère can be written by $(x, y, z) \in \mathscr{H}$ (Blachère 2003).

4.2 Word Length in $\mathscr{H}$

Word lengths in the discrete Heisenberg group have been detailed explicitly in the work of Blachère, and form the basis of the proofs found both here and in the work of Bishop and Elder (Bishop and Elder 2020; Blachère 2003).

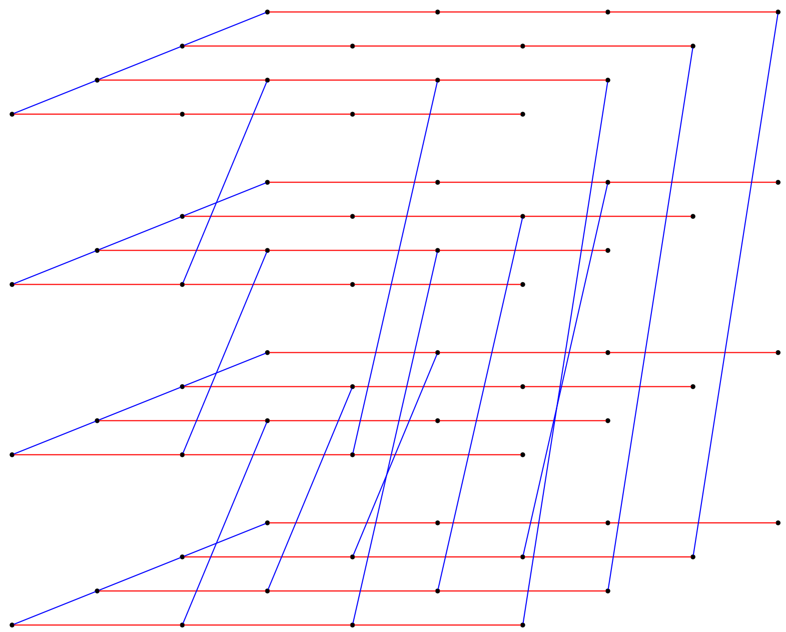

Figure 1: Cayley graph for $\mathscr{H}$, with respect to generating set $S = \{a, a^{-1}, b, b^{-1}\}$. The generators a, a^{-1} are shown by the red edges, and similarly the generators b, b^{-1} are shown by the blue.

5 Virtually Heisenberg Group $v\mathscr{H}$

5.1 Definition

Here we follow the work of Bishop and Elder in their construction of a virtually Heisenberg group $v\mathscr{H}$, with presentation

$$v\mathscr{H} = \left\langle a, b, t \,\middle|\, [a, [a, b]] = 1, [b, [a, b]] = 1, t^2 = 1, tat = b \right\rangle. \tag{1}$$

From the above presentation, we see that

$$v\mathscr{H} = \langle a, b, t\rangle = \langle a, t\rangle$$

since the element b may be generated through $tat = b$. A Tietze transform is applied to remove the generator b, yielding the following presentation

$$v\mathscr{H} = \left\langle a, t \,\middle|\, [a, [a, tat]] = 1, [tat, [a, tat]] = 1, t^2 = 1 \right\rangle. \tag{2}$$

5.2 Upper Bound of Geodesic Growth

Bishop and Elder previously found that if a word w contains > 7 t's, then w is not geodesic. Thus any geodesic of $v\mathscr{H}$ with respect to the generating set $S = \{a, a^{-1}, t\}$ must have the form

$$w = a^{m_1} t a^{m_2} t \dots t a^{m_{k+1}},\ k \le 7, m_i \in \mathbb{Z}.$$

As a result of these findings, a degree 8 polynomial upper bound was demonstrated for the geodesic growth function in $v\mathscr{H}$.

5.3 Lower Bound of Geodesic Growth

In finding a lower bound for the geodesic growth function, one can try to find and count an infinite family of geodesics, which was the approach employed here.

Proposition 1 (Northcote)**.** *For all integers i, n with $1 \leq i \leq n$, words of the form $w = a^i ta^n ta^{-n} ta^{-n} ta^{n-i}$ are geodesics of $v\mathscr{H}$ with respect to the generating set $S = \{a, a^{-1}, t\}$.*

Proof. Let $w \in S^*$ be a word of the form

$$w = a^i ta^n ta^{-n} ta^{-n} ta^{n-i},$$

where $i, n \in \mathbb{Z}, i \leq n$. Notice that since w has an even number of t letters, then $\overline{w}$ belongs to the subgroup $\mathscr{H}$. The Tietze transform given by $b = tat$ which was applied to obtain the presentation (1) from (2) yields an automorphism $\varphi : v\mathscr{H} \to v\mathscr{H}$ given by $\varphi(a) = a$, $\varphi(t) = t$, $\varphi(b) = tat$, and since $t^2 = 1$ we have $\varphi\left(b^k\right) = ta^k t$ for $k \in \mathbb{Z}$. Let $X = \{a,\, a^{-1},\, b,\, b^{-1}\}$ be a generating set for the subgroup $\mathscr{H}$. Then from the word $w \in S^*$ we may construct a word

$$w_2 = a^i b^n a^{-n} b^{-n} a^{n-i} \in X^*$$

where $\overline{w_2} = \overline{w}$ since $\varphi(w_2) = w$, and $|w|_S = |w_2|_X + 4$. We can verify using (Blachère 2003, Remark 2.4) that $\overline{w_2}$ evaluates to a group element of form

$$\overline{w_2} = (0, 0, n^2) \in \mathscr{H},\, n > 0,$$

and from (Blachère 2003, Theorem 2.2) we can see that w_2 is geodesic since

$$2\lceil 2\sqrt{n^2}\rceil = 4n = |w_2|_X.$$

As a result of (Blachère 2003, Remark 2.4) we can also see that any geodesic representative of a group element

$$g = (0, 0, z) \in \mathscr{H},\, z > 0$$

contains at least one b letter and one b^{-1} letter, giving

$$\ell(\overline{w_2})_S = |w_2| + 4 = |w|_S$$

Thus, $w = a^i ta^n ta^{-n} ta^{-n} ta^{n-i}$ is geodesic of $v\mathscr{H}$ for all $i, n \in \mathbb{Z}$, $i \leq n$. □

Corollary 1.1. *The geodesic growth function of $v\mathscr{H}$ with respect to $S = \{a, a^{-1}, t\}$ is bounded from below by a polynomial of degree 2.*

Proof. From Proposition 1, we see that there are $n+1$ words of length $4n+4$, giving $\sum_{r=2}^{n+1} r$ words of length up to $4n+4$ for $n \geq 1$. Thus, the number of geodesics in $v\mathscr{H}$ with respect to S is bounded below by $O(n^2)$ □

6 Discussion

The proof for a lower bound of the geodesic growth function of $v\mathscr{H}$ represents a first step in a complete description of the language of geodesics in that group. Many directions were considered for this project, with each revealing further the complexity of the problem. An initial attempt to explicitly characterise the language of geodesics in $v\mathscr{H}$ proved to be both too difficult and time-consuming given the constraints of the research grant, but allowed for the identification of open questions and directions for future work.

7 Open Questions

Through our exploration of geodesic words in $v\mathscr{H}$, we identified the following possible areas for further work.

7.1 Description of an Infinite Family of Geodesics with $\gamma_X(n) > 2$

While we have shown that the number of geodesics in a virtually Heisenberg group is bounded from below by a polynomial of degree 2, we believe that the bounds can be sharpened further using similar techniques. This may be possible through the use of a larger set of experimental data, or more time analysing the word lengths in $\mathscr{H}$ as described by (Blachère 2003). Further, experimental results from (Bishop 2020) and verified by our own experimental work suggest that the geodesic growth function of $v\mathscr{H}$ is bounded from above and below by polynomials of degree 6, though this has yet to be shown.

7.2 Explicit Description of the Language of Geodesics in $v\mathscr{H}$

During the course of this research project, one of our goals was to provide an explicit description of the language of geodesics in $v\mathscr{H}$, though progress was limited by the depth of the problem. Based on the work of (Blachère 2003), an explicit description of word lengths in $v\mathscr{H}$ should be possible to construct given appropriate time.

8 Conclusion

A lower bound was proved for the geodesic growth function of a virtually Heisenberg group $v\mathscr{H}$ with respect to a certain generating set through the description of an infinite family of geodesic words. This result further sharpens the bounds on the group's geodesic growth function previously identified in the work of (Bishop and Elder 2020), and is a step towards an explicit description of the language of geodesics in $v\mathscr{H}$.

References

Bishop, Alex (2020). *A virtually 2-step nilpotent group with polynomial geodesic growth (data and code)*. `https://doi.org/10.5281/zenodo.3941381`.

— (2021). "Geodesic growth in virtually abelian groups". In: *Journal of Algebra* 573, pp. 760–786. ISSN: 0021-8693. DOI: `10.1016/j.jalgebra.2020.12.003`. URL: `http://dx.doi.org/10.1016/j.jalgebra.2020.12.003`.

Bishop, Alex and Murray Elder (2020). "A virtually 2-step nilpotent group with polynomial geodesic growth". In: *arXiv preprint arXiv:2007.06834*.

Blachère, Sébastien (2003). "Word distance on the discrete Heisenberg group". In: *Colloq. Math.* 95.1, pp. 21–36. ISSN: 0010-1354. DOI: `10.4064/cm95-1-2`. URL: `https://doi.org/10.4064/cm95-1-2`.

Bridson, Martin R. et al. (2012). "On groups whose geodesic growth is polynomial." In: *International Journal of Algebra and Computation* 22.05, p. 1250048. ISSN: 1793-6500. DOI: `10.1142/s0218196712500488`. URL: `http://dx.doi.org/10.1142/S0218196712500488`.

Coxeter, Harold SM and William OJ Moser (2013). *Generators and relations for discrete groups*. Vol. 14. Springer Science & Business Media.

Lang, Serge (2012). *Algebra*. Vol. 211. Springer Science & Business Media.

Lauritzen, Niels (2003). *Concrete Abstract Algebra: From Numbers to Gröbner Bases*. Cambridge University Press. DOI: `10.1017/CBO9780511804229`.

Peifer, David (1997). "An Introduction to Combinatorial Group Theory and the Word Problem". In: *Mathematics Magazine* 70.1, pp. 3–10. DOI: `10.1080/0025570X.1997.11996491`. eprint: `https://doi.org/10.1080/0025570X.1997.11996491`. URL: `https://doi.org/10.1080/0025570X.1997.11996491`.