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Escher Configurations On Triangles

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1 Abstract

In his work on tessellations of the plane, the famous artist Maurits Cornelis Escher was led to consider configurations on a triangle across which was drawn three concurrent lies. Escher used the standard (5,4,3) right-angled triangle, where each of the three concurrent lines intersect the edges at integer distance from adjacent vertices. Escher obtained 12 such configurations. In this project we investigate the existence of such *Escher* configurations in a more general context. We consider triangles whose edges are divided into segments of equal length, but while the segments are of equal length on each edge, they may vary in length from edge to edge. The three concurrent lines must intersect the edges at the points determined by these subdivisions. In general terms, we found in this project that for some subdivisions Escher configurations do not exist, but on the other hand, in certain cases Escher configurations are by no means rare. Indeed, we find infinitely many such configurations.

2 Introduction

In his work on tessellations of the plane, Escher examined configurations on a triangle in the following sense (see [3]). For integers $p, q, r > 1$, consider a triangle ABC with edges a, b, c. Divide the edge a (resp. b, resp. c) into p (resp. q, resp. r) segments of equal length. Now consider points $A', A'' \in a$, $B', B'' \in b$, $C', C'' \in c$ that each lie at the end of one of the segments just defined, as in Figure 1. Escher was interested in the case where the lines $A''B', B''C', C''A'$ are concurrent.

Definition. We say that the different points $A', A'', B', B'', C', C''$ are an *Escher solution for the triple* (p, q, r) if the lines $A''B', B''C', C''A'$ are concurrent.

Note that we allow some of the points $A', A'', B', B'', C', C''$ to coincide with some of the vertices of the triangle ABC. However, the condition that these points are different guarantees that the point of intersection of the lines $A''B'$, $B''C'$ and $C''A'$ always lies in the interior of the triangle.

Escher studied the case $(p, q, r) = (5, 4, 3)$, envisaged as a right angled triangle of side lengths $5, 4, 3$. He found that there were 12 solutions. Klamkin and Liu verified analytically

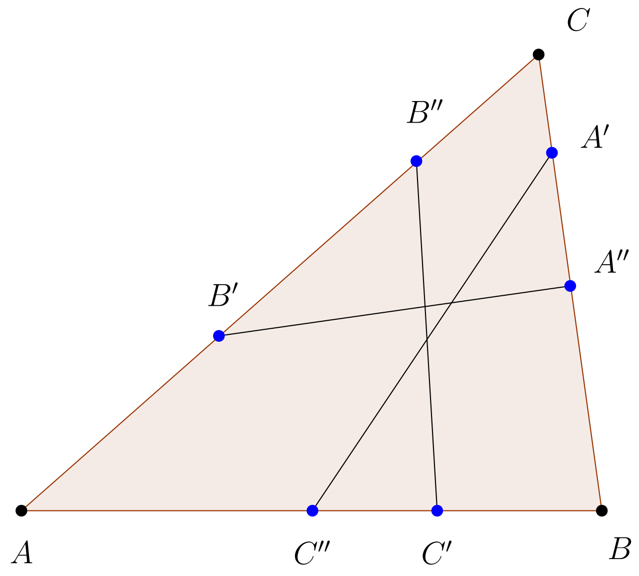

Figure 1: The general case

that Escher's list is indeed correct and complete [2]. To do this, they first established the following result. Let

$$\overrightarrow{AB'} = x_b\,\overrightarrow{CB'}, \qquad \overrightarrow{CA'} = x_a\,\overrightarrow{BA'}, \qquad \overrightarrow{BC'} = x_c\,\overrightarrow{AC'},$$
$$y_b\,\overrightarrow{AB''} = \overrightarrow{CB''}, \qquad y_a\,\overrightarrow{CA''} = \overrightarrow{BA''}, \qquad y_c\,\overrightarrow{BC''} = \overrightarrow{AC''}.$$

Klamkin and Liu's Theorem. *The lines $A''B', B''C', C''A'$ are concurrent if and only if*

$$x_a x_b x_c + y_a y_b y_c - x_a y_a - x_b y_b - x_c y_c + 1 = 0. \tag{1}$$

The object of our study is to commence a general investigation by examining the cases $(p, 2, 2)$ and $(p, 3, 2)$. Here are our results

Theorem 1. *For an integer $p \geq 2$, there exists an Escher solution for the triple $(p, 2, 2)$ if and only if p^2 has a divisor n with $n < p \leq 2n$.*

Corollary 1. *If p is an odd prime, then there exists no Escher solution for the triple $(p, 2, 2)$.*

Theorem 2. *If $p > 5$ is a prime, then there exists no Escher solution for the triple $(p, 3, 2)$.*

3 The $(p, 2, 2)$ case.

Let us consider the $(p, 2, 2)$ case, where p is an integer with $p \geq 2$. Here the edges c and b are each divided into two equal segments and the edge a is divided into p equal segments. By symmetry, we may assume without loss of generality that $B' = A$ and $C' = B$, and thus necessarily B'' is the midpoint of b and C'' is the midpoint of c.

First consider the case where p is even, say $p = 2n$. In this case an obvious Escher solution is obtained by taking $A' = C$ and A'' to be the midpoint of a; the three concurrent lines are thus the medians of the triangle. (The case $p = 6$ is shown in Figure 2.) Moreover, the condition $n < p \leq 2n$ is obviously satisfied. Hence Theorem 1 holds when p is even. So we may assume that p is odd.

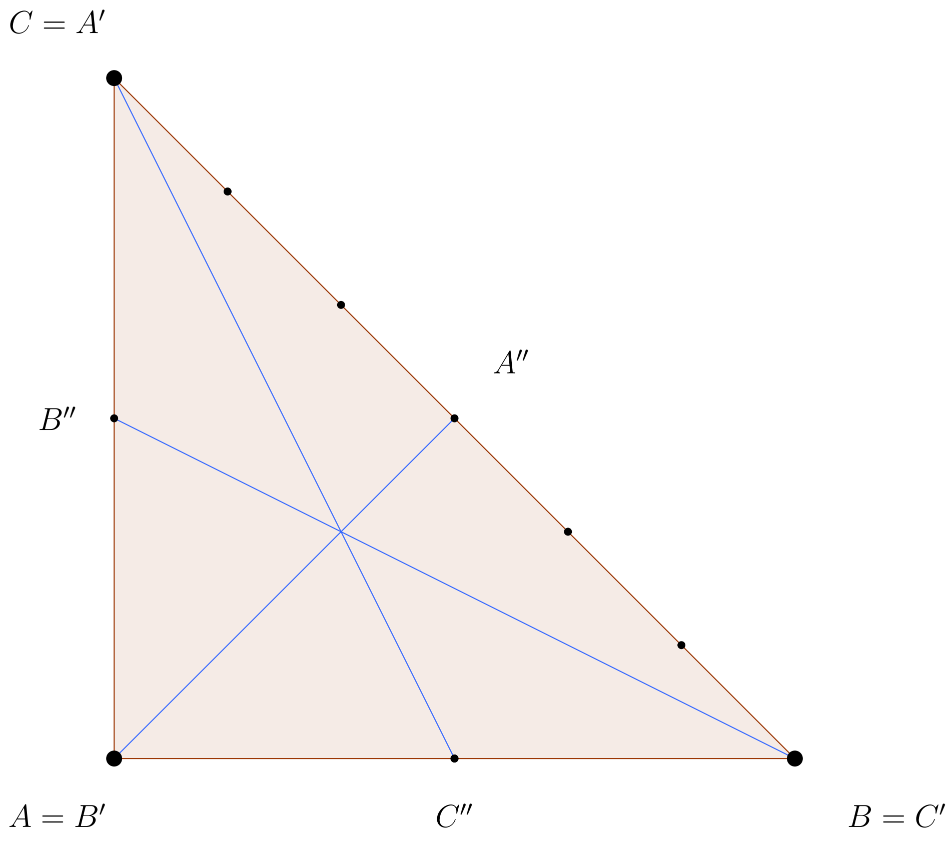

Figure 2: The $(6, 2, 2)$ case

To facilitate calculations, let us fix the vertices by taking A to be the point $(0,0)$, B to be $(2,0)$ and C to be $(0,2)$. So $B'' = (0,1)$ and $C'' = (1,0)$. Now consider points A', A'' on edge a. Their general form is as follows:

$$A' = \left(\frac{2m}{p}, 2 - \frac{2m}{p}\right), \qquad A'' = \left(\frac{2n}{p}, 2 - \frac{2n}{p}\right),$$

where m and n are integers between 0 and $p-1$ (inclusive) and $m < n$. It is then easy to determine the variables in the Klamkin–Liu Theorem:

$$x_b = x_c = 0,$$

$$y_b = y_c = -1,$$

$$x_a = \frac{-m}{(p-m)}, \qquad y_a = \frac{-(p-n)}{n}.$$

Now putting all these values into Equation (1), a few parts cancel out and we are left with the following:

$$\frac{-(p-n)}{n} - \frac{m(p-n)}{n(p-m)} + 1 = 0,$$

which nicely simplifies down to:

$$(2n-p)p = mn. \tag{2}$$

Thus, in summary, by the Klamkin–Liu Theorem, there is an Escher solution to the $(p,2,2)$ case if and only if there exists integers m, n with $0 \le m < n \le p-1$ such that (2) holds. Notice that on the one hand, if such integers exist, then from (2), n is a divisor of p^2, and since the right-hand-side of (2) is nonnegative, we have $n < p \le 2n$. Conversely, suppose that there is a positive integer n such that n is a divisor of p^2 and $n < p \le 2n$. Setting

$$m = \frac{(2n-p)p}{n} = 2p - \frac{p^2}{n}$$

we obtains nonnegative integers m, n satisfying (2). In order to conclude that an Escher solution exists, it remains to show that $m < n$. Consider the quadratic polynomial

$$n^2 - 2np + p^2 = (n-p)^2$$

in the variable n. Since $n < p$ by hypothesis, we have $n^2 - 2np + p^2 > 0$ and so $(2n-p)p < n^2$ and thus

$$m = \frac{(2n-p)p}{n} < n,$$

and an Escher solution exists. This completes the proof of Theorem 1.

Corollary 1 follows easily from Theorem 1. Indeed, suppose that p is an odd prime and that p^2 has a divisor n with $n < p \leq 2n$. Then as p^2 only has divisors $1, p, p^2$ and $n < p$, we necessarily have $n = 1$. But then $p \leq 2n$ gives $p \leq 2$, contrary to the assumption that p is an odd prime.

Examples. In the case of the composite number $p = 15$, there is an Escher solution. Indeed, taking $n = 9$ we see that $p^2 = 225$ is divisible by n and $2n = 18$ is greater than p.

However, for the composite number $p = 21$, there is no Escher solution. Indeed, the divisors of p^2 are $1, 3, 7, 9, 21, 49, 63, 147, 441$, so the divisor are less than 21 are $1, 3, 7, 9$. But for none of these possible values of n does one have $21 \leq 2n$.

Remark. From Theorem 1, there are Escher solutions for each of the cases $(p, 2, 2)$ where p is even. So there are infinitely many Escher solutions. Note that there are also infinitely many cases $(p, 2, 2)$ where p is odd that have Escher solutions. Indeed, consider $p = 3^i 5$ and let $n = 3^{i+1}$. So p^2 is divisible by n and furthermore, $n < p$ and $2n = 6p/5 > p$. So an Escher solution exists for $(p, 2, 2)$.

More generally, if an Escher solution exists for a triple $(p, 2, 2)$, it also exists for any triple $(kp, 2, 2)$, where $k \in \mathbb{N}$. It follows that the set of those p for which an Escher solution exists for a triple $(p, 2, 2)$ is the set of all multiples of the elements from a certain subset $\mathbb{S}$ of $\mathbb{N}$. It is not hard to see that this subset is infinite; for example, for any odd prime number i, take a prime j strictly between i and $2i$ (such a j exists by Bertrand's postulate [1, Chapter 2]). Then (2) has a solution $p = ij$, $n = i^2$, $m = (2i - j)j$. Using Theorem 1 we find that $\mathbb{S} = \{2, 15, 35, 63, 77, 91, 99, \dots\}$.

4 The $(p, 3, 2)$ case.

Having dealt with the $(p, 2, 2)$ cases, we will now examine the $(p, 3, 2)$ cases. The $(p, 3, 2)$ cases are a little trickier as there are more ways we are able to label our triangle. Indeed, there are 7 different ways we can configure our triangle so that the lines $A''B'$, $B''C'$, $C''A'$ all cross through our triangle and not along any edges. The possible configurations are shown in Figure 4.

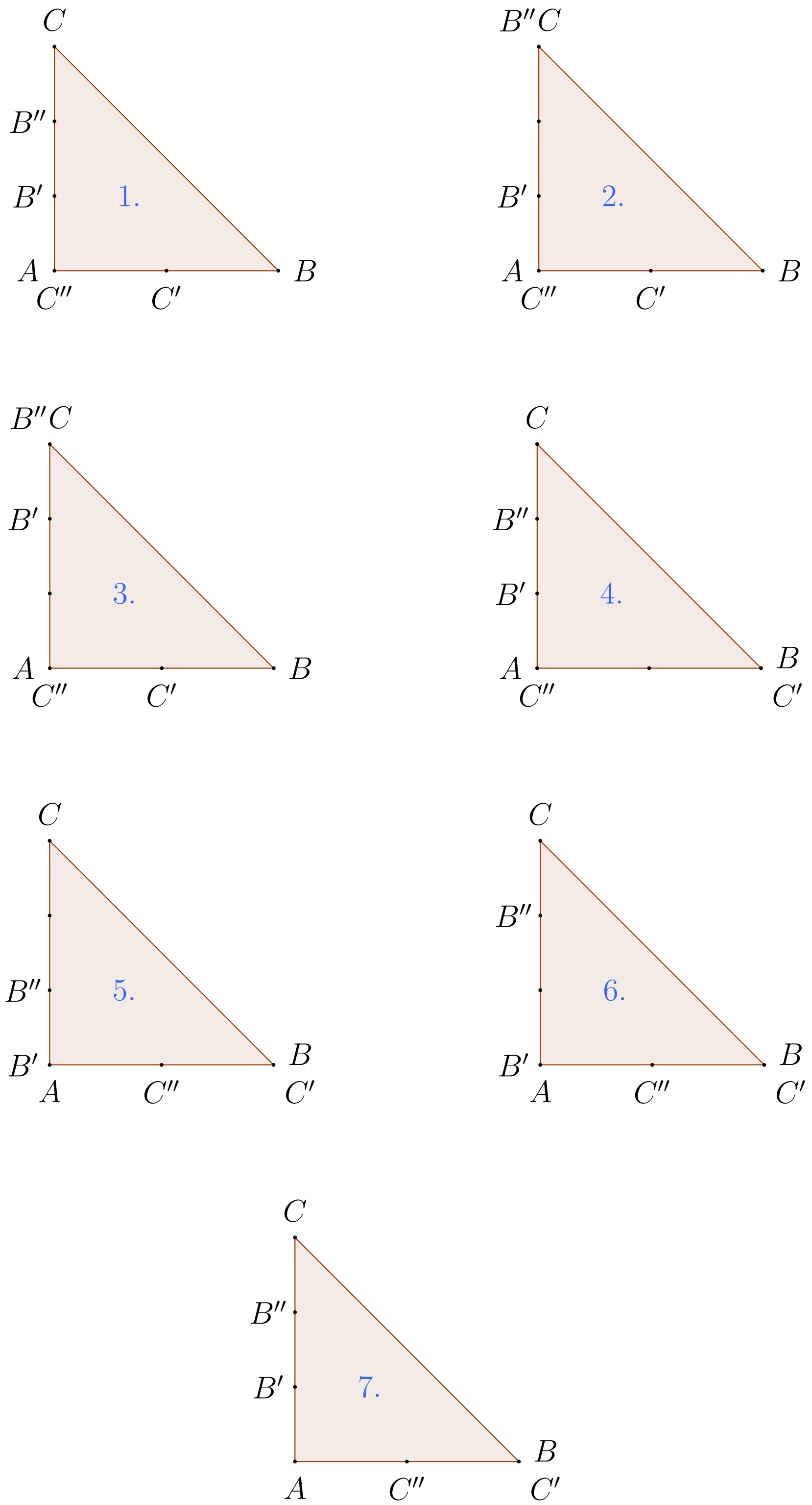

Figure 3: The seven $(p, 3, 2)$ triangles

We will start with triangle 1. We will only go into the details for this triangle; the other triangles are treated in exactly the same manner. Similar to how we solved the $(p, 2, 2)$ case, we take A to be the point $(0, 0)$, B to be $(2, 0)$, C to be $(0, 3)$, and we set the points A' and A'' to be the following:

$$A' = \left(\frac{2m}{p}, 3 - \frac{3m}{p}\right),$$
$$A'' = \left(\frac{2n}{p}, 3 - \frac{3n}{p}\right),$$

where m and n are integers between 0 and $p-1$ (inclusive) and $m < n$. For triangle 1, we have $C' = (1, 0)$, $B' = (0, 1)$ and $B'' = (0, 2)$. This gives us the following values:

$$x_a = \frac{-m}{(p-m)} \qquad y_a = \frac{-(p-n)}{n}$$
$$x_b = \frac{-1}{2} \qquad y_b = \frac{-1}{2}$$
$$x_c = -1 \qquad y_c = 0.$$

After substituting these values into Equation (1) and simplifying we get:

$$3np - 4mp = mn. \tag{3}$$

In the same fashion, for the other triangles, Equation (1) reduces to the following respective conditions:

$$2np - 2mp = mn \tag{4}$$
$$\frac{np - mp}{2} = mn \tag{5}$$
$$4mp - 3np = mn \tag{6}$$
$$\frac{p}{2}(3n + m - 2p) = mn \tag{7}$$
$$3np - mp - p^2 = mn \tag{8}$$
$$p(5n - 2m - 2p) = mn. \tag{9}$$

We first examine the case when $m = 0$. It is easy to see that in this case, Equations (3), (4), (5) and (6) have no solution with $0 < n < p$. The solution sets of Equations (7), (8) and (9) are given by $(m, n, p) = (0, 2k, 3k)$, $(m, n, p) = (0, k, 3k)$ and $(m, n, p) = (0, 2k, 5k)$ respectively, where $k \in \mathbb{N}$.

Now suppose that p is prime. Each of the Equations (3-9) imply that p is a divisor of mn. But this is impossible if $0 < m < n < p$, and for $m = 0$, from the previous paragraph, no solution exists if the prime p is greater than 5.

This concludes the proof of Theorem 2.

Examples. We now give examples showing that in each of the above 7 cases there is a composite value of p for which there is an Escher solution.

1. For triangle 1, let $p = 21$, $m = 9$ and $n = 14$. Equation (3) holds.
2. For triangle 2, let $p = 15$, $m = 5$ and $n = 6$. Equation (4) holds.
3. For triangle 3, let $p = 15$, $m = 3$ and $n = 5$. Equation (5) holds.
4. For triangle 4, let $p = 15$, $m = 5$ and $n = 6$. Equation (6) holds.
5. For triangle 5, let $p = 15$, $m = 10$ and $n = 12$. Equation (7) holds.
6. For triangle 6, let $p = 15$, $m = 9$ and $n = 10$. Equation (8) holds.
7. For triangle 7, let $p = 21$, $m = 7$ and $n = 12$. Equation (9) holds.

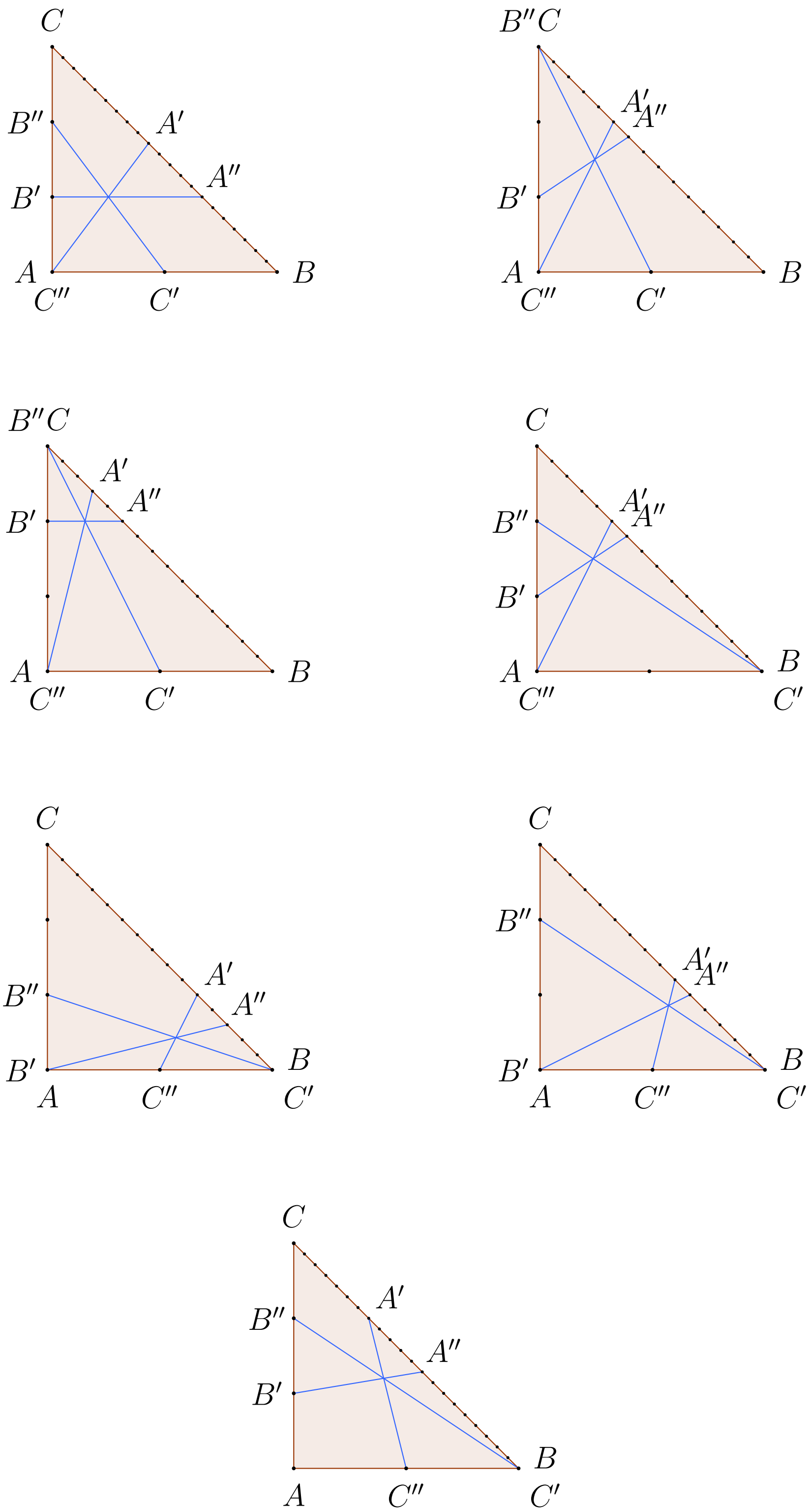

Figure 4: The seven $(p, 3, 2)$ example triangles

5 Discussion and Conclusion

We have determined the values of p for which the $(p, 2, 2)$ case has Escher solutions. In particular, we have shown that there are no Escher solutions for $(p, 2, 2)$ when p is prime and > 2. We have also shown that there are no Escher solutions for $(p, 3, 2)$ when p is prime and > 5. Of course, much remains to be explored. Specific questions that remain open include:

1. Is it true that for all primes $q > 2$, the triple (p, p, q) has no Escher solution for p a sufficiently large prime?

2. Is the smallest element of a triple having no Escher solution always prime?

Statement of Authorship

The supervisors Grant Cairns and Yuri Nikolayevsky proposed this project and outlined how the $(p, 2, 2)$ case might be approached using the Klamkin–Liu Theorem.

The vacation scholar, Aidan Brohm, first conducted the $(p, 2, 2)$ case and then systematically examined the $(p, 3, 2)$, performing the calculations for each of the subcases.

Cairns and Nikolayevsky carefully checked the calculations of Brohm, and read and provided specific feedback on the project report.

References

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[3] Doris Schattschneider, *M. C. Escher: Visions of Symmetry*, Harry N. Abrams 2004.