AMSI
VACATION
RESEARCH
SCHOLARSHIPS
2018-2019

Rigid spheres and CR structures in general relativity

Daniel Sykes

Supervised by Professor Gerd Schmalz

University of New England

Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Australian Government
Department of Education and Training

Abstract

We undertake the study of foundational material of CR (Cauchy-Riemann) geometry and shear-free Lorentzian geometry. We highlight the intrinsic nature of their relation and state, following previously completed work of Trautman, Robinson and others, that given an initial CR structure, one can lift the structure to Lorentzian space, which may produce a spacetime. In view of the classification of a 4-parametric family of CR spherical structures, called rigid spheres, we study a paper where Wilson lifts CR spherical structures and produces corresponding spacetimes.

Contents

1 Introduction **2**

2 CR structures **2**

2.1 Examples . . . 6

2.1.1 Example 1 . . . 6

2.1.2 Example 2 . . . 6

2.1.3 Example 3 . . . 7

2.2 Levi form . . . 7

3 Rigid Spheres **8**

3.1 Contact structures . . . 9

3.2 Sasakian structures . . . 10

4 Shearfree Congruences and Lorentzian geometry **10**

4.1 Lorentzian geometry . . . 11

4.2 Shearfree vector fields . . . 11

5 CR structures and Lorentzian space **14**

5.1 Lifting CR structures . . . 14

6 Discussion **15**

1 Introduction

It is well known that there is an intrinsic relation between CR geometry and the framework of general relativity, Lorentzian geometry. A CR structure can be constructed from a Lorentzian structure with a shearfree congruence. Conversely, and more involved in our work, is that by making a choice of a CR structure, one can construct a class of conformal Lorentzian metrics, with a shearfree congruence [11], [10], [8].

It turns out that spherical CR structures with a special choice of 1-form that is determined up to some particular functional change, relate to famous spacetimes, for example the Taub-NUT solution, of interest to physicists [11]. However, the entire class of CR spherical structures has not been lifted to their corresponding spacetimes. We possess sufficiently large 4-parametric family of spherical CR structures, called rigid spheres [6]. With such a restriction, we reduce the freedom from a functional change to only a 4-parmetric family. Using this family one may be able to induce a spacetime, or multiple spacetimes, which may by of particular interest to the physics community, where the spacetime fits particular observational data.

2 CR structures

In this section we outline some foundational CR geometry. We shall give the definition of an almost complex manifold, which is then used to define our CR manifold. Our main construct is based on structure imposed on the tangent bundle of a submanifold embedded in complex euclidean space. We then go on to translate this definition into the language of differential forms, which shall be used repeatedly throughout the report. All proofs provided are merely outlines, to which the interested reader can refer to any of the following resources, among many others, to fill in the details [9], [4] and [3]. However, standard conventions are reversed in [9].

Firstly, we provide two equivalent definitions of an almost complex manifold.

Definition 1 *Consider some manifold M endowed with a smoothly varying field of endomorphisms $\mathcal{J}_p : T_pM \to T_pM$, satisfying $\mathcal{J}_p^2 = -Id$ for each point $p \in M$. We call the pair $(M, \mathcal{J})$*

an almost complex manifold*, where for each $p \in M$, $\mathcal{J}_p$ is referred to as the* complex structure map *for T_pM.*

Definition 2 *Consider some manifold M and a distribution $V \subset \mathbb{C}TM$ (complexified tangent bundle), where $\mathbb{C}T_pM = T_pM \oplus iT_pM$ for all points in M. We call the pair (M, V) an* almost complex manifold *given*

1. $V \oplus \overline{V} = \mathbb{C}TM$
2. $V \cap \overline{V} = \{0\}$

Proposition 1 *Consider (M, $\mathcal{J}$) and (M, V) as above. Both structures are equivalent.*

Proof. First consider (M, V), then define a map $\mathcal{J}^{\mathbb{C}} : \mathbb{C}TM \to \mathbb{C}TM$ by $\mathcal{J}^{\mathbb{C}}|_V = i\,Id$ and $\mathcal{J}^{\mathbb{C}}|_{\bar{V}} = -i\,Id$ for which $(\mathcal{J}^C)^2 = -1$. Then, one can see that $\mathcal{J}^{\mathbb{C}}|_{TM} \subseteq TM$, so (M, $\mathcal{J} = \mathcal{J}^{\mathbb{C}}|_{TM}$) = (M, $\mathcal{J}$). Conversely, given (M, $\mathcal{J}$), extend $\mathcal{J}$ to $\mathcal{J}^{\mathbb{C}}$ and generate the eigendistributions V and $\overline{V}$ for the eigenvalues i and -i respectively. □

These equivalent definitions now provide two alternative paths to defining a CR structure, which too, turn out to be equivalent. However, we will first, prior to defining the CR structures, state a useful lemma without proof, which states involutivity with respect to the distribution V is equivalent to involutivity in terms of the endomorphism $\mathcal{J}$.

Lemma 1 *Given (M, V) or equivalently (M, $\mathcal{J}$), with $X, Y \in TM$*

$$[\Gamma(V), \Gamma(V)] \subset \Gamma(V) \Leftrightarrow [X, \mathcal{J}Y] + [\mathcal{J}X, Y] = \mathcal{J}([X, Y] - [\mathcal{J}X, \mathcal{J}Y])$$

where $\Gamma(V)$ is the set of sections on V, and $[\cdot, \cdot]$ is the usual Lie bracket.

Now, moving on to define our CR structures.

Definition 3 *Let M be a smooth manifold of dimension 2n+1 and suppose we have some smooth distribution $H \subset TM$. We call the triple $(M, H, \mathcal{J})$ a* CR manifold *of hypersurface type, if the following properties are satisfied*

1. $dim_{\mathbb{R}} H = 2n$

2. *$\mathcal{J}$ is a smooth field of endomorphisms, that is, $\mathcal{J}: H \to H$, where $\mathcal{J}^2 = -Id$*

3. *If $X, Y \in \Gamma(H)$, then $[\mathcal{J}X, Y] + [X, \mathcal{J}Y] \in \Gamma(H)$ and $\mathcal{J}([\mathcal{J}X, Y] + [X, \mathcal{J}Y]) = [\mathcal{J}X, \mathcal{J}Y] - [X, Y]$.*

Alternatively, we can define the CR structure as follows.

Definition 4 *Let M be a smooth manifold of dimension 2n+1. We designate (M,V) to be a* CR manifold *of hypersurface type, if $V \subset \mathbb{C}TM$ is a rank n distribution of complex subspaces and the following properties are satisfied*

1. *$V \cap \overline{V} = \{0\}$*

2. *V is involutive, meaning $[\Gamma(V), \Gamma(V)] \subset \Gamma(V)$, where $\Gamma(V)$ is the set of all sections on V.*

Proposition 2 *The constructs of definition 3 and 4 are equivalent.*

Proof. Consider $(M, H, \mathcal{J})$, and extend, via complex linearity, $\mathcal{J}$ to the complexification of H, $H^{\mathbb{C}}$. Now identify V with the eigendistribution for i and likewise $\overline{V}$ for -i. By definition, we have $V \cap \overline{V} = \{0\}$ and by lemma 1, we have the appropriate involutivity property. Conversely, given (M, V), consider a frame $\{L_k\}$ on V and note that since $V \cap \overline{V} = \{0\}$ and $\{L_k\}$ is linearly independent, $\{Re(L_k), Im(L_k)\}$ is linearly independent. Set $H = span_{\mathbb{R}}\{Re(L_k), Im(L_k)\}$ and define $\mathcal{J}$ such that $\mathcal{J}(Re(L_k)) = Im(L_k)$ and $\mathcal{J}(Im(L_k)) = -Re(L_k)$. Extend $\mathcal{J}$ to $H^{\mathbb{C}}$ and assign V and $\overline{V}$ to i and -i eigenvalues, respectively. Again, by lemma 1 we have the necessary involutivity condition. □

We now present an equivalent definition of CR structures in terms of differential forms, as by E. Cartan [9]. However, for our purposes we restrict our definition to a 3-dimensional CR manifold in $\mathbb{C}^2$. It should be noted that the integrability condition of the CR structure becomes vacuous in real dimension 3.

Definition 5 *Suppose (M, H, $\mathcal{J}$) is a CR-manifold, then the CR-structure can be locally encoded in a choice of a real 1-form and complex 1-form, λ and μ respectively, abiding the following conditions*

1. λ is an annihilator of our distribution, i.e. ker(λ) $= H$

2. The forms λ, μ and $\overline{\mu}$ are linearly independent, i.e $\lambda \wedge \mu \wedge \overline{\mu} \neq 0$

3. $\mu\big|_H \circ \mathcal{J} = i\mu\big|_H$.

Any other pair of 1-forms (λ', μ') defines the same CR structure if the following relations are satisfied

$$\lambda' = a\lambda,\ \mu' = b\mu + c\lambda$$

where $a \neq 0$ is a real function, b and c are complex funtions with $b \neq 0$.

Since all of the CR structures to be considered are in $\mathbb{C}^2$, we specialise our definition to suit this case. Note that the mapping $\mathcal{J}_p : T_p\mathbb{C}^2 \to T_p\mathbb{C}^2$ is the ambient complex structure map for $\mathbb{C}^2$ at every point $p \in M$, naturally defined with respect to the coordinates $(z, w) = (x+\mathrm{i}y, u+\mathrm{i}v)$ by

$$\mathcal{J}(\partial_x) = \partial_y,\ \mathcal{J}(\partial_y) = -\partial_x,\ \mathcal{J}(\partial_u) = \partial_v,\ \mathcal{J}(\partial_v) = -\partial_u.$$

Definition 6 *For any $p \in M$, we define the complex tangent space as*

$$H_pM = T_pM \cap \mathcal{J}_p(T_pM)$$

which is the largest $\mathcal{J}_p$- invariant subspace of T_pM, which in $\mathbb{C}^2$ is a complex line.

We now define the complementary part of the tangent space.

Definition 7 *A totally real complement X_pM of T_pM is any complementary subspace of H_pM. That is, $H_pM \oplus X_pM = T_pM$. Consequently, X_pM isomorphic to the quotient space T_pM modulo H_pM, that is*

$$X_pM \cong \frac{T_pM}{H_pM}.$$

By our construction $\mathcal{J}|_H$ is well-defined and due to H_pM being $\mathcal{J}$-invariant, we can extend to the complexification, $H_p{}^{\mathbb{C}}M$, which is $\mathcal{J}^{\mathbb{C}}$-invariant, where $\mathcal{J}^{\mathbb{C}}$ is the extension of $\mathcal{J}$ to the complexified tangent bundle of the ambient space $\mathbb{C}^2$. Consequently, we naturally decompose

${H_p}^{\mathbb{C}}M$ into V_p and $\overline{V}_p$, which form the i and -i eigenspaces with respect to $\mathcal{J}^{\mathbb{C}}|_{H^{\mathbb{C}}}$, respectively. That is,

$$ {H_p}^{\mathbb{C}}M = V_p \oplus \overline{V}_p, $$

where, given $T^{1,0}\mathbb{C}^2$ and $T^{0,1}\mathbb{C}^2$ are the i and $-$i eigenspaces of $\mathcal{J}^{\mathbb{C}}$, we have

$$ V = T^{1,0}\mathbb{C}^2 \cap \mathbb{C}TM \text{ and } \overline{V} = T^{1,0}\mathbb{C}^2 \cap \mathbb{C}TM $$

2.1 Examples

Presently we introduce some examples of CR structures. Our CR structure $M \subset \mathbb{C}^2$ can be locally described, with respect to some chosen coordinates by its defining function

$$ M = \{(z, u + \mathrm{i}v) \in \mathbb{C}^2 : r(x, y, u, v) = 0\} = \{(z, u + \mathrm{i}v) \in \mathbb{C}^2 : v = h(z, u)\} $$

A central notion in our work is that of rigidity of a CR structure. We call M *rigid* with respect to chosen coordinates, if the defining function is independent of u. That is,

$$ M = \{(z, u + \mathrm{i}v) \in \mathbb{C}^2 : v = h(z)\}. $$

2.1.1 Example 1

The simplest example is the following

$$ \mathbb{R}^3 = \{(z, u + \mathrm{i}v) \in \mathbb{C}^2 : v = 0\} $$

to which it can be shown that $H = span\{\partial_x, \partial y\} \cong \mathbb{R}^2 \cong \mathbb{C}$. The associated forms are

$$ \lambda = du, \ \mu = dz $$

2.1.2 Example 2

This example allocates a CR structure to the Heisenberg Group.

$$ \mathbb{H} = \{(z, u + \mathrm{i}v) : v = |z|^2\} = \{(z, u + \mathrm{i}v) : r(z, u, v) = Im(w) - z\overline{z} = \frac{w - \overline{w}}{2\mathrm{i}} - z\overline{z}\} $$

Consider the decomposition of $H^{\mathbb{C}} = V \oplus \overline{V}$, then one can show that $V = span\{\frac{\partial}{\partial z} + \mathrm{i}z\frac{\partial}{\partial u}\}$. It is a tautology that

$$ [\frac{\partial}{\partial \overline{z}} - \mathrm{i}z\frac{\partial}{\partial u}, \frac{\partial}{\partial \overline{z}} - \mathrm{i}z\frac{\partial}{\partial u}] = 0, $$

which reiterates the vacuous nature of the integrability condition for CR structures with complex distribution of rank 1.

Thus, $(\mathbb{H}, V)$ provides a CR structure on the Heisenberg group. Furthermore, the CR structure can be described by

$$\lambda = du - \mathrm{i}\bar{z}dz + \mathrm{i}zd\bar{z},\ \mu = dz.$$

2.1.3 Example 3

Our final example is the sphere $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, which can be endowed with a CR structure with

$$H = span\{u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y} - x\frac{\partial}{\partial u} - y\frac{\partial}{\partial v}, v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y} - y\frac{\partial}{\partial u} - x\frac{\partial}{\partial v}\}$$

2.2 Levi form

We now introduce a particular bilinear form which elicits structural information about the CR manifold.

Definition 8 *Consider some CR manifold $(M, H, \mathcal{J})$ then we define the Levi form of the CR structure at some point $p \in M$ as*

$$\mathcal{L}_p : H_p \times H_p \longrightarrow \frac{T_pM}{H_pM} \cong X_pM$$

$$(X_p, Y_p) \longmapsto \pi_p([X, Y])$$

where $X, Y \in \Gamma(H)$ are extensions of X_p and Y_p respectively, and $\pi_p : T_pM \longrightarrow \frac{T_pM}{H_pM}$ is the canonical projection mapping under the factorisation.

Considering the definition of the Levi form, and seeing that the form maps into a totally real complement, X, of the complex distribution H, and coupling this with the fact that any $\omega \in X^*M$ annihilates H, we see that the Levi form is designed to measure the degree to which our complex distribution fails to be involutive.

We now introduce some structural notions involving the Levi form.

Definition 9 *Consider our CR manifold $(M, H, \mathcal{J})$ we say that the CR structure is*

(a) nondegenerate *at $p \in M$, if $\mathcal{L}_p(X_p, Y_p) = 0$ for all $Y \in \Gamma(H)$, then $X_p = 0$.*

(b) strictly pseudoconvex *at $p \in M$, if $\forall X_p \neq 0 \in H_p$ $\mathcal{L}_p(X_p, \mathcal{J}X_p) > 0$ or $\forall X_p \neq 0 \in H_p$ $\mathcal{L}_p(X_p, \mathcal{J}X_p) < 0$.*

3 Rigid Spheres

In this section we introduce the notion of rigid CR structures. Geometrically, these structures can be characterised by translational symmetry in some transversal direction to the complex tangent space $H_pM \subset T_pM$.

Considering $(z, w) = (x + iy, u + iv) \in \mathbb{C}^2$, then under appropriate choice of coordinates we can write the defining equation of the CR structure as $v = h(z)$. Then, the translational symmetry can be seen to be generated as the flow of the ∂_u, evident from the defining functions independence of u.

Definition 10 *Consider any CR manifold M in $\mathbb{C}^2$, then we say that M is a* Rigid sphere *if M satifies the following*

(a) The defining function is independent of u. i.e. $v = h(z)$

(b) The CR structure is locally equivalent to the CR structure of the sphere.

A complete classification of all rigid spheres was provided in 2015 by Ezhov and Schmalz [5].

We shall not restate the theorem, but highlight the crux. They presented a closed formula for every rigid sphere, which is described by a 4-parameter family. The formula is stated as follows

$$(1 + 2\phi|z|^2)\frac{\sin 2rv}{2r} - e^{-2\theta v}|z|^2 - (\phi + \overline{a}z + a\overline{z} + 4\phi(\phi - \theta)|z|^2)\frac{e^{-2\theta v} - \cos 2rv + \frac{\theta \sin 2rv}{r}}{r^2 + \theta^2} = 0$$

where a is a complex parameter and r^2, θ, ϕ are real parameters, which are related by a cubic equation.

Rigid structures are all naturally related to a special type of CR structure called a Sasakian structure. We shall now present our rigid structures through the framework of this conceptualisation.

We have already encountered a rigid sphere, namely the Heisenberg group. Clearly the Heisenberg group's defining function, $Im(w) = |z|^2$, is independent of u and thus satifies the first condition. Moreover, the Heisenberg group is locally equivalent to the sphere. One such mapping that attains such a relation is the following, which is a generalisation of the Cayley mappings from undergraduate complex analysis,

$$z' = \frac{2z}{w + \mathrm{i}},\ w' = \frac{w - \mathrm{i}}{w + \mathrm{i}}.$$

3.1 Contact structures

Intimately related to CR structures and forming an underlying foundation for the Sasakian structure are those called contact structures.

Definition 11 *A smooth manifold M^{2n+1} equipped with a corank-1 distribution $H \subseteq TM$ is called a contact structure, if there is a 1-form λ, called a contact form, satisfying*

(a) λ is the annihilator of our distribution, i.e. $ker(\lambda) = H$

(b) $\lambda \wedge d\lambda \wedge \cdots \wedge d\lambda = \lambda \wedge (d\lambda)^n \neq 0$.

An interesting point to note, is that one can show $\lambda \wedge d\lambda \equiv 0$, with λ the defining 1-form of a distribution H in TM is equivalent to the integrability of the distribution. Consequently, a contact structure, by the very definition, is maximally non-integrable.

Furthermore, and quite importantly, every contact manifold (M, H) with a choice of contact form λ has a distinguished vector field, the Reeb vector field.

Definition 12 *For any contact manifold (M, H), with λ the corresponding contact form, there exists a unique vector field $Z \in \Gamma(TM)$, such that*

$$Z \lrcorner\, d\lambda = 0,\ Z \lrcorner\, \lambda = 1.$$

A particularly important fact is the following.

Proposition 3 *If $(M^3, [(\lambda, \mu)])$ is a CR structure, with nondegenerate Levi form, then the complex distribution H is a contact structure, with corresponding contact form λ.*

Proof. One can define a nondegenerate CR structure by the generators of $V = span\{\partial = X - \mathrm{i}\mathcal{J}X\}$, $\overline{V} = span\{\overline{\partial} = X + \mathrm{i}\mathcal{J}X\}$, and some complementary vector field $\partial_0 = \mathrm{i}[\partial, \overline{\partial}] = -2[X, \mathcal{J}X]$, where $(\partial, \overline{\partial}, \partial_0)$ is a frame for $\mathbb{C}TM$, with corresponding coframe $(\lambda, \mu, \overline{\mu})$. If $\lambda \wedge d\lambda = 0$, then $d\lambda = c\mu \wedge \lambda + \overline{c\mu} \wedge \lambda$. But $d\lambda(\partial, \overline{\partial}) = \frac{1}{2}(\partial\lambda(\overline{\partial}) - \overline{\partial}\lambda(\partial) - \lambda([\partial, \overline{\partial}])) = -\frac{1}{2}\lambda([\partial, \overline{\partial}]) = \mathcal{L}_p(X, \mathcal{J}X) = 0$. □

3.2 Sasakian structures

Usually Sasakian structures are derived via Kählerian structures, which are integrable, almost complex, Hermitian structures. However, under the condition of strict pseudoconvexity, the Sasakian notion becomes far more transparent. We will state a theorem from [1], which drastically simplifies the characterisation of Sasakian structures under these conditions. Firstly, we present the notion of an infinitesimal CR-automorphism and then present the theorem.

Definition 13 *Consider $Z \in \Gamma(TM)$, then we call Z an* infinitesimal CR-automorphism *if and only if $\mathscr{L}_Z\Gamma(V) \subseteq \Gamma(V)$.*

Theorem 1 *Let $(M, [(\lambda, \mu)])$ be a strictly pseudoconvex CR manifold of dimension $2n + 1$, where H is a contact structure with contact form λ and corresponding Reeb vector field Z. If Z is an infinitesimal CR-automorphism, then M is Sasakian.*

4 Shearfree Congruences and Lorentzian geometry

It is well known in the literature that the condition of shearfreeness of the Lorentzian metric is strong enough to make solving Einstein's equations tractable, yet is not too strong, as to yield nothing but trivial solutions.

4.1 Lorentzian geometry

The mathematical framework with which general relativity is set, is that of Lorentzian geometry. For the sake of pragmatism, we restrict our discussion to 4-dimensional Lorentzian structures.

Definition 14 *Take a 4-dimensional smooth manifold, $\mathcal{M}$, with a choice of a smoothly varying Lorentz metric, g, which assigns to every tangent space T_pM a symmetric, nondegenerate bilinear form in which the signature of the metric is (3,1). We call the pair $(\mathcal{M}, g)$ a Lorentzian manifold.*

The simplest example is that of Minkowski space, that is $\mathbb{R}^4$ with coordinates (x, y, u, r) and Lorentzian metric

$$g = dx^2 + dy^2 + du^2 - dr^2.$$

Furthermore, an interesting consequence by removing the positivity condition from the metric, is that lengths no longer need be positive. In particular, Lorentzian spaces possess nonzero vectors with zero length with respect to the metric, which are called null vectors.

Definition 15 *Consider some vector $v \in T_p\mathcal{M}$ on some Lorentzian manifold $(\mathcal{M}, g)$, then this vector is called* null, *if $g(v, v) = 0$. This notion naturally extends to a vector field $k \in \Gamma(TM)$ and thus we call a vector field* null *if $g_p(k_p, k_p) = 0, \forall\ p \in \mathcal{M}$, where $k_p \in T_p\mathcal{M}$*

We now introduce the following proposition as a matter of notational convenience for when we introduce a particular class of metrics.

Proposition 4 *For any Lorentzian manifold $(\mathcal{M}, g)$ of signature (3,1), there exists a particular choice of frame for the complexified tangent bundle, and corresponding complex coframe $(\omega^1, \omega^2, \omega^3, \omega^4)$, in which the Lorentzian metric takes the following form*

$$g = 2(\omega^1\omega^2 + \omega^3\omega^4)$$

4.2 Shearfree vector fields

In this section we define the notion of a vector field and metric being shearfree, and define a related conformal class of Lorentzian metrics, which are unique up to scaling by a postive real function called a conformal factor.

Definition 16 *Consider a smooth manifold M endowed with some Lorentzian metric g, and furthermore consider any vector field $k \in \Gamma(TM)$. Then k is said to be* conformal Killing *if*

$$\mathscr{L}_k g = \rho g$$

where ρ is some real valued function on M, and Killing *if $\rho = 0$, i.e.*

$$\mathscr{L}_k g = 0.$$

We now generalise this notion to that which rests as a fundamental assumption about the attributes of light rays in general relativity.

Definition 17 *Take (M, g) and consider some nonzero $k \in \Gamma(TM)$. We call k a* shearfree *vector field if it satisfies the following conditions.*

(i) The vector field k is null, i.e. $g(k, k) = 0$.

(ii) The metric g changes conformally under the flow of k if restricted to the subspaces of

$$k^{\perp} = \{X \in \Gamma(TM) : g(k, X) = 0\}.$$

More succinctly, $\mathscr{L}_k g = \rho g + g(k, \cdot) \vee \psi$, where ρ is a real function on $\mathcal{M}$ and ψ is a 1-form.

A shearfree foliation by integral curves of some shearfree vector field on a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$ is called a shearfree congruence in physics. This foliation can be physically interpreted as the propagation of light rays.

The next theorem yields an important property to the way that null, shearfree integral curves propagate through Lorentzian space.

Proposition 5 *Any null, shearfree vector field k on a Lorentzian manifold is geodetic, that is*

$$\nabla_k k = f k$$

where f is a real valued function and ∇ is the Levi-Civita connection. Furthermore, k can be rescaled so that

$$\nabla_k k = 0$$

which we shall call autoparallel.

Firstly we state a lemma required for the proof of the above statement. We omit the respective proof, but refer the reader to [2] or [7].

Lemma 2 *Take $(\mathcal{M}, g)$ a Lorentzian manifold, then for every $X \in \Gamma(TM)$, there exists an operator called the* Nomizu operator

$$L_X : \Gamma(TM) \longrightarrow \Gamma(TM),\ Y \longmapsto L_X Y := -\nabla_X Y$$

for any $Y \in \Gamma(TM)$, that satisfies the following properties

(a) *If k is a null vector field, then $L_k^* k = 0$*

(b) *Given $X, Y, V \in \Gamma(TM)$, then $\mathscr{L}_V g(X, Y) = -2g(X, L_V^s Y)$*

where L_X^* is the g-adjoint operator defined by

$$g(L_X^* V, W) = g(V, L_X W),\ \forall X, Y, W \in \Gamma(TM)$$

and L_X decomposes into its symmetric and anti-symmetric parts

$$L_X^s := \frac{1}{2}(L_X + L_X^*) \text{ and } L_X^a := \frac{1}{2}(L_X - L_X^*)$$

Utilising this lemma, we shall now prove the first part of proposition 5. We shall omit part of the proof which shows rescaling, once again referring the reader to the [2] and [7]

Proof. Since k is null, $L_k^* k = 0$, and regarding Lemma 2 (b), letting $V = Y = k$,

$$\mathscr{L}_k g(X, k) = -2g(X, \frac{1}{2}(L_k k + L_k^* k)) = g(X, \nabla_k k).$$

However, k is shearfree and thus there exists a nonzero real ρ and a 1-form ψ, such that

$$\mathscr{L}_k g = (\rho g + g(k, \cdot) \vee \psi)(X, k)$$

$$= \rho g(X, k) + g(X, k)\psi(k) = g(X, (\rho + \psi(k))k).$$

Thus,

$$g(X, \nabla_k k) = g(X, (\rho + \psi(k))k),$$

$$\nabla_k k = (\rho + \psi(k))k = fk$$

where $f = \rho + \psi(k)$. □

5 CR structures and Lorentzian space

The relation between 3-dimensional CR structures and 4-dimensional Lorentzian space equipped with a shearfree congruence has been thoroughly studied from both directions. That is, constructing CR manifolds from a Lorentzian manifold with shearfree congruence, and conversely constructing Lorentzian space from an underlying CR structure. Our focus is on the latter, in which we start with a particular CR structure.

5.1 Lifting CR structures

It is well known [8] that starting with a 3-dimensional CR structure $(M, [(\lambda, \mu)])$, one can construct a conformal class of Lorentzian metrics on the line bundle $\mathcal{M} = M \times \mathbb{R}$, possessing a shearfree congruence.

We introduce the conformal class of Lorentzian metrics on $\mathcal{M}$ to be

$$g = 2P^2(\mu\overline{\mu} + \lambda(dr + W\mu + \overline{W}\overline{\mu} + D\lambda))$$

where r is the coordinate function of the fibres $\mathbb{R}$ in $\mathcal{M}$, P and D are a real functions, with P nonzero, and W is a complex valued function on $\mathcal{M}$. The function P^2 is called the conformal factor of the metric, and each class of metrics is defined up to some conformal change via P^2.

One can readily find suitable choices of pullbacks of the differential forms, that show the above class of metrics, is in the form presented in proposition 4.

The next proposition indicates a natural choice, k, of shearfree vector field to attain (g, k) as a shearfree metric. It also states that a family of shearfree metrics is CR invariant meaning each class is uniquely generated up to the transformations

$$\lambda' = a\lambda, \; \mu' = b\mu + c\lambda.$$

Proposition 6 *The metric*

$$g = 2P^2(\mu\overline{\mu} + \lambda(dr + W\mu + \overline{W}\overline{\mu} + H\lambda))$$

satisfies the following properties

(a) $k = \partial_r$ *is shearfree, and thus* (g, k) *is a shearfree metric*

(b) The family of shearfree metrics is CR invariant.

We now introduce the notion which interrelates a particular choice of representative (λ, μ) of a CR structure to a family of Lorentzian spaces.

Definition 18 *Take* $(M, [(\lambda, \mu)])$ *to be a CR structure and fix a representative* (λ, μ) *for the structure. Then the pair* (g, ∂_r) *defined on* $\mathcal{M}$ *is called a lift of a CR structure into Lorentzian space.*

Finally, we state a theorem given in [8] which wraps up our discussion of lifting of CR structures into Lorentzian space.

Theorem 2 *Let* $(\mathcal{M}, g)$ *be a 4-dimensional Lorentzian manifold. Suppose* $\mathcal{M}$ *is equipped with a shearfree congruence and* $\mathcal{M}$ *is locally the trivial line bundle of a 3-dimensional CR manifold, then the manifold* $(\mathcal{M}, g)$*, alongside the shearfree congruence, uniquely determines the CR structure* $(M, [(\lambda, \mu)])$ *up to choice of representative. Furthermore, if* r *is the real coordinate corresponding to* $\mathbb{R}$ *in the line bundle, such that* $k = \partial_r$ *is tangent to the shearfree congruence, then the Lorentzian metric* g *on* $\mathcal{M}$ *can be locally represented by*

$$g = 2P^2(\mu\overline{\mu} + \lambda(dr + W\mu + \overline{W}\overline{\mu} + H\lambda))$$

with specific functions P, W, H *depending on the choice of representative* (μ, λ) *for the underlying CR structure.*

6 Discussion

Wilson (1989) [13] studied in his thesis, a family of spacetimes which all possess an underlying spherical CR structure. In particular, he lifts rigid spheres to generate the Lorentzian metric. Wilson used different realisations of the sphere with differing coframes, and since a complete classification of rigid spheres was then unknown, he used a list of CR spherical structures developed by E. Cartan [5], which impeded the extent of his results. Consequently, instead of an exhaustive classification of spacetimes related to CR spherical structures, has on potentially only shown a small subclass of structures are related to the spacetimes.

Subsequent work, by Robinson and Wilson (1991) [12], took a spherical CR structure and generated a new spacetime called the generalised Taub-NUT solution. This backs up the claim that Wilson's thesis is not exhaustive, and provides us with an interesting template with which to study.

As noted, Ezhov and Schmalz (2015) [6] gave a complete classification of the rigid spheres in $\mathbb{C}^2$, and thus we have at our disposal a large family of CR structures which have never been related to a corresponding spacetime, and consequently could generate new spacetimes, which may yield interest from both the mathematics and physics communities.

It is our goal to study these results and make transparent exactly which CR structures, in particular rigid spheres, have been related to their respective spacetimes, and then to extend upon this work by deriving new spacetimes. In particular, the aim of this current research period was to lay the foundations of this problem.

References

[1] D. V. Alekseevsky, V. Cortés, K. Hasegawa, and Y. Kamishima. "Homogeneous locally conformally Kähler and Sasaki manifolds". In: *Internat. J. Math.* 26.6 (2015), pp. 1541001, 29.

[2] Dmitri V. Alekseevsky, Masoud Ganji, and Gerd Schmalz. "CR-Geometry and Shearfree Lorentzian Geometry". In: *Geometric Complex Analysis*. Ed. by Jisoo Byun, Hong Rae Cho, Sung Yeon Kim, Kang-Hyurk Lee, and Jong-Do Park. Singapore: Springer Singapore, 2018, pp. 11–22.

[3] M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild. *Real submanifolds in complex space and their mappings*. Vol. 47. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1999, pp. xii+404.

[4] Albert Boggess. *CR manifolds and the tangential Cauchy-Riemann complex*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991, pp. xviii+364.

[5] Élie Cartan. "Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes II". In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2)* 1.4 (1932), pp. 333–354.

[6] Vladimir Ezhov and Gerd Schmalz. "Explicit description of spherical rigid hypersurfaces in". In: *Complex Analysis and its Synergies* 1.1 (2015), p. 2.

[7] Masoud Ganji. "Shearfree Lorentzian geometry and CR geometry". In: (Sept. 2018).

[8] C. Denson Hill, Jerzy Lewandowski, and Paweł Nurowski. "Einstein's equations and the embedding of 3-dimensional CR manifolds". In: *Indiana Univ. Math. J.* 57.7 (2008), pp. 3131–3176.

[9] Howard Jacobowitz. *An introduction to CR structures*. Vol. 32. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1990, pp. x+237.

[10] I. Robinson and A. Trautman. *Optical geometry*. New Theories in Physics: Proceedings of the XI Warsaw Symposium on Elementary Particle Physics. World Scientific, 1989, pp. 457–497.

[11] Ivor Robinson and Andrzej Trautman. "Cauchy-Riemann structures in optical geometry". In: *Proceedings of the fourth Marcel Grossmann meeting on general relativity, Part A, B (Rome, 1985)*. North-Holland, Amsterdam, 1986, pp. 317–324.

[12] Ivor Robinson and Edward P. Wilson. "The generalized Taub-NUT congruence in Minkowski space". In: *Gen. Relativity Gravitation* 25.3 (1993), pp. 225–244.

[13] E. P. Wilson. *Some lifting of Cauchy Riemann geometry to Minkowski space*. PhD Dissertation, The University of Texas, Dallas, 1988.