AMSI VACATION RESEARCH SCHOLARSHIPS 2018-2019

Closed Geodesics on Euclidean Homogeneous Spaces

Rohin Berichon

Supervised by Ramiro Lafuente

The University of Queensland

Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

AMSI Australian Mathematical Sciences Institute

Australian Government Department of Education and Training

1 Introduction

We study the existence of closed geodesics on *Euclidean* homogeneous Riemannian manifolds, which are homogeneous Riemannian manifolds diffeomorphic to $\mathbb{R}^n$. Our goal is to give a partial answer to the problem stated on page 503 of Bohm and Lafuente (2017) in the case of a k-step nilpotent Lie group. Namely, we study the following

Problem 1.1 (Bohm and Lafuente)**.** *Is it possible for a Homogeneous space* $(\mathbb{R}^n, g)$ *to have a closed geodesic?*

We derive a particularly nice form of the geodesic equation for a left-invariant Riemannian metric on a Lie group G in Section 3 as an evolution equation on the associated Lie algebra of G, known as the Arnold equation. Then, we use the Arnold equation in Section 4 to provide solutions of the existence problem 1.1. Our main result is the following:

Theorem. *The geodesics of any left-invariant metric on any simply connected real 2-step nilpotent Lie group diffeomorphic to* $\mathbb{R}^n$ *are not closed.*

2 Preliminaries

This is a project in Riemannian geometry. For an introduction to the topics of Riemannian manifolds and geodesics, the reader is encouraged to refer to Chapters 0-3 of Do-Carmo (1992). For a broader introduction to Lie theory, the reader is directed to Varadarajan (1984).

A homogeneous Riemmanian manifold is a Riemannian manifold $\mathcal{M}$ on which a Lie group G acts on transitively by isometries. The simplest examples of homogeneous Riemannian manifolds are Lie groups with Riemannian metrics, known as *Riemannian Lie groups*, on which G acts on itself by left-multiplication. We may associate to the tangent space $\mathfrak{g} \coloneqq T_e\mathsf{G}$ a positive definite inner product $\langle\ ,\ \rangle_e$ and use it to define a *left-invariant metric* on G,

$$\langle X, Y\rangle_g \coloneqq \left\langle (\mathrm{d}L_{g^{-1}})_g X, (\mathrm{d}L_{g^{-1}})_g Y\right\rangle_e, \quad X, Y \in T_g\mathsf{G} \tag{1}$$

where $L_g x = gx$ for any $x, g \in \mathsf{G}$. It is clear that the left-invariant metric associated to a positive definite inner product is the metric that declares that every left-translation is an isometry of G. A left-invariant Riemannian Lie group is therefore precisely a Riemannian manifold which locally resembles the identity element $e \in \mathsf{G}$ at all points in the manifold. Because of the high amount of symmetry in these spaces, equations regarding geodesics become incredibly simple, making them easier to deal with.

We use Lie groups diffeomorphic to $\mathbb{R}^n$ to study the existence problem of closed geodesics on Euclidean homogeneous manifolds. This class of homogeneous space is particularly well studied in literature, which allows us to write a physical evolution equation for geodesics in terms of the Lie algebra $\mathfrak{g}$ associated to the Lie group. In Section 3 we use results from Arnold (1966) to derive the so-called Arnold Equation associated to a Lie group. In Section 4, we prove the non-existence of closed geodesics in the 3-dimensional Heisenberg Lie group and the Lie group of rigid motions of the Minkowski plane $\mathsf{E}(1,1)$. Finally, in Section 5, we prove non-existence of closed geodesics in any simply connected 2-step nilpotent left-invariant Riemannian Lie group.

3 Geodesics & the Arnold Equation

We owe a great deal of the content in this section to Bryant (1991). In order to develop the theory of geodesics on Riemannian Lie groups, we need to approach the notion of a locally distance minimising curve from a more general point of view.

3.1 Basics Results from Lagrangian Mechanics

We start with a fixed (but arbitrary) smooth manifold $\mathcal{M}$. We may associate to $\mathcal{M}$ some smooth function $\mathcal{L} : T\mathcal{M} \to \mathbb{R}$, which we call a *Lagrangian* of $\mathcal{M}$. Then, we let $\mathcal{F}_{\mathcal{L}}$ be a map which associates to a curve section $\gamma : [a,b] \to \mathcal{M}$ the real number

$$\mathcal{F}_{\mathcal{L}}(\gamma) \coloneqq \int_a^b \mathcal{L}(\gamma(t), \dot{\gamma}(t))\mathrm{d}t.$$

We shall call this association the *functional* associated to the Lagrangian $\mathcal{L}$.

We can see how this relates to our purposes if we require that $\mathcal{L}$ restricts to each $T_x\mathcal{M}$ a positive definite quadratic form. In this case, $\mathcal{L}$ defines precisely a Riemannian metric on $\mathcal{M}$. For our purposes, the Lagrangian is thus a generalisation of a metric on a Manifold. It is clear then that if we want to study the properties of geodesics, we should have a firm understanding of those curves which minimise $\mathcal{F}_{\mathcal{L}}$. Given a curve $\gamma : [a,b] \to \mathcal{M}$, we shall call the map $\Gamma : [a,b] \times (-\varepsilon, \varepsilon) \to \mathcal{M}$ for which $\Gamma(t,0) = \gamma(t)$, $\Gamma(a,s) = \gamma(a)$, and $\Gamma(b,s) = \gamma(b)$ a *smooth variation with fixed endpoints*, or *variation* for short. We will also define the mapping $\mathcal{F}_{\mathcal{L},\Gamma} : (-\varepsilon, \varepsilon) \to \mathbb{R}$, by setting

$$\mathcal{F}_{\mathcal{L},\Gamma}(s) = \mathcal{F}_{\mathcal{L}}(\Gamma(\cdot, s)).$$

We say a curve $\gamma : [a,b] \to \mathcal{M}$ is $\mathcal{L}$-critical if $\mathcal{F}'_{\mathcal{L},\Gamma}(0) = 0$ for any variation Γ. These curves may be shown to satisfy a special system of partial differential equations, known as the Euler-Lagrange Equations[1]. In the interpretation of the Riemannian geometric Lagrangian g associated to $\mathcal{M}$, these curves are precisely the curves which minimise the distance between $\gamma(a)$ and $\gamma(b)$. Therefore, the Euler-Lagrange equations simplify to the geodesic equations in the Lagrangian g.

In this more general physical theory, we can see that the geodesics on a Riemannian manifold $(\mathcal{M}, g)$ are energy minimising curves on the manifold. It turns out we can use this more general view of geodesics to derive a particularly efficient system of equations on a left-invariant Riemannian metric on a Lie group G.

3.2 Arnold's Equation and Associated Euler-Lagrange System

Our interest in the geodesics on left-invariant Riemannian metrics on a Lie group G leads us to the task of writing the geodesic equation in a more concise way, which relies only on the metric restricted to the identity[2]. Furthermore, this theory will allow us to write the geodesic equation on the left-invariant Lie group (G, g) as an evolution equation on $\mathfrak{g}$, therefore making the task of finding geodesics a linear-algebraic problem. Let $\gamma : [a,b] \to \mathsf{G}$ be a geodesic segment, and recall that γ is the curve which minimises

$$\mathcal{F}_g(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}\left(\gamma'(t), \gamma'(t)\right)} \mathrm{d}t.$$

We would like to study these curves. To do this, we need a number of definitions, which we will state and elaborate on here.

The first object of interest is a so-called *inertia operator*; $\pi : \mathfrak{g} \to \mathfrak{g}^*$, which is defined as

$$\langle X, Y \rangle_e = \pi(X)Y.$$

[1] See Bryant (1991) for details.

[2] Note: We attempt to use Arnold's original construction as much as possible (Arnold 1966), and we have used (Kolev 2004) as a translation reference.

If we choose an orthonormal basis $\{e^i\}$ with corresponding dual basis $\{e_i\}$, then $\pi(e^i) = e_i$, and so π becomes the natural pairing of elements in the dual. We may make this inertia operator a left invariant tensor $\pi_x : T_x\mathsf{G} \to T_x\mathsf{G}^*$ by defining $\pi_x = \mathrm{d}L^*_{x^{-1}}\pi\mathrm{d}L_{x^{-1}}$.

We then define the map $\mathcal{T} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\mathcal{T}(X,Y) = \pi^{-1}\mathrm{ad}^*_Y(\pi(X)), \quad \text{equivalently,} \quad \langle [X,Y],Z\rangle = \langle \mathcal{T}(Z,X),Y\rangle.$$

Let us now define the following maps associated to γ;

$$V(t) = \mathrm{d}L_{\gamma^{-1}(t)}\gamma'(t) \in \mathfrak{g}, \quad M(t) = \pi_{\gamma(t)}(\gamma'(t)) \in \mathfrak{g}^*, \quad M_L(t) = \mathrm{d}L^*_{\gamma(t)}M(t) \in \mathfrak{g}^*, \quad M_R(t) = \mathrm{d}R^*_{\gamma(t)}M(t) \in \mathfrak{g}^*.$$

We have the relations $M_L = \pi(V)$, and $M_R = \mathrm{Ad}^*_{\gamma(t)}M_L$, where $(\mathrm{Ad}^*_x f)(V) = f(\mathrm{Ad}_x V)$ is the coadjoint representation[3] of G. Now, we observe that we may also write

$$g_{\gamma(t)}(\gamma'(t),\gamma'(t)) = \langle \gamma',\gamma'\rangle_\gamma = \langle V,V\rangle_e = M_L(V).$$

At this point, we have the tools we need to construct the Arnold equation. For the most part, we shall skim over the details involving Noether's theorem, since fully developing the theory for this requires a more rigorous treatment of the general theory of Lagrangian mechanics and conservation laws[4]. Again, we shall denote $\gamma : [a,b] \to \mathcal{M}$ a geodesic on (G, g) with g left-invariant. According to Noether's theorem, the map M_R is constant along the geodesic γ. This allows us to write

$$\frac{\mathrm{d}M_R}{\mathrm{d}t} \equiv 0.$$

Using $M_R = \mathrm{Ad}^*_\gamma M_L$, we obtain

$$\frac{\mathrm{d}M_L}{\mathrm{d}t} = \mathrm{ad}^*_V M_L.$$

Finally, using $V = \pi^{-1}M_L$ we acquire the evolution equation on the Lie algebra $\mathfrak{g}$,

$$V' = \mathcal{T}(V,V), \iff \langle V',X\rangle = \langle [V,X],V\rangle, \quad \forall X \in \mathfrak{g}$$

by the definition of $\mathcal{T}$. The former of which is called the Arnold equation. The latter form will be especially helpful for our purposes. We can see that the system defined implicitly by

[3] Note that the adjoint representation of G is defined $\mathrm{Ad}_{(\exp(X))} = \exp(\mathrm{ad}_X)$, see (Varadarajan 1984)

[4] Both Kolev (2004), and Bryant (1991) develop the theory in two distinct and effective ways.

the second equation and the definition of V gives us, given an orthonormal basis $\{e_i\}$ with $V = \sum_i V_i e_i$;

$$\begin{cases} \gamma' = \mathrm{d}L_\gamma V, & \text{(2a)} \\ V_i' = \langle V', e_i \rangle = \langle [V, e_i], V \rangle. & \text{(2b)} \end{cases}$$

Together, we shall call (2) the Arnold-Euler-Lagrange (AEL) system for a geodesic γ on G. In what follows, we shall show how to use the AEL system to determine the existence of closed geodesics on the Lie groups equipped with a left-invariant Riemannian metric.

3.3 Using The AEL System to Locate Closed Geodesics

The upshot of the above derivation is two-fold. On one hand, the Arnold equation is an explicit autonomous system on the Lie algebra $\mathfrak{g}$ of G, which depends entirely on the choice of inner product $\langle\ ,\ \rangle_e$. It allows us, in some sense to by-pass superfluous details of the metric when computing the geodesics in G with the AEL system. Furthermore, the Arnold equation tells us it suffices to know the local behaviour of geodesics at the identity of G to be able to describe their behaviour anywhere in the Lie group. On the other hand, the Arnold equation, and associated AEL system gives us a robust system which we may practically use to derive formulae for geodesics of any Left-invariant metric on G. We can also use the AEL system to determine qualitative behaviour about geodesics. Observe the following

Lemma 3.1. *Suppose $\gamma : \mathbb{R} \to \mathsf{G}$ is a closed geodesic on a Riemannian Lie group (G, g) with left-invariant metric. Then, the field $V : \mathbb{R} \to \mathfrak{g}$ defined $V(t) = \mathrm{d}L_{\gamma(t)^{-1}}\dot\gamma(t)$ is periodic.*

Proof. Suppose $\gamma(t) = \gamma(t+T)$ for some $T \in \mathbb{R}$. Then,

$$\begin{aligned} V(t+T) = \mathrm{d}L_{\gamma(t+T)^{-1}\gamma(t+T)}V(t+T) &= \mathrm{d}L_{\gamma(t+T)^{-1}}\mathrm{d}L_{\gamma(t+T)}V(t+T) \\ &= \mathrm{d}L_{\gamma(t+T)^{-1}}\dot\gamma(t+T) = \mathrm{d}L_{\gamma(t)^{-1}}\dot\gamma(t) = V(t) \end{aligned}$$

□

Therefore, if V is not periodic, then we have that the geodesic γ associated to V is not a closed curve. Hence, if we want to show the geodesics of a left-invariant metric of a Lie group are not closed, we may assume without loss of generality that V is a periodic field. Unfortunately, this is the best we can hope for, as the converse is in general false.

4 Existence of Closed Geodesics on Homogeneous 3-Manifolds

4.1 Existence of Closed Geodesics in $\mathcal{H}_3$

We prove there are no closed geodesics in any left-invariant Riemannian metric on the 3-dimensional Heisenberg Lie group. Recall the following

Definition 4.1. *The Heisenberg group $\mathcal{H}_3$ is the matrix Lie group*

$$\mathcal{H}_3 \coloneqq \left\{ \left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix} \right) \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

with defining action

$$(x, y, z) \cdot (x', y', z') \mapsto (x + x', y + y', z + z' + xy').$$

The associated Lie algebra $\mathfrak{h}_3 \coloneqq \mathrm{Lie}(\mathcal{H}_3)$ is the matrix Lie algebra of strictly upper-triangular matrices. By fixing an arbitrary inner product $\langle\ ,\ \rangle_e : \mathfrak{h}_3 \times \mathfrak{h}_3 \to \mathbb{R}$, we may define a left-invariant Riemannian metric on $\mathcal{H}_3$, as explained in Equation (1). According to Milnor (1976), there is an orthonormal basis $\{e_1, e_2, e_3\}$ on $\mathfrak{h}_3$ for which the commutation relations

$$[e_2, e_3] = \lambda e_1, \qquad \lambda > 0$$

are satisfied. Clearly, distinct values of λ give rise to different metrics on $\mathcal{H}_3$.

Let us denote the moduli space of left-invariant metrics on G by $\mathfrak{B}\mathfrak{M}(\mathsf{G})$. That is, we consider two left-invariant metrics g_1 and g_2 of G to be equivalent if they are isometric up to a scalar. Clearly, γ is a closed geodesic in the metric g_1 if and only if it is closed in the metric g_2. We have Lauret's theorem in its partial form,

Theorem 4.2. *(Lauret 2003) Every non-abelian Lie group $\mathcal{H}$ with Lie algebra of the form $\mathfrak{h}_3 \oplus \mathbb{R}^{n-3}$ has moduli space of left invariant metrics up to isometry and scaling equivalent to a singleton set. That is, $\mathfrak{B}\mathfrak{M}(\mathcal{H}) = \{\mathrm{pt.}\}$.*

Therefore, all Left-invariant Riemannian metrics on $\mathcal{H}_3$ are equivalent under the equivalence relation of isometry and scaling. Since closed geodesics are preserved in this equivalence relation, it is enough to consider the existence problem of closed geodesics on a single left-invariant Riemannian metric, specifically that which is generated by fixing $\lambda = 1$.

We make the following,

Definition 4.3 (Canonical Coordinates of the 2nd Kind). *Let G^n be a simply connected Lie group with Lie algebra $\mathfrak{g}$, and define*

$$\varphi : U \overset{\circ}{\subseteq} \mathfrak{g} \to \mathsf{G}, \qquad \varphi(x^1, \ldots, x^n) = \prod_{i=1}^{n} \exp(x^i e_i).$$

We call the coordinates (U, φ^{-1}) the canonical coordinates of the 2nd kind.

We need the following result appearing as Theorem 19 from Graner (2018):

Theorem 4.4. *Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$, and let $\{e_1, \ldots e_n\}$ be a basis of $\mathfrak{g}$ for which $\mathfrak{h}_i \coloneqq \operatorname{span}\{e_1, \ldots, e_i\}$ is a subalgebra of $\mathfrak{g}$ and an ideal of $\mathfrak{h}_{i+1}$.*[5] *Then, 4.3 with $U = \mathfrak{g}$ is a diffeomorphism.*

The upshot of this is a global coordinate system

$$\psi : \mathfrak{h}_3 \to \mathcal{H}_3, \qquad \psi(X_1, X_2, X_3) = \exp(X_1 e_1)\exp(X_2 e_2)\exp(X_3 e_3),$$

since $\{e_1, e_2, e_3\}$ is clearly an adapted basis.

We turn our minds to the existence problem. Let $\gamma : \mathbb{R} \to \mathcal{H}_3$ be some geodesic with $\gamma(t) = \exp(\Gamma_1 e_1)\exp(\Gamma_2 e_2)\exp(\Gamma_3 e_3), \Gamma_i : \mathbb{R} \to \mathbb{R}$. Define V as in Equation (2a), and observe that Equation (2b) gives us

$$V_i' = \langle [V, e_i], V \rangle .$$

By fixing values of $i = 1, 2, 3$, we get

$$V_1' = 0, \qquad V_2' = -V_3 V_1, \qquad V_3' = V_2 V_1.$$

Immediately we get $V_1(t) = k \in \mathbb{R}$. If any of V_2, V_3 are not periodic, we may use Lemma 3.1 to prove γ is not closed. Assume therefore that V is a periodic solution to the equation above.

Using Equation (2a), and identifying[6] $\mathrm{d}L_\gamma = L_\gamma$ we have

$$V(t) = \gamma^{-1}\gamma'(t).$$

Now by using the Baker Campbell Haussdorff formula (Hall 2003), we see that

$$\begin{aligned}\gamma(t) = \exp(\Gamma_1(t)e_1)\exp(\Gamma_2(t)e_2)\exp(\Gamma_3(t)e_3) &= \exp(\Gamma_1 e_1 + \Gamma_2 e_2)\exp(\Gamma_3 e_3) \\ &= \exp(\Gamma_1 e_1 + \Gamma_2 e_2 + \Gamma_3 e_3 + 1/2\Gamma_2\Gamma_3 e_1).\end{aligned}$$

Then, due to F. Schur (Rossmann 2006), we have the following

[5]Note: we call such a basis an *adapted basis*.

[6]Since $\mathcal{H}_3$ is a matrix Lie group

Theorem 4.5. *If $X(t)$ is a curve on $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ with G simply connected, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp X(t) = \exp X(t) \frac{1 - \exp(-\mathrm{ad}_X)}{\mathrm{ad}_X} \frac{\mathrm{d}X}{\mathrm{d}t}.$$

Furthermore, if G is a matrix Lie group, then

$$\frac{1 - \exp(-\mathrm{ad}_X)}{\mathrm{ad}_X} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (\mathrm{ad}_X)^k.$$

This gives us

$$\begin{aligned} Y(t) = \gamma^{-1}\gamma'(t) &= \gamma^{-1} \exp(\overbrace{\Gamma_1 e_1 + \Gamma_2 e_2 + \Gamma_3 e_3 + 1/2\Gamma_2\Gamma_3 e_1}^{\mathbf{\Gamma}}) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (\mathrm{ad}_{\mathbf{\Gamma}})^k \right) \begin{pmatrix} \Gamma_1' + 1/2(\Gamma_2'\Gamma_3 + \Gamma_2\Gamma_3') \\ \Gamma_2' \\ \Gamma_3' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1/2\Gamma_3 & -1/2\Gamma_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_1' + 1/2(\Gamma_2'\Gamma_3 + \Gamma_2\Gamma_3') \\ \Gamma_2' \\ \Gamma_3' \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_1' + \Gamma_2'\Gamma_3 \\ \Gamma_2' \\ \Gamma_3' \end{pmatrix}. \end{aligned}$$

Therefore, we have proven the following

Lemma 4.6. *The Arnold-Euler-Lagrange system for the geodesics on the 3-dimensional Heisenberg Lie group $\mathcal{H}_3$ is equivalent to the system*

$$\begin{cases} \Gamma_1' = V_1 - V_2\Gamma_3, & V_1 = k \in \mathbb{R}, \\ \Gamma_2' = V_2, & V_2' = -V_1V_3, \\ \Gamma_3' = V_3, & V_3' = V_1V_2. \end{cases} \tag{3}$$

If $k = 0$, then $V_1 = 0$, and we have the explicit solution $\Gamma_2 = V_2(0)t + \Gamma_2(0)$, which is strictly monotone, so γ isn't closed. Assuming $k \neq 0$, we get the linear system in V_2 and V_3,

$$\begin{pmatrix} V_2 \\ V_3 \end{pmatrix}' = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \begin{pmatrix} V_2 \\ V_3 \end{pmatrix}$$

which in general yields periodic solutions. Let us assume γ is T-periodic for the sake of contradiction. Then, V_3 and V_2 are both T-periodic functions since $\Gamma_i, i = 1,2,3$ are T-periodic. Then, observe that Lemma 4.6 gives us

$$\int_0^T \Gamma_3 V_3' \mathrm{d}t = \int_0^T k V_2 \Gamma_3 \mathrm{d}t.$$

Now, since Γ_1 is T periodic, we have

$$\begin{aligned} 0 = \int_0^T k\Gamma_1' \mathrm{d}t &= k^2 T - \int_0^T k V_2 \Gamma_3 \\ &= k^2 T - \int_0^T \Gamma_2 V_3' \mathrm{d}t. \end{aligned}$$

Integrating by parts and using that $V_3\Gamma_3$ is periodic gives,

$$0 = \underbrace{k^2 T}_{>0} - \underbrace{\int_0^T (\Gamma_2 V_2)' \mathrm{d}t}_{=0} + \underbrace{\int_0^T V_2^2 \mathrm{d}t}_{\geq 0} > 0.$$

This is a contradiction. Hence, γ is not closed, and we are done.

This process is useful in the sense we do not need to explicitly solve the geodesic equation to determine global behaviour of geodesics in $\mathcal{H}_3$. Unfortunately, it is not always possible to extract all the information we need to use this process effectively. As we shall see, when we consider Lie algebras whose lower central series does not terminate, the Baker Campbell Haussdorff formula doesn't have an algebraic closed form. The case of the Lie group of the rigid motions of the Minkowski plane $\mathsf{E}(1,1)$ is a solvable group, which will allow us to see this in play, without losing the global coordinate system of Theorem 4.4.

4.2 Existence of Closed Geodesics on $\mathsf{E}(1,1)$

Whilst some Lie groups attribute highly periodic solutions to their field V defined in Equation (2b), others are particularly well behaved. $\mathsf{E}(1,1)$ is almost completely non-periodic in its solution space to Equation (2a), which we shall exploit proving the geodesics in any left-invariant metric of $\mathsf{E}(1,1)$ are not closed. We make the following

Definition 4.7. *The group* $\mathsf{E}(1,1)$ *is the Lie group* $\mathbb{R} \ltimes \mathbb{R}^2$ *where* $\mathbb{R}$ *acts on* $\mathbb{R}^2$ *as* $((x,y),z) \mapsto (e^z x, e^{-z} y)$ *with defining actions*

$$(x,y,z)\cdot(x',y',z') = (x'e^z + x, y'e^{-z} + y, z + z').$$

$\mathsf{E}(1,1)$ *may be faithfully represented as the solvable matrix Lie Group*

$$\mathsf{E}(1,1) = \left\{ \left(\begin{smallmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{smallmatrix}\right) \,\middle|\, x,y,z \in \mathbb{R} \right\},$$

under the natural identification. It follows that the associated Lie algebra is faithfully represented as

$$\mathfrak{e}(1,1) \coloneqq \left\{ \left(\begin{smallmatrix} z & 0 & x \\ 0 & -z & y \\ 0 & 0 & 0 \end{smallmatrix}\right) \,\middle|\, x,y,z \in \mathbb{R} \right\}.$$

Let us again fix a positive definite inner product $\langle\ ,\ \rangle_e : \mathfrak{e}(1,1) \times \mathfrak{e}(1,1) \to \mathbb{R}$. This gives rise to a left-invariant metric on $\mathsf{E}(1,1)$ under Equation (1). Fixing an orthonormal adapted basis

$\{e_1, e_2, e_3\}$ for $\mathfrak{e}(1,1)$, the commutation relations are

$$[e_3, e_1]\,\lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1, \qquad \lambda_1 > 0, \lambda_2 < 0.$$

As with the Heisenberg case, exploring the space of left-invariant metrics equivalent up to isometry and scaling $\mathfrak{BM}(\mathsf{E}(1,1))$, we get $\dim\mathfrak{BM}(\mathsf{E}(1,1)) = 1$, by Kodama, Takahara, and Tamaru (2011). This allows us to fix a single parameter in the Milnor frame above, since there is a one parameter group of metrics equivalent up to isometry and scaling. Choosing $\lambda_1 = 1$, and denoting $\lambda = \lambda_2$, it follows that every left-invariant metric on $\mathsf{E}(1,1)$ up to isometry and scaling may be generated by fixing a value of λ.

Let $\gamma : \mathbb{R} \to \mathsf{E}(1,1)$ be a geodesic, and $X = \mathrm{d}L_\gamma \dot{\gamma}$ be the associated left-translated angular velocity field written as $X = X_1 e_1 + X_2 e_2 + X_3 e_3$ in $\mathfrak{e}(1,1)$. Then, writing the Arnold equation;

$$\langle X_1' e_1 + X_2' e_2 + X_3' e_3, e_i \rangle = \langle [X_1 e_1, e_i] + [X_2 e_2, e_i] + [X_3 e_3, e_i], X_1 + X_2 + X_3 \rangle,$$

which, by picking values for i, yields the system

$$X_1' = \lambda X_2 X_3, \quad X_2' = -X_1 X_3, \quad X_3' = (1-\lambda) X_1 X_2.$$

Setting $Z = X_1 X_2$, this gives us

$$Z' \coloneqq (X_1 X_2)' = X_1' X_2 + X_2 X_1' = (\lambda - 1) Z X_3, \quad X_3' = (1-\lambda) Z.$$

Substituting in the expression for X_3' gives

$$\begin{aligned} Z'(t) &= -X_3' X_3 \\ \Rightarrow Z(t) - Z(0) &= -\int_{X_3(0)}^{X_3(t)} X_3 \mathrm{d}X_3 \\ \Rightarrow (X_1 X_2)(t) = Z(t) &= X_1(0) X_2(0) + \frac{X_3(0)^2}{2} - \frac{X_3(t)^2}{2}. \end{aligned}$$

This allows us to rewrite X_3' as

$$X_3'(t) = (1-\lambda)\left[X_1(0) X_2(0) + \frac{X_3(0)^2}{2} - \frac{X_3(t)^2}{2}\right].$$

Setting $k = X_1(0) X_2(0) + \frac{X_3(0)^2}{2}$, we may see that $X_3' > 0$ whenever $\frac{X_3(t)}{2} \in \left(-\sqrt{k}, \sqrt{k}\right)$, and that $X_3' < 0$ whenever $\frac{X_3(t)}{2} \in \mathbb{R} \setminus \left[-\sqrt{k}, \sqrt{k}\right]$. Finally, whenever $X_3' = 0$, we must have that $\frac{X_3(t)}{2} \equiv \pm\sqrt{k}$. By the existence and uniqueness of first order ordinary differential equations, the solution space of X_3' is partitioned into the three infinite rectangles defined by $X_3 = \pm 2\sqrt{k}$. Therefore, either $X_3' > 0$ for all t, or $X_3' < 0$ for all t or $X_3(t) \equiv \pm 2\sqrt{k}$. In the first two cases, X' has a strictly monotone coordinate, so is not periodic, so its corresponding geodesics are not closed by Lemma 3.1. We have therefore proven the following

Lemma 4.8. *If $X_3(t) \neq \pm 2\sqrt{k}$ for some $t \in \mathbb{R}$, the geodesic γ as defined above is not closed.*

Thus, we are left with the cases where $X_3(t) = \pm 2\sqrt{k}$. In these cases, define global coordinates around $\gamma(t)$ in using Theorem 5.2. In global coordinates therefore, $\gamma(t) = \exp(\Gamma_1 e_1)\exp(\Gamma_2 e_2)\exp(\Gamma_3 e$ for some $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) \in \mathfrak{e}(1,1)$. Using this, and the Baker-Campbell-Haussdorff formula, we get

$$\begin{aligned}\gamma = \exp(\Gamma_1 e_1)\exp(\Gamma_2 e_2)\exp(\Gamma_3 e_3) &= \exp(\Gamma_1 e_1 + \Gamma_2 e_2)\exp(\Gamma_3 e_3)\\ &= \exp\underbrace{\left(\left(\Gamma_1 + \frac{1}{2}\Gamma_2\Gamma_3\right)e_1 + \left(\Gamma_2 - \frac{1}{2}\Gamma_1\Gamma_3\right)e_2 + \Gamma_3 e_3\right)}_{\mathbf{\Gamma}}.\end{aligned}$$

Then, as with $\mathcal{H}_3$, since $\mathsf{E}(1,1)$ is a linear Lie group, we have

$$\begin{pmatrix} * \\ * \\ \pm 2\sqrt{k} \end{pmatrix} = \gamma^{-1}\gamma'(t) = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}(\mathrm{ad}_{\mathbf{\Gamma}})^k\right)\frac{\mathrm{d}\mathbf{\Gamma}}{\mathrm{d}t}$$
$$\Leftrightarrow \begin{pmatrix} * \\ * \\ \pm 2\sqrt{k} \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} * \\ * \\ \Gamma_3' \end{pmatrix},$$

and so $\Gamma_3' = \pm 2\sqrt{k}$. Therefore, Γ_3 is strictly monotone, so γ is not a closed geodesic.

5 Existence of Closed Geodesics on 2-Step Nilpotent Lie Groups

Using the Heisenberg Lie group for inspiration, we were interested in the existence problem 1.1 in the case of a general finite dimensional *2-step nilpotent* Lie group diffeomorphic to $\mathbb{R}^n$. A 2-step nilpotent Lie group is a Lie group with an associated Lie algebra $\mathfrak{n}$, which satisfies $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$. We make the immediate observation that this definition implies $[\mathfrak{n}, \mathfrak{n}]$ lies in the centre of $\mathfrak{n}$.

In this section, we prove the following

Theorem 5.1. *The geodesics of any left-invariant metric g on a simply connected real 2-step nilpotent Lie group N diffeomorphic to $\mathbb{R}^n$ are not closed.*

We spend the rest of this section proving this fact. Fixing an adapted orthonormal basis $\{f_j; e_i\}_{i,j}$ for some set inner product $\langle\ ,\ \rangle_e$ on $\mathfrak{n}$, we may write the lower triangular orthogonal decomposition $\mathfrak{n} = \mathfrak{v} \oplus [\mathfrak{n}, \mathfrak{n}]$, with $\mathfrak{v} = [\mathfrak{n}, \mathfrak{n}]^\perp$. We know by 4.3 that the coordinates of the second kind in this frame are a diffeomorphism onto N. That is, $\varphi : \mathfrak{n} \to \mathsf{N}$ defined by

$$\varphi(v^i; u^j) \coloneqq \varphi(\sum_j u^j f_j + \sum_i v^i e_i) = \prod_j \exp(u^j f_j) \prod_i \exp(v^i e_i)$$

is a diffeomorphism onto N. From this, and using the Baker-Campbell-Haussdorff formula, we get that

$$\varphi(v^i; u^j) = \exp\left(\sum_i v^i e_i + \sum_j \left(u^j + \sum_{\ell,k} \frac{1}{2} \lambda^j_{\ell k} v^\ell v^k\right) f_j\right), \quad \lambda^j_{\ell k} = \langle [e_\ell, e_k], f_j \rangle .$$

Since $\mathfrak{v}$ and $[\mathfrak{n}, \mathfrak{n}]$ commute, this reduces to

$$\varphi(v^i; u^j) = \exp\left(\sum_i v^i e_i\right) \exp\left(\sum_j \left(u^j + \sum_{\ell,k} \frac{1}{2} \lambda^j_{\ell k} v^\ell v^k\right) f_j\right).$$

In particular, this tells us that $\mathfrak{v} \oplus [\mathfrak{n}, \mathfrak{n}] \mapsto \exp(\mathfrak{v}) \exp([\mathfrak{n}, \mathfrak{n}]) = \mathsf{N}$ is a diffeomorphism onto its image. That is, we get the following

Corollary 5.2. *If $\mathfrak{v} \oplus [\mathfrak{n}, \mathfrak{n}] \eqqcolon \mathfrak{n} \coloneqq \mathrm{Lie}(\mathsf{N})$ is the Lie algebra associated to a simply connected real 2-step nilpotent Riemannian Lie group N with orthogonal decomposition $\mathfrak{v} \oplus [\mathfrak{n}, \mathfrak{n}]$, then the map $\psi : \mathfrak{n} \to \mathsf{N}$ defined by*

$$\psi(V + Z) = \exp(V) \exp(Z), \quad V \in \mathfrak{v}, Z \in [\mathfrak{n}, \mathfrak{n}]$$

is a diffeomorphism.

The result of this is a global coordinate system for N defined in terms of the mapping ψ, so any curve defined on this coordinate system may be extended to the whole space N. With the stage set, we shall prove Theorem 5.1.

Proof. Let $\gamma(t) \coloneqq \exp(\Gamma(t)) \exp(\Lambda(t)) = \exp(\Omega(t))$, with $\Omega(t) \coloneqq \Gamma(t) + \Lambda(t)$ be a geodesic on N. Denote $V(t) + Z(t) \eqqcolon X(t) \coloneqq \mathrm{d}L_{\gamma(t)^{-1}} \gamma'(t) = \gamma^{-1}(t)\gamma'(t)$ as in Equation (2a). We make the immediate observation that, by Equation (2b),

$$\langle X', Y \rangle = \langle V', Y \rangle + \langle Z', Y \rangle = \langle [V, Y], Z \rangle .$$

Setting $Y \in [\mathfrak{n}, \mathfrak{n}]$ gives $Z' \equiv 0$, so

$$Z = Z_0 \in \mathfrak{n}. \tag{4}$$

Furthermore, we get

$$\langle V', Y \rangle = \langle \mathrm{ad}_V Y, Z_0 \rangle$$
$$\Leftrightarrow \langle V', Y \rangle = \left\langle \mathrm{ad}_V^{\mathrm{t}} Z_0, Y \right\rangle,$$

and so

$$V' = \mathrm{ad}_V^{\mathrm{t}} Z_0 = MV, \tag{5}$$

for some $M \in M_n(\mathbb{R})$ since the equation $\mathrm{ad}_V^{\mathrm{t}} Z_0$ is linear in V. We shall break up into three cases from here.

Case 1: $Z(0) = 0 \Rightarrow V(0) \neq 0$

By (5), $V' = 0$, so $V(t) = V_0 \neq 0$. By the linearity of N, we identify $\mathrm{d}L_{\gamma^{-1}}$ with $L_{\gamma^{-1}}$ in (2a), so that $\gamma'(t) = \gamma(t)V_0$. By the existence and uniqueness of geodesics, $\gamma(t) = \exp(tV_0)$. Since N is 2-step nilpotent, we just have $\gamma(t) = \mathrm{Id}_{\mathsf{N}} + tV_0$. Clearly γ never self-intersects, so γ is not a closed geodesic, as desired.

Case 2: $V(0) = 0 \Rightarrow Z(0) \neq 0$

By (5), we get the general solution $V(t) = \exp(tM)V_0 = 0$, so $X(t) = Z_0$. From this, $\gamma'(t) = \gamma(t)Z_0$. Again, by existence and uniqueness of geodesics, we get $\gamma(t) = \exp(tZ_0)$, which simplifies to $\gamma(t) = \mathrm{Id}_{\mathsf{N}} + tZ_0$, which is clearly not a closed curve.

Case 3: $V(0) \neq 0$, **and** $Z(0) \neq 0$

Observe that since (4) is constant, by Lemma 3.1, if V is not periodic, γ is not a closed curve. Therefore, we assume that V is a T-periodic field for some $T \in \mathbb{R}_+$. Then, using Theorem 4.5 and the 2-step nilpotency of $\mathfrak{n}$, we get

$$V + Z_0 = \gamma^{-1}\gamma' = \sum_{k=0}^{1} \frac{(-1)^k}{(k+1)!} (\mathrm{ad}_\Omega)^k \left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right) = \Gamma' + \left(\Lambda' - \frac{1}{2}[\Gamma, \Gamma']\right),$$

which yields the Arnold-Euler-Lagrange system

$$Z' = 0, \quad V' = \mathrm{ad}_V^{\mathrm{t}} Z_0 = MV, \quad \Gamma' = V, \quad \Lambda' = Z_0 + \frac{1}{2}[\Gamma, V].$$

We proceed by contradiction. Suppose γ is a closed geodesic. Then, Λ is a periodic vector field with period $T \in \mathbb{R}_+$. In particular, this gives us $\langle \Lambda, Z_0 \rangle$ is a T-periodic function. Furthermore, it is also clear that $\langle \Gamma, V \rangle$ is also a T-periodic function. This gives us[7]

$$\int_0^T \langle \Lambda, Z_0 \rangle' \, \mathrm{d}t = \int_0^T \langle \Gamma, V \rangle' \, \mathrm{d}t = 0.$$

Then,

$$\begin{aligned}
0 = \int_0^T \langle \Lambda, Z_0 \rangle' \, \mathrm{d}t &= \int_0^T \langle \Lambda', Z_0 \rangle \, \mathrm{d}t \\
&= \int_0^T \|Z_0\|^2 + \frac{1}{2} \langle [\Gamma, V], Z_0 \rangle \, \mathrm{d}t \\
&= \int_0^T \|Z_0\|^2 - \frac{1}{2} \left\langle \Gamma, \mathrm{ad}_V^{\mathrm{t}} Z_0 \right\rangle \mathrm{d}t \\
&= \int_0^T \|Z_0\|^2 \, \mathrm{d}t - \frac{1}{2} \int_0^T \langle \Gamma, V' \rangle \, \mathrm{d}t \\
&= \int_0^T \|Z_0\|^2 \, \mathrm{d}t - \frac{1}{2} \underbrace{\int_0^T \langle \Gamma, V \rangle' \, \mathrm{d}t}_{0} + \frac{1}{2} \int_0^T \langle \Gamma', V \rangle \, \mathrm{d}t \\
&= \underbrace{T \|Z_0\|^2}_{>0} + \underbrace{\frac{1}{2} \int_0^T \|V\|^2 \, \mathrm{d}t}_{\geq 0} > 0.
\end{aligned}$$

This is clearly a contradiction, so γ is not a closed curve.

Therefore, any geodesic γ on a simply connected 2-step nilpotent Lie group with left-invariant Riemannian metric is not closed. □

6 Current Progress

Whilst the resolution of Problem 1.1 in the case of the 2-step nilpotent Lie group indicates a general proof for the k-step nilpotent Lie group N may be within grasp, the non-commutativity of the terms of the lower central series of the Lie algebra $\mathfrak{n} := \mathrm{Lie}(\mathsf{N})$ for $k \geq 3$ destroys the simplicity of the Arnold equation. In particular, the general method we used no longer holds in the 3-step nilpotent case. Whilst we can no longer use coordinates of the second kind[8], the fact that the exponential map is still a diffeomorphism onto N allows us to still use the so-called coordinates of the first kind. We therefore make the following in good hopes;

Conjecture 6.1. *The geodesics on any Nilpotent, left-invariant Riemannian Lie group* N *are not closed.*

[7] By the Fundamental Theorem of Calculus

[8] Since, for $k \geq 3$, the coordinates of the second kind are not necessarily a global coordinate system

The ramifications of a proof of this would be two-fold. On the one hand, resolving Problem 1.1 for the case of the nilpotent Lie group with left-invariant metric provides the first step for a solution for the general case. On the other hand, the field of compact nilpotent Left-invariant Riemannian Lie groups is largely more well understood than the non-compact counterpart, and so would enrich the theory of simply connected nilpotent left-invariant Riemannian Lie groups on the whole.

References

Arnold, Vladimir (1966). "Sur la geometrie differentialle des groups de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits". In: *Annales De L'Institut Fourier*, pp. 322–324.

Bohm and Lafuente (Nov. 2017). "Immortal homogeneous Ricci flows". In: *Invent. math.* P. 503. ISSN: 212. DOI: 10.1007/s00222-017-0771-z.

Bryant, Robert L. (July 1991). *An Introduction to Lie Groups and Symplectic Geometry.* URL: https://services.math.duke.edu/~bryant/ParkCityLectures.pdf. 24-20.

Do-Carmo, Manfredo (1992). *Riemannian Geometry.* Trans. by Francis Flaherty. Chap. 0, 1, 2, 3. ISBN: 0-8176-3490-8.

Graner, Nicholas (2018). "Canonical Coordinates on Lie Groups and the Baker-Campbell-Haussdorff Formula". In: *Utah State University Digital Commons @ USU*, p. 32.

Hall, Brian C. (2003). *Lie Groups, Lie Algebras and Representations. An Elementary Introduction.* Springer-Verlag New York, Inc, p. 51. ISBN: 978-1-4419-2313-4. DOI: 10.1007/978-0-387-21554-9.

Kodama, Hiroshi, Atsushi Takahara, and Hiroshi Tamaru (2011). "The space of left-invariant metrics on a Lie group up to isometry and scaling". In: *manuscripta mathematica*, pp. 229–243. ISSN: 135. DOI: 10.1007/s00229-010-0419-4.

Kolev, Boris (Mar. 2004). "Lie Groups and mechanics, an Introduction". Version 2. In: *Wave Motion: Proceedings of the Conference at the Oberwolfach Mathematical Research Institute*, pp. 480–498. DOI: 10.2991/jnmp.2004.11.4.5. URL: https://arxiv.org/abs/math-ph/0402052.

Lauret, J. (2003). "Degenerations of Lie algebras and geometry of Lie group". In: *Differential Geom. Appl.* 2, 18, pp. 177–194.

Milnor, John (Sept. 1976). "Curvatures of Left Invariant Metrics on Lie Groups". In: *Advances in Mathematics* 21 (3), pp. 293–329. DOI: 10.1016/S0001-8708(76)80002-3. New Jersey.

Rossmann, Wulf (2006). *Lie Groups: An Introduction to Linear Groups.* Oxforrd University Press. Chap. 1. The Exponential Map, p. 15. URL: https://books.google.com.au/books?id=bAjulQ65W-UC&pg=PA1&source=gbs_toc_r&cad=4#v=onepage&q&f=false.

Varadarajan, V.S. (1984). *Lie Groups, Lie Algebras, and Their Representations.* Ed. by S. Axler. Ed. by F.W. Gehring. Ed. by K.A. Ribet. Springer. DOI: 10.1007/978-1-4612-1126-6.