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Optimal Exit Times in a Declining Industry

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Contents

Abstract

In this paper, we study the optimal exit strategy for a firm in a monopoly. We then consider the industry with two firms as a duopoly and examine the existence and construction of a Nash equilibrium for the resultant nonzero-sum Dynkin exit game.

1 Introduction

Economic decline of an industry can be caused by a plethora of different reasons: replacement by a competitor, its product no longer seeing high demand and lack of innovation are a few examples. A natural question arises: when should a company which observes declining profits exit the market?

This paper considered the optimal exit times for companies in a declining industry. We first studied the case where the firm is a monopoly. The exit problem is an optimal stopping problem which may be solved via dynammic programming methods. We then studied two competing firms in a duopoly. One firm's decision to leave the market will give the other firm a monopolistic advantage, boosting its profits. This is a nonzero-sum Dynkin game and we examined the existence and construction of its Nash Equilibrium Point.

For both problems, we assumed that the profits were expected to decrease over time. We considered time periods up to a finite time horizon T. A firm can stop at each discrete time period (but not in between). The decision to stop is non-anticipative, meaning that their decision to stop depends on the historical data, not future data.

The investigation into the monopoly problem revealed that there is a boundary value for the profit process, which was computed using backward induction on Python. Once the profit went below that boundary value, it is then optimal for the firm to exit the market. Intuition may suggest that this boundary value is zero yet it can be negative for many time periods. It increases to zero only when we are close to the time horizon T.

In the duopoly problem, we confirmed the existence of a Nash Equilibrium Point (NEP) for the particular game of interest using the results from Hamadene and Hassani (2014). We then constructed a Nash Equilibria under the assumption of one firm being stronger or more patient than the other, following the methods outlined in Fine and Li (1989). We observed that there are multiple exit time equilibria, corresponding to the situation where both firms jump from being viable duopolists to unviable duopolists yet viable monopolists. The exit of either firm in this context would not violate the definition of a Nash Equilibrium Point.

Statement of Authorship

Dr Zhou conceived the project idea, supervised the work and proofread this report. Fine and Li studied a similar problem and their special case Nash Equilibria construction was replicated. Hamadene and Hassani provided required theorems and proofs for the general case Nash Equilibrium's existence, and they provided the approximating scheme. I created the Python code and images via matplotlib, performed numerical simulations, interpreted the results and wrote this report.

2 Optimal Exit Times for a Single Firm Monopoly

2.1 Framing the Problem

To begin with, we introduce a notion of profit which the firm receives at every time period n.

Definition 2.1 (Profit Process)**.** A profit process is a sequence of random variables $(X_n)_{n\geq 0}$ so that $X_0 = x$ and

$$X_n = x + \xi_1 + \xi_2 + \ldots + \xi_n$$

where $\xi_1, \xi_2, \ldots$ are i.i.d. random variables satisfying

$$\mathbb{P}(\xi_i = 1) = p$$
$$\mathbb{P}(\xi_i = -1) = 1 - p$$

We assume that $p \in (0, \frac{1}{2})$ so that $\mathbb{E}[\xi_i] < 0$, and as a consequence, the profit is decreasing on average.

Next, we define the *discounted accumulative profit* which is the overall money that the firm receives, discounted by a factor δ

$$L(n) = \sum_{k=0}^{n} \delta^k X_k$$

The discount factor $\delta \in (0, 1)$ can be interpreted as the company's valuation of receiving money in the future compared to receiving money today. It depends on factors such as the company's patience and the risk free interest rate.

We assume that $T < \infty$ is a finite time horizon when the company is forced to stop.

Let $\mathbb{F} := (\mathcal{F}_n)_{0\leq t\leq T}$ be a *filtration*. A filtration is a family of σ-algebras on a measure space $(\Omega, \mathcal{F})$ such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s \leq t$.

Moreover, let the nonnegative random variable τ be a *stopping time w.r.t. filtration* $\mathbb{F}$. This means that τ satisfies the condition

$$\{\tau = n\} \in \mathcal{F}_n, \qquad \forall n = 0, 1, 2, \ldots$$

Our problem can then be formulated as follows:

Find $\hat{\tau}$ such that

$$\mathbb{E}[L(\hat{\tau})] = \sup_{0\leq\tau\leq T} \mathbb{E}[L(\tau)]$$

2.2 Method

One intuitive "solution" which stands out is to stop once $X_n \leq 0$. The stopping set would be characterised by $\tau = \inf\{n \geq 0 : X_n \leq 0\}$. This seems intuitive because the profit is expected to decrease in the next time period so there is a high risk of loss. We shall soon see that this intuition is wrong.

The method to solve the classical optimal stopping problem is to perform a backward induction. We first define an optimal value process

$$V_n(X_n) = \sup_{n \leq \tau \leq T} \mathbb{E}[Z_\tau | \mathcal{F}_n]$$

where $Z_t = X_n + \delta X_{n+1} + \ldots + \delta^t X_t$. Then $V_n(x)$ can be found by letting

1. $V_T(x) = x$
2. $V_n(x) = x + \delta \mathbb{E}[V_{n+1}(X_{n+1})|\mathcal{F}_n]^+$

Proof. Assume that we are in time $n < T$ and it is optimal to stop now. Then

$$V_n(x) = x \geq x + \delta \mathbb{E}[V_{n+1}|\mathcal{F}_n]$$

Assume that it is not optimal to stop now. Then

$$V_n(x) = x + \delta \mathbb{E}[V_{n+1}|\mathcal{F}_n] > x$$

As $V_n(x)$ is the maximum of x and $x + \delta \mathbb{E}[V_{n+1}|\mathcal{F}_n]$, we have

$$\begin{aligned} V_n(x) &= max(x, x + \delta \mathbb{E}[V_{n+1}|\mathcal{F}_n]) \\ &= x + max(0, \delta \mathbb{E}[V_{n+1}|\mathcal{F}_n]) \\ &= x + \delta \mathbb{E}[V_{n+1}(X_{n+1})|\mathcal{F}_n]^+ \end{aligned}$$

□

2.3 Results

Using computers to perform some numerical simulations, we can visualise the optimal exit strategy via a stopping boundary. The optimal strategy is to stop once (x, t) falls below this stopping boundary.

Setting $p = 0.49, \delta = 0.99, x = 0.5$ and varying T gives the following:

We can notice a few patterns. It appears that as T increases, the amount of time where the stopping barrier remains at -3.5 increases. Moreover, as t approaches T, the stopping barrier increases until it reaches 0.

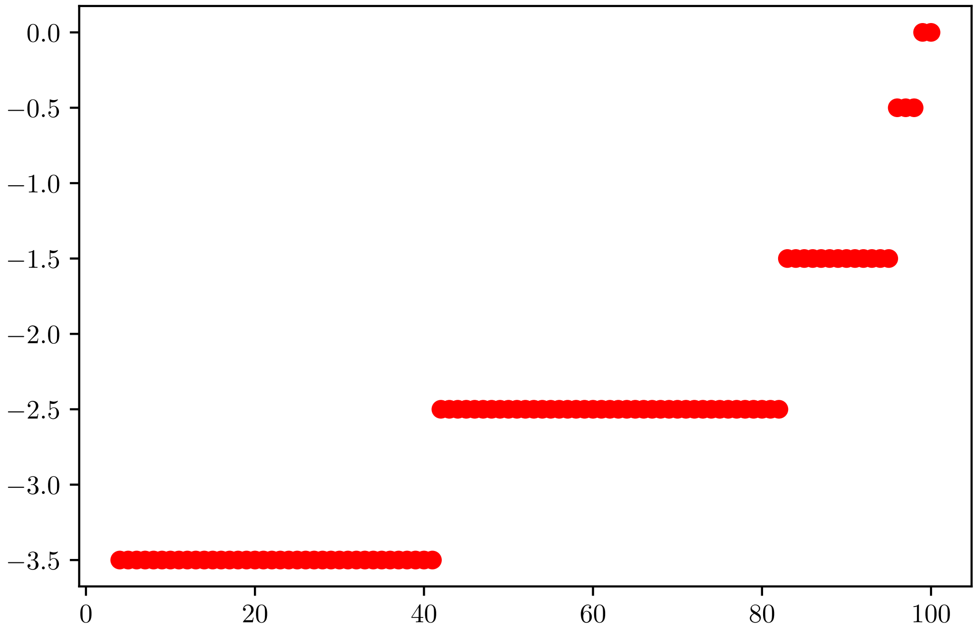

Figure 1: Stopping Barrier for T = 100.

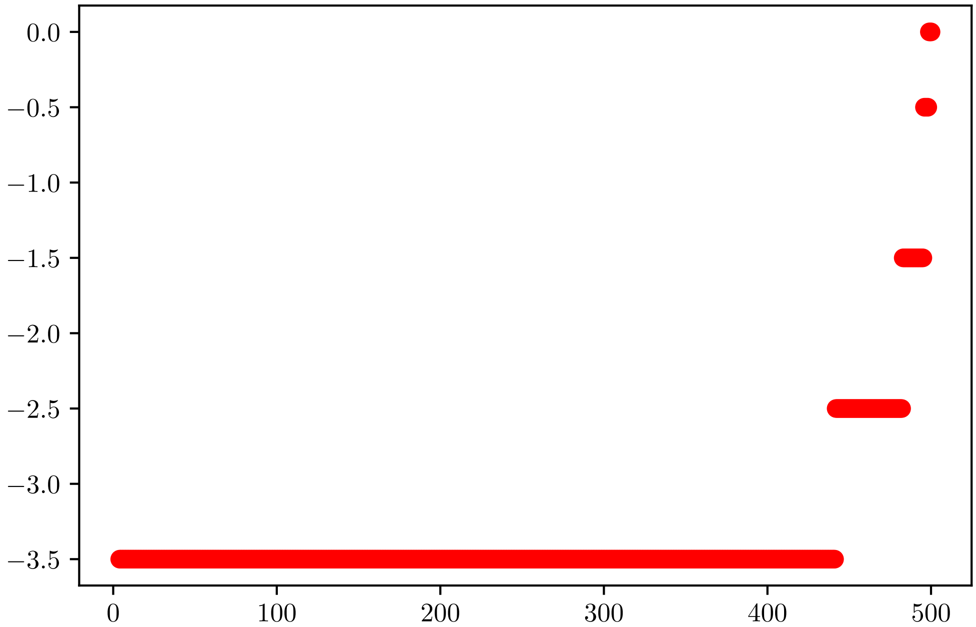

Figure 2: Stopping Barrier for T = 500.

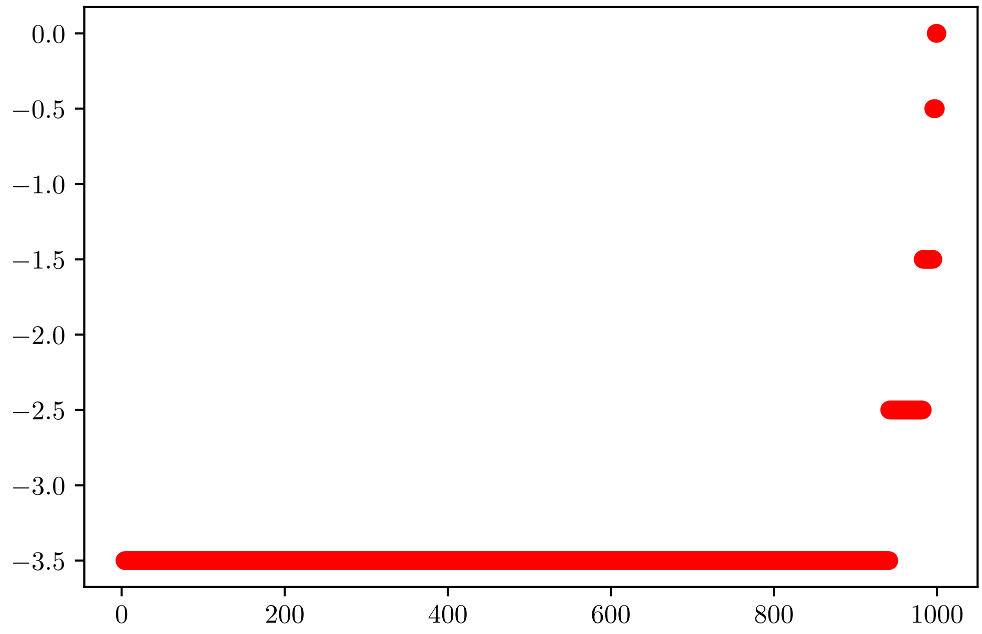

Figure 3: Stopping Barrier for T = 1000.

3 Optimal Exit Times for a Two Firm Duopoly

A monopoly is rare in the real world and we would ideally want to analyse what happens when there are multiple competing firms. To accomplish this goal, we will begin by considering a duopoly. As such, let the profit process of firm i be defined as

$$X_n^i = x^i + \xi_1^i + \xi_2^i + \ldots + \xi_n^i$$

where $i = 1, 2$.

Moreover, let τ^i be the stopping time of firm i and $R = \min(\tau_1, \tau_2)$ be the time when the first firm stops. When one firm exits, the remaining firms has a monopoly and this gives that firm a boosted profit $X_{R+1}^i + a^i$, where $a > 0$. As a consequence, it is generally better to be the last remaining firm.

We'll define $Y_R^{i,I}$ as the payoff for firm i given that it stops at R ($\tau_i = R$) and I is the set of firms that stop at R. Then if firm i stops at time R,

$$Y_R^{i,\{i\}} = Y_R^{i,\{1,2\}} = \sum_{k=0}^{R} (\delta^i)^k X_k^i$$

Now let's look at the payoff if firm i stops after time R. We notice that from time R onwards, the problem is reduced to the original monopoly problem with an initial profit $x = X^i_{R+1} + a^i$. This means

$$Y_R^{i,\{j\}} = \sum_{k=0}^{R} (\delta^i)^k X^i_k + \delta^{R+1} V^i(X^i_{R+1} + a^i)$$

where $j \neq i$ and $V^i(x)$ is the value the remaining monopolising firm receives before exiting.

We note that if $R = \infty$ (neither firm stops) then

$$Y_R^{i,\{1,2\}} = -\infty$$

since the profit is decreasing on average.

3.1 Dynkin Games

This situation is now a *two player nonzero-sum Dynkin Game.* A Dynkin Game is one where the players π_1, π_2 use stopping times as a means to stop the evolution of a system. If the player π_i decides to stop the system at τ_i, then the payoff that they receive is given by

$$J_i(\tau_1, \tau_2) = X^i 1_{\{\tau_i < \tau_j\}} + Q^i 1_{\{\tau_i = \tau_j\}} + Y^i 1_{\{\tau_i > \tau_j\}}$$

where $i \neq j$.

In our case,

$$\begin{aligned} J_i(\tau_1, \tau_2) &= Y_R^{i,\{i\}} 1_{\{\tau_i < \tau_j\}} + Y_R^{i,\{1,2\}} 1_{\{\tau_i = \tau_j\}} + Y_R^{i,\{j\}} 1_{\{\tau_i > \tau_j\}} \\ &= Y_R^{i,\{i\}} 1_{\{\tau_i \leq R\}} + Y_R^{i,\{j\}} 1_{\{\tau_i > R\}} \end{aligned}$$

The nonzero-sum description refers to the fact that the game is one where one player's payoff is not the other player's loss. Indeed, both companies may exit while making an overall profit.

3.2 Existence of Nash Equilibrium Point

To find the optimal strategy, we want to examine the *Nash Equilibrium Point* (NEP) of this game. A NEP is a strategy of stopping times (τ_1^*, τ_2^*) where one player's unilateral decision to change their strategy will not lead to profit. That is,

$$\mathbb{E}[J_1(\tau_1^*, \tau_2^*)] \geq \mathbb{E}[J_1(\tau_1, \tau_2^*)]$$
$$\mathbb{E}[J_2(\tau_1^*, \tau_2^*)] \geq \mathbb{E}[J_1(\tau_1^*, \tau_2)]$$

where τ_1, τ_2 are arbitrary stopping strategies.

A theorem from paper (reference 2) states that the NEP as long as the following two assumptions are met:

1.

$$\limsup_{t \in \mathbb{N}} Y_t^{i,\{i\}} \leq Y_\infty^{i,\{1,2\}}, \mathbb{P} - a.s.$$

2.

$$Y_t^{i,I\cup\{i\}} \leq Y_t^{i,I}, \mathbb{P} - a.s.$$

Intuitively, this means that the payoff for π_i if they are among the stopping group, is less than the payoff for π_i if they are not necessarily in the stopping group. In other words, it is more favourable to continue than to stop at R.

Using this theorem, we prove that there is a NEP by showing that our problem meets the assumptions. The first assumption is easily met since $\limsup_{t\in\mathbb{N}} Y_t^{i,\{i\}} = -\infty = Y_\infty^{i,\{1,2\}}$. The second assumption is also met (noting that $V(X_{R+1}^i + a^i) \geq 0$ - the company would have stopped if it was negative) because

$$\begin{aligned} Y_t^{i,I\cup\{i\}} &= \sum_{k=0}^{R} (\delta^i)^k X_k^i \\ &\leq \sum_{k=0}^{R} (\delta^i)^k X_k^i + \delta^{R+1} V^i(X_{R+1}^i + a^i) \\ &= Y_t^{i,I} \end{aligned}$$

As such, a NEP exists in this problem.

3.3 Construction of Nash Equilibrium Point

The most natural question to ask is now how do we construct this NEP. In the same paper, they introduce sequences of stopping times which will converge to a NEP of the game. This approximating scheme is performed by considering a sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\geq 1}$ defined y induction:

Let $i_n = n \mod 2$. Also, set

1. $\tau_1 = \tau_2 = \infty$
2. For $n \geq 3$, let

   (a)

$$U_t^n = Y_t^{i_n,\{i_n\}} 1_{\{t<\tau_{n-1}\}} + Y_{\tau_{n-1}}^{i_n,\{\tau_{n-1}\}} 1_{\{t\geq\tau_{n-1}\}} 1_{\{\tau_{n-1}<\infty\}} + Y_\infty^{i_n,\{1,2\}} 1_{\{t=\tau_{n-1}=\infty\}}, \forall t \in \bar{\mathbb{N}}$$

   (b)

$$W_t^n = \sup_\nu \mathbb{E}[U_\nu^n | \mathcal{F}_t], \forall t \in \bar{\mathbb{N}}$$

(c)

$$\mu_n = \inf\{s \in \bar{\mathbb{N}} : W_s^n = U_s^n\}$$

(d)

$$\tau_n = \mu_n \wedge \tau_{n-2} 1_{\{\mu_n \wedge \tau_{n-2} < \tau_{n-1}\}} + \tau_{n-2} 1_{\mu_n \wedge \tau_{n-2} \geq \tau_{n-1}\}}$$

where $\mu_n \wedge \tau_{n-2} = \min(\mu_n, \tau_{n-2})$.

Now we may define

$$\tau_1^* = \lim_{n \to \infty} \tau_{2n+1}$$

$$\tau_2^* = \lim_{n \to \infty} \tau_{2n+2}$$

Note that these limits exist since $0 \leq \tau_n \leq \tau_{n-2}$, meaning that the stopping times are non-increasing for any fixed i. These two stopping strategies form the Nash Equilibrium Point.

This NEP is not necessarily unique - in fact, there could be multiple NEPs.

3.4 Special Case of Nash Equilibria

We illustrate this multiplicity with a special case. We say that firm 1 is more patient than firm 2 if $\delta_1 > \delta_2$ and we say firm 1 is stronger than firm 2 if $x^1 > x^2$ and $a^1 > a^2$.

Assume that firm 1 is stronger/more patient than firm 2. Moreover, assume that at time n, both firms are making positive profits. However, at time $n + 1$, they both start to make a loss as a duopolist but they are still viable monopolists. That is, if the other firm exits, the monopolistic advantage boosts the firm's profit sufficiently to make it positive.

In this scenario, either firm's exit will be consistent with a Nash Equilibria.

More formally, (τ_1^*, τ_2^*) is a subgame perfect equilibrium iff for each $s \geq 0$, $\tau_i^* = \inf\{t \geq s : X_t \in B_{it}\}$ for $i = 1, 2$. The exit sets B_{it} are define recursively as follows:

1. Let $\pi_{i1}(x) = x$ and $\pi_{i2}(x) = x + a^i$. Then define

$$V_t^{ij}(x) = \pi_{ij}(x) + \delta^i \mathbb{E}[V_{t+1}^{ij}(X_{t+1}) | X_t = x]^+$$

where $i, j \in \{1, 2\}$.

2. Let $y_t^{ij} = \inf\{z \in \mathbb{R} : V_t^{ij}(z) \geq 0\}$ represent the smallest profit so that the company still has a positive value.

3. Let

$$A_{it} = \{x : \pi_{i2}(x) + \delta^i \mathbb{E}[V_{t+1}^i(X_{t+1}) | X_t = x]^+ < 0\}, \qquad i = 1, 2$$

where

$$V_t^i = (\pi_{i2}(x) + \delta^i \mathbb{E}[V_{t+1}^i(x)|X_t = x]^+)1_{B_{j,t+1}^c}(x) + (\pi_{i1}(x) + \delta^i \mathbb{E}[V_{t+1}^{i1}(x)|X_t = x]^+)1_{B_{j,t+1}}(x)$$

with $j \neq i$.

4. Let A_t be a Borel set contained in the set $A_{1t} \cap A_{2t} \cap [y_t^{21}, \infty)$

5. Define

$$B_{1t} = (-\infty, y_t^{11}) \cup (A_{2t}^c \cap A_{1t}) \cup A_t$$
$$B_{2t} = A_{2t} \cap A_t^c$$

Now, suppose that $A_t \subset A_{1t} \cap A_{2t} \cap [y_t^{21}, \infty)$ consists of the states where firm 1 exits and notice that the remaining states in $A_{1t} \cap A_{2t} \cap [y_t^{21}, \infty)$ are when firm 2 exits.

To see this in more detail, assume that $X_t^i \in A_t$. Then the NEP is $\tau_1^* = t$ (firm 1 exits) and $\tau_2^* = \infty$ (firm 2 doesn't exit). Firm 1's decision to unilaterally change their strategy means they will exit in a future time period. However, they will see a loss in the next time period and end up with a lower payoff. If firm 2 decides to stay then they fail to use monopolistic advantage and will have a lower payoff. Hence this is a NEP.

A similar argument can be made when considering $X_t^i \notin A_t$. As such, we have constructed a scenario where multiple NEPs may arise.

4 Discussion and Conclusion

4.1 Summary

We investigated dynamic programming methods to find the optimal stopping time for a monopolising firm which sees declining profits. We observed the interesting fact that under specific circumstances, a company may not exit even when profit is negative.

We also investigated the nonzero-sum Dynkin Game which resulted when there were two competing firms. Here we saw a construction of the general case NEP. We also looked at a special case of the problem which had multiple NEPs.

4.2 Possible Future Direction

In the future, we may want to examine a ξ_i which follows the binomial distribution so that the profit can increase/decrease by more than one in each time step.

It also makes sense to examine more firms $n > 2$ since there are usually many competitors in any given market.

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Thomas Bjork's Arbitrage Theory in Continuous Time gave techniques for handling classical optimal stopping problems.

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