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Curve Shortening Flow and Grayson's Theorem

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1 Abstract

This report aims to give an overview of the curve shortening flow in two dimensions, a geometric partial differential equation that pertains to evolving curves in the plane; as well as Grayson's theorem (1987) which states that any closed simple curve exposed to curve shortening flow will in finite time shrink to a round point, and then disappear.

2 Introduction

The curve shortening flow, abbreviated CSF, is a one dimensional geometric partial differential equation (PDE), and Grayson's theorem [4] concerns this flow, stating that any closed, simple and compact curve that is subjected to this flow will in finite time collapse into a perfectly round singularity, and not develop any intersections or other singularities until this final time. The reasons for studying this flow and the theorems developed for it include its physical applications, which include modelling the boundaries in annealing metals and two immiscible liquids; as well as its theoretical applications, such as in the theorem of three geodesics and the isoperimetric inequality.

The main concern of this report is Grayson's theorem [4], and will examine the CSF, properties of relevance for the CSF, and how this proof was developed, and give an outline of proving Grayson's theorem [4].

3 Statement of Authorship

Assisted by my supervisor, I read papers about the subject matter and relevant background material in order to come to an understanding of Grayson's theorem [4], which I have presented here using proofs. To the best of my knowledge, all sources used by this report have been attributed to their original author. This report was proof read by Matthew Cooper. This research project was funded by AMSI.

4 Definitions and notation

This paper defines a curve to be simple if it does not intersect itself and closed if it ends where it starts.

For the purposes of this report, a curve is defined using a parametrisation of it, so it is defined using two functions that for any input, will return the x and y coordinates of the corresponding point on the curve, and smooth changes in the input parametrisation variable will correspond with smooth changes in the output point. This allows for differentiation over the parametrisation variable along the curve. Curves are denoted as $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^2, v \mapsto (x(v), y(v))$, where $x, y : I \subseteq \mathbb{R} \mapsto \mathbb{R}$. Additionally, as curves here are considered under the influence of a PDE, the ones examined in this report are all sufficiently differentiable, often infinitely so, but regularly at least C^2.

Many different parametrisations of a curve are possible, but of particular interest is the arc length parameterisation, which is indicated by the use of s as the parametrisation variable. This arc length parametrisation starts at zero at the beginning of the curve (if the curve is closed, this is an arbitrary point on it), and increases equivalently to the length travelled by the point indicated by the x and y functions. This parametrisation therefore will take the value of the length of the curve once it has traversed the entire curve.

In this report, the unit tangent and normal vectors are used. The tangent vector is $\frac{d\gamma}{ds}$, and is denoted by T_* (used rather than T to avoid confusion with the time variable). The normal vector is simply defined as being perpendicular to T_*, and a unit in length, denoted N. Additionally, N is defined to consistently point to one side of the curve specifically, which ensures that in a closed simple curve, N can be defined to always point to the inside of the curve.

Curvature, written as κ, is defined to satisfy the equation $\frac{d^2\gamma}{ds^2} = \kappa N$. Note that this definition implies negative curvature where a curve bends in the opposite direction to the normal (concave parts of the curve if it is simple and closed).

Since CSF concerns curves over time, a time variable will be frequently used, and denoted by t. Finally, a solution of CSF may only exist for a finite time, and this final time is denoted by T.

5 Curve shortening flow and known solutions

The curve shortening flow acts on a curve over time, and to represent this the solutions of it are functions which take a given time and return the state of the evolved curve at that time. Specifically:

$$X : I \times [0, T) \mapsto \mathbb{R}^2 \tag{1}$$

where T is the final time that the curve exists under the flow and $I \subseteq \mathbb{R}$ is the domain of the parametrisation variable of the curve.

With this construct given, CSF can be defined:

$$\frac{dX}{dt} = \kappa N \tag{2}$$

This simple equation states that any point on the curve will move in the direction of the normal with speed of the curvature at that point. This also intuitively gives insight into the necessity of a finite time of existence for CSF; as for closed simple curves the normal points inwards it could be expected that in some cases the curve will come to shrink, which increases curvature, causing it to shrink faster, and potentially develop a singularity with infinite curvature for which CSF is not defined. Further study will show that this is indeed possible, and a finite of time existence can be necessary.

There are some known solutions to this equation; starting with the simplest, the line. This has no curvature and thus will never move under CSF and will exist for all time. Similarly simple is a circle, this has an inward pointing normal and equal curvature for all points, namely $\frac{1}{r}$, where r is the radius of the circle. By observing that all points are equivalent by symmetry, an initial guess for the solution might be a shrinking circle, with an ODE for the radius given by:

$$\frac{dr}{dt} = -\frac{1}{r}$$
$$r = \sqrt{C - 2t}$$

where the constant can be calculated to be the initial radius squared. Allowing r_0 to denote the initial radius, this hypothesis is tested:

$$\begin{aligned} X : [0, 2\pi] \times [0, \frac{r_0^2}{2}),\ X(v,t) &= (\sqrt{r_0^2 - 2t}\cos v, \sqrt{r_0^2 - 2t}\sin v) \\ \frac{dX}{dt} &= -\frac{1}{\sqrt{r_0^2 - 2t}} \times (\cos v, \sin v) \\ &= -\frac{1}{r} \times -N \\ &= \kappa N \end{aligned}$$

Therefore a shrinking circle of radius $r_0^2 - 2t$ is the solution when the initial curve is a circle.

Another curve without such a simple solution is given by looking for solutions of the form $X(v,t) = \gamma_0(v) + ut$, where u here is some constant velocity and γ_0 is the initial curve. This amounts to finding an initial graph with curvature such that it will translate upwards without otherwise distorting itself. The curve that is found is named the grim reaper curve, and has the general form

$$y = -ut \log \cos(u(x - x_0)) + y_0 \tag{3}$$

where x and y are the variables for the Cartesian plane, and x_0 and y_0 give the initial coordinates of the vertex of the curve.

From this equation, two more closed form solutions for CSF may be derived; through manipulation of the equation the hairclip and paperclip solutions arise. The hairclip is comprised of an endless row of alternating upwards and downwards grim reapers connected, and as curve evolves it flattens into a horizontal line. The paperclip solution resembles two horizontal grim reapers connected to each other at the ends, and will shrink down into a point under CSF.

6 Properties of Curve shortening flow

In order to prove the results of Grayson's theorem [4], it is necessary to examine and show some properties of closed and simple curves under CSF in order to both better understand the process, and hopefully develop some useful identities for later use.

6.1 Length decrease

The rate of decrease in length for a curve can be expressed as $\frac{d}{dt}L = \frac{d}{dt}\int_I \|\frac{dX(v,t)}{dv}\|dv$ which is evaluated by the following:

$$\begin{aligned}\frac{d}{dt}\int_I \|\frac{dX(v,t)}{dv}\|dv &= \int_I \frac{d}{dt}\|\frac{dX(v,t)}{dv}\|dv \\ &= \int_I \frac{\langle \frac{d^2X}{dvdt}, \frac{dX}{dv}\rangle}{\|\frac{dX}{dv}\|}dv \\ &= \int_I \langle \frac{d^2X}{dvdt}, T_*\rangle dv \\ &= \int_I \langle \frac{d(\kappa N)}{dv}, T_*\rangle dv \\ &= \int_I \langle \frac{d\kappa}{dv}N, T_*\rangle + \kappa\langle \frac{dN}{dv}, T_*\rangle dv \\ &= \int_I \kappa\langle \frac{dN}{dv}, T_*\rangle dv\end{aligned}$$

Using the Frenet equations, it may be determined that $\langle \frac{dN}{dv}, T\rangle = \langle -\kappa\frac{ds}{dv}T, T\rangle = -\kappa\frac{ds}{dv}$. Therefore,

$$\frac{dL}{dt} = -\int_{\gamma_t} \kappa^2 ds$$

This expression for length shows that the length of the curve will always decrease so long as some segments of the curve have curvature. Note that all closed, simple curves must have positive curvature somewhere, as it must double back on itself. Therefore, while a curve remains closed and simple, it must strictly have decreasing length.

6.2 Area decrease

An expression may also be found for the rate of change in area where the curve is closed and simple. This area loss may be modelled by considering the small approximate rectangles that would be removed or added from the enclosed space by the movements of two close points over a short period of time. Note that this method of area measurement assumes that the normal points to the interior, which can be arranged for the initial curve, but if the curve happens to intersect with itself at a later time could come to be false. However, this report will later observe that this will never happen, and with this assumption the rate of area decrease may be calculated.

For a sufficiently small step in time Δt and a sufficiently small change in parametrisation Δv, it is expected that the height of the rectangle in question can be well approximated by $\|\frac{dX}{dt} \times \Delta t\| = \kappa\Delta t$, and the width of such a rectangle to be $\|\frac{dX}{dv} \times \Delta v\| = \frac{ds}{dv}\Delta v$. As such to determine the change in area, the following integral is evaluated over the entire curve:

$$\begin{aligned}\frac{dA}{dt} &= -\int_I \kappa \frac{ds}{dv} dv \\ &= -\int_{\gamma_t} \kappa ds\end{aligned}$$

This expression may be simplified further by observing that by the theorem of turning tangents, a closed simple curve has $\int_{\gamma_t} \kappa ds = 2\pi$, and hence

$$\therefore \frac{dA}{dt} = -2\pi$$

Given that the closed simple curve in question has finite area, and does not come to intersect itself as assumed above, this constant rate of area decrease implies that if the curve remains under CSF for $\frac{A_0}{2\pi}$ time where A_0 is its initial area, it will reach zero area and come to stop existing. The curve may develop infinite curvature before this time, so this is not a guarantee that the curve will exist long enough to come to zero area, but does put a maximal existence time on closed simple curves with finite area.

6.3 Curvature evolution

The time derivative of curvature of a curve under CSF can also be computed; using $\kappa = \frac{\langle N, \frac{d^2X}{ds^2}\rangle}{\|\frac{dX}{ds}\|^2}$, (the addition of the denominator in the previous equation is necessary because the arc length parameter may not be consistent across all times for the curve due to its shortening, and as such s here denotes the arc length parameter for the curve only at the time being considered)

$$\begin{aligned}\frac{d\kappa}{dt} &= \frac{d}{dt}\frac{\langle N, \frac{d^2X}{ds^2}\rangle}{\|\frac{dX}{ds}\|^2} \\ &= \frac{\|\frac{dX}{ds}\|^2\langle \frac{dN}{dt}, \frac{d^2X}{ds^2}\rangle + \|\frac{dX}{ds}\|^2\langle N, \frac{d^3X}{d^2sdt}\rangle - 2\|\frac{dX}{ds}\|\frac{d}{dt}(\|\frac{dX}{ds}\|)\langle N, \frac{d^2X}{ds^2}\rangle}{\|\frac{dX}{ds}\|^4}\end{aligned}$$

Since N is a unit vector, its derivative is perpendicular to it, and $\frac{d^2X}{ds^2}$ is normal. Therefore the first term is zero, and since $\|\frac{dX}{ds}\| = 1$ at the time of interest:

$$\begin{aligned}\frac{d\kappa}{dt} &= \langle N, \frac{d^3X}{d^2sdt}\rangle - 2\frac{d}{dt}(\|\frac{dX}{ds}\|)\langle N, \frac{d^2X}{ds^2}\rangle \\ &= \langle N, \frac{d^2}{ds^2}(\kappa N)\rangle - 2\langle \frac{dX}{ds}, \frac{d^2X}{dsdt}\rangle \times \kappa \\ &= \kappa\langle N, \frac{d}{ds^2}N\rangle + \frac{d^2\kappa}{ds^2} + 2\kappa^3\end{aligned}$$

In order to solve this, use $0 = \langle N, N' \rangle' = \langle N, N'' \rangle + \langle N', N' \rangle = \langle N, N'' \rangle + \kappa^2$. Therefore,

$$\frac{d\kappa}{dt} = \kappa^3 + \frac{d^2\kappa}{ds^2}$$

This result not only gives an expression for the rate of curvature change at any given time, but also allows for placing a maximal time on the existence of a closed curve if it is strictly convex, that is it has positive curvature everywhere. Denoting κ_t as the minimum curvature for the curve at time t, since at a minimum $\frac{d^2\kappa}{ds^2} \geq 0$, the maximum principle gives $\kappa_t \leq \kappa_t^3$, an ODE. Solving:

$$\begin{aligned} \frac{1}{\kappa_t^3}\frac{d\kappa_t}{dt} &\leq 1 \\ \frac{d}{dt}\left(-\frac{1}{2}\frac{1}{\kappa_t^2}\right) &\leq 1 \\ \frac{1}{\kappa_t^2} - \frac{1}{\kappa_0^2} &\leq -2t \\ \kappa_t^2 &\geq \frac{\kappa_0^2}{1 - 2\kappa_0^2 t} \end{aligned}$$

As the above shows, the minimum curvature anywhere on the curve is increasing (given the initial conditions of a convex closed curve), and will approach infinity as time approaches $\frac{1}{2\kappa_0^2}$. This of course will prevent CSF, hence giving our maximal existence time for these types of curves.

6.4 Avoidance principle

The avoidance principle is an interesting property of CSF which states that any two closed and simple curves in the same plane which do not initially intersect, will never come to intersect so long as they exist under CSF. In order to prove this, it can be shown that the minimal distance between the two curves is non-decreasing in time.

One intuitive approach to proving this is to observe that at any two points on the different curves that are minimally distant from each other, they must have parallel tangents that are perpendicular to their difference, otherwise there would locally be a point that is closer to the other curve. Thereby, the two points must have parallel normals, that are additionally parallel to the difference between the two points. Roughly speaking, in order for the points to come closer over time, the curvature of one must pull it to the other point faster than that other point's curvature pulls it away. This should not happen, as the minimality will locally prevent curvature of this variety.

By denoting D as the distance between the two minimal points and the i^{th} curve X_i, its curvature κ_i, and the point in question s_i (with arc length parametrisation at the given time for both curves such that their tangents point in the same direction):

$$D = \|X_1(s_1) - X_2(s_2)\|$$
$$\frac{dD}{dt} = \frac{\langle (X_1(s_1) - X_2(s_2)), \frac{d}{dt}(X_1(s_1) - X_2(s_2))\rangle}{\|X_1(s_1) - X_2(s_2)\|}$$
$$\frac{dD}{dt} = \frac{\langle (X_1(s_1) - X_2(s_2)), (\kappa_1 N_1 - \kappa_2 N_2)\rangle}{\|X_1(s_1) - X_2(s_2)\|}$$

Therefore it must be shown that the difference between the two points is directed in the same way as their difference in time derivatives, as since the normals and differences are parallel this would force $\frac{dD}{dt} \geq 0$. Assuming that the two curves are sufficiently differentiable, their Taylor series may be examined. Using Δs as a small change in the arc length parameter for both, and T_* as the shared tangent vector at the points,

$$\|X_1(s_1 + \Delta s) - X_2(s_2 + \Delta s)\| \geq \|X_1(s_1) - X_2(s_2)\| \quad \text{as distance is minimal at } s_1,\ s_2, \text{ but}$$
$$\|X_1(s_1 + \Delta s) - X_2(s_2 + \Delta s)\| = \|X_1(s_1) + T_*\Delta s + \frac{\kappa_1 N_1}{2}\Delta s^2 - (X_2(s_2) + T_*\Delta s + \frac{\kappa_2 N_2}{2}\Delta s^2) + \mathrm{O}(s^3)\| \quad \text{therefore}$$

$$\|X_1(s_1) + T_*\Delta s + \frac{\kappa_1 N_1}{2}\Delta s^2 - (X_2(s_2) + T_*\Delta s + \frac{\kappa_2 N_2}{2}\Delta s^2) + \mathrm{O}(s^3)\| \geq \|X_1(s_1) - X_2(s_2)\|$$

However, as the difference between the two points and the normals are parallel, for sufficiently small Δs the only way this can be possible is if $\frac{\kappa_1 N_1}{2} - \frac{\kappa_2 N_2}{2}$ is oriented the same way as $X_1(s_1) - X_2(s_2)$.

An entirely separate way of proving this principle is to treat D as a function of one time and two spatial variables (the two parametrisations) and use the derivatives of D to form a sort of heat equation, which allows for the application of the maximum principle, although this is not shown here.

One use of this principle is to observe that encircling a given curve within a circle that does not touch it allows the conclusion that as the circle must shrink down in a point in $\frac{r_0^2}{2}$ (by its closed form solution), the curve encapsulated within will never intersect and so must come to stop existing before this time, whether this be by shrinking down into a point itself, or developing a singularity and disappearing.

6.5 Preservation of embeddedness

Related to the avoidance principle, embeddedness, the property of having no self-intersections, can be shown to be preserved under CSF. Much of this argument follows similarly to that of avoidance, however, the caveat that two points on the same curve may not have arbitrarily small distance from one another forces the proof to first find a lower bound for distance given a small arc length.

For any two points on the curve, $X(s_1)$ and $X(s_2)$ ($X(s,t)$ parametrised by arc length for convenience), a function is defined $l : I \times I \to \mathbb{R}$, which given the parameters of two points returns the arc length between them. This is used to assign the notation s_α to the point such that $l(s_1, s_\alpha) = l(s_\alpha, s_2)$, that is, the midpoint of the two points in terms of arc length, and T_{s_α} gives the tangent at this point. Additionally, $\theta : I \to \mathbb{R}$ is defined such that it gives the angle between the tangents of a given point and that of T_{s_α}. Then if the curve has maximum curvature K at the time in question, the aim is to prove $l(s_1, s_2) \leq \frac{\pi}{K} \implies d \geq \frac{2}{K}\sin(\frac{Kl(s_1,s_2)}{2})$, where d is the distance between the two points. Noting that the bound on curvature implies that $\theta(s) \leq Kl(s, s_\alpha)$:

$$
\begin{aligned}
d &\geq \langle T_{s_\alpha}, X(s_1) - X(s_2)\rangle \\
&= \int_{s_2}^{s_1} \cos(\theta(s))ds \\
&\geq \int_{s_2}^{s_1} \cos(Kl(s, s_\alpha))ds \\
&= 2\int_{s_\alpha}^{s_1} \cos(K(s - s_\alpha))ds \\
&= 2\left[\frac{1}{K}\sin(K(s - s_\alpha))\right]_{s_\alpha}^{s_1} \\
&= \frac{2}{K}\sin(K(s_1 - s_\alpha)) \\
&= \frac{2}{K}\sin(\frac{Kl(s_1, s_2)}{2})
\end{aligned}
$$

Having established that two points must have finite distance apart given that they are some finite arc length apart so long as the curve exists under CSF, the proof of the avoidance principle can come into effect for the points of the curve not within a $\frac{\pi}{K}$ arc length of each other. For a given point, the curve can be divided into points with arc length from it less than or equal to $\frac{\pi}{K}$, and those strictly greater than. Any point belonging to the greater than region, must have a small region around it such that the avoidance principle argument can be applied, so any minimal point in the strictly greater than region will move away from the given point over time. The limit of points approaching the boundary of this region, since the distance function is smooth, will be bounded by the same distance as applies to the boundary of the less than or equal to region. All the points that are less than or equal to are bounded away from zero so long as there is finite arc length, therefore we have that while the curve has finite curvature, it will never come to intersect itself.

As shown, an embedded curve in the plane will never come to intersect itself under CSF. This is an extremely important property, as it ensures that a simple curve will remain simple, allowing much of the further work on CSF to continue to apply to a curve for its whole existence, rather than a fragment of. Additionally, this property was already alluded to in the context of area decrease, as non-intersection means that a curve will not develop negative area in the calculation used for area, allows that property to ensure that any curve which exists for long enough will come to have zero area in the traditional sense, although this report has yet to prove that the curve will exist long enough.

7 Measures of roundness

As Grayson's theorem [4] not only states that a closed simple curve will come to a point in finite time, but explicitly states that the curve will converge to a circle as it approaches this final time. Therefore, in order to prove this methods of measuring roundness must be introduced and evaluated.

7.1 Isoperimetric ratio

The isoperimetric ratio is a classic method to measure the roundness of a curve, and in addition it has many crucial properties that will prove useful in the analysis of CSF. This ratio is expressed as $\frac{L^2}{A}$, where L is the length of the curve in question, and A is its area.

Of its properties, of primary interest is it having a minimum value for a circle, 4π, with a strictly larger value for all other curves, but it additionally is scale invariant, allowing the scaling of the curves that are examined, a property that is useful in the face of curves that may be expected to shrink to singularities.

Perhaps the most important property of the isoperimetric ratio concerning CSF is that, with certain conditions on the curves, namely convexity, it can be proved that the ratio is strictly decreasing. The proof of this was provided by Gage [2]. Gage and Hamilton [3] built upon this in the case of convex curves to prove Grayson's theorem [4] by showing that in finite time the a convex curve would shrink to a round point. Although it does not allow for the full proof of the theorem, this development was critical in the final proof used by Grayson, as he showed that any closed simple curve would become convex in finite time, at which point the proof of Gage and Hamilton [3] would ensure the result. This is not the method this report will outline to demonstrate Grayson's theorem [4], but the isoperimetric ratio is clearly invaluable in the study of CSF.

7.2 Huisken's distance comparison estimate

Huisken's distance comparison estimate [6] is a measure of roundness developed by Huisken in his studies of CSF, and shares many of the most crucial properties of the isoperimetric ratio: strictly minimal value for circles, with their value being π, scale invariance, and in fact improves on the the effectiveness of the isoperimetric ratio in the aspect of decreasing under CSF. More explicitly, this distance comparison quantity is strictly decreasing under CSF for closed, simple curves, and does not require convexity.

Huisken's distance comparison estimate [6] uses the function $Z = \frac{L}{\pi}\sin(\frac{\pi l}{L})$, where L is the length of the entire curve, and l is the arc length between any two points s_1, s_2. Huisken's distance comparison estimate [6] itself is given by $\sup_{s_1,s_2\in\gamma} Z$. In order to reach the conclusion that this expression was strictly decreasing over time, Huisken considered that for any given time for the CSF, a time after this for which a new supremum of the function occured must have $\frac{dZ}{dt} \geq 0$ and $\left(\frac{d}{dx} - \frac{d}{dy}\right)^2 Z \leq 0$, with x and y a set of points which attained this supremum for Z when $s_1 = x, s_2 = y$. By careful manipulation of these derivatives, it can be concluded that $\left(\frac{d}{dt} - \left(\frac{d}{dx} - \frac{d}{dy}\right)^2\right) Z < 0$, which is clearly a contradiction.

Although this measure does not allow the conclusion of Grayson's theorem [4] on its own, it will play a principle role in the outline of Grayson's theorem [4] this report will provide.

8 Bounds on curvature

Before proving the roundness part of Grayson's theorem [4], Huisken's distance comparison estimate [6] is used to develop bounds on curvature based on the length of a curve. These bounds are, in part, used to show that any closed, simple curve will continue under CSF until the length of the curve is zero by bounding the curvature away from infinity.

The centrepiece of the proof for curvature boundedness is the chord arc profile, ψ; a function of arc length which gives the shortest distance between any two points on a curve with the given arc length, that is $\min_{s_1,s_2\in\gamma} d$ with d the distance between s_1 and s_2, and s_1 and s_2 the given arc length apart. This is because it can be shown that $\psi(l) \leq l - \frac{K^2}{24}l^3 + \mathrm{O}(l^5)$, where K is the maximum curvature for the curve that ψ models. This allows for the problem of finding an upper bound for curvature for finite length to be solved by finding a sufficiently strong lower bound for ψ in the same situations.

Finding a lower bound for the chord arc profile amounts to attempting to find a function φ that depends on the relative arc length between the two points, $\frac{l}{L}$ with l arc length and L total length (denoted z hereafter), that ensures that $d \geq L\varphi$. A function which satisfies this requirement will not be unique in this property, as $\varphi = 0$ trivially solves this problem, but in order to create a sufficient bound for the chord arc profile, φ must be carefully constructed, a process which begins by calculating and extrapolating from the time derivative and second spatial derivatives of $d - L\varphi$ to ensure that this function will not reach an interior maximum, a restriction that results in the condition:

$$L^2\frac{d\varphi}{dt} \leq 4\frac{d^2\varphi}{dz^2} + \left(\frac{4\arccos^2(\frac{d\varphi}{dz})}{z} - 4\pi^2 z\right)\frac{d\varphi}{dz} + 4\pi^2\varphi \tag{4}$$

This inequality can be rewritten and modified into another differential equation, and by doing so along with making assumptions on the form of the final solution, the suitability of the resultant solutions can be evaluated, and a sufficient form for φ can finally be found:

$$\varphi = \frac{e^{4\pi^2\tau}}{a}\arctan\left(\frac{a}{\pi e^{4\pi^2\tau}}\sin\left(\frac{\pi l}{L}\right)\right) \tag{5}$$

where a is some constant set so that $d \geq L\varphi$ for time 0, and $\tau = \int_0^t L^2 dt$. In addition to ensuring the $d \geq L\varphi$ will remain true for all time, using a Taylor expansion allows a bound to be put on the chord arc profile, and hence a bound on curvature for all time such that curve length is non-zero:

$$\kappa^2 \leq \left(\frac{2\pi}{L}\right)^2 + \frac{8a^2e^{-8\pi^2\tau}}{L^2} \tag{6}$$

By the bound developed above, the curvature will remain finite while the length of the curve is non-zero, and hence if the curve stops existing, which finite area enclosing curves will, it will be as the curve shrinks to a point. This is again an important step towards Grayson's theorem [4], leaving essentially just the convergence to a round point to be proven.

9 Convergence of curves by Grayson's theorem

Using the theory developed in the prior sections, this report now aims to give an outline of the proof of Grayson's theorem [4]. By the theorem of turning tangents, the total curvature of any closed simple curve is 2π, which in turn indicates that the average curvature of such a curve is $\frac{2\pi}{L}$. Noting that in the curvature bound, $\lim_{L\to 0,\tau\to\infty} \kappa^2 = \left(\frac{2\pi}{L}\right)^2$, if it can be shown that $L \to 0$ and $\tau \to \infty$ as $t \to \infty$, then the curvature everywhere must limit with respect to this equation in order to satisfy the theorem of turning tangents.

As curvature will remain bounded while length is non-zero, the CSF will not stop existing until the length approaches zero, and as closed simple curves have a finite existence time, this final time must coincide with a length of zero.

In order to conclude that $\tau = \int_0^t L^2 dt \to \infty$, an inequality is developed from the curvature bound:

$$\begin{aligned}
\int_\gamma \kappa^2 ds &\leq \int_\gamma \left(\left(\frac{2\pi}{L}\right)^2 + \frac{8a^2}{L^2}\right) ds \\
-L\frac{dL}{dt} &\leq 4\pi^2 + 8a^2 \\
\int_t^T -L\frac{dL}{dt}dt &\leq \int_t^T 4\pi^2 + 8a^2 dt \\
L^2 &\leq 8(\pi^2 + 2a^2)(T-t)
\end{aligned}$$

Substituting this for τ:

$$\begin{aligned}
\tau &= \int_0^t \frac{1}{L^2}dt \\
&\geq \int_0^t \frac{1}{8(\pi^2+2a^2)(T-t)}dt \\
&= \frac{1}{8(\pi^2+2a^2)} \log\left(\frac{T^2}{T^2-t^2}\right)
\end{aligned}$$

As this shows, τ must approach ∞ as $t \to T$, and hence, the curvature must approach the same limit at every point across the curve, and which forces the curve to converge to a circular point as it approaches its final time.

There are still some final details to rigorously complete this argument, such as showing boundedness for the derivatives of curvature and the curve itself, as well as to show full convergence of a rescaled version of the curve to a circle, however, this sketch of the proof should give intuitive insight into the mechanisms which cause Grayson's theorem [4].

10 Discussion and Conclusion

Throughout the research project, and this report designed from that project, Grayson's theorem [4] and the methods used to prove it were studied in the aim of developing a useful and valuable understanding of the topic. Many fundamental properties of CSF were explored in the process, nearly all of which eventually becoming involved in order to reach the final result, providing a broad view of how the subject functions as a whole, and ideally presenting to a broad audience an intuitive understanding of how the mathematical theory reaches the conclusion of the proof.

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12 References

References

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