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Non-Commutative Harmonic Analysis on Compact Groups

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Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

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Contents

1 Prelude

1.1 Acknowledgements

I would like to thank Anthony Dooley for his mentorship both before and throughout this project, it is something I have come to feel incredibly lucky to have.

Peter McLean, thank you for letting me sit in your lab over this last summer (and beyond). You allowed me to talk incessantly about the ins-and-outs of mathematics, and beared with me whilst I tell you how I figured something out, only to later listen to me say how I had it all wrong.

And to my family, who have shown me their endless and endless and unwaivering support.

1.2 Abstract

We elucidate the established field of abstract harmonic analysis, building up the theory from an understanding of the Fourier transform. The progression starts with a derivation of the Fourier transform, details of its properties, some miscellaneous notes on linear algebra and other miscellaneous pieces of mathematics.

From there we build up some knowledge of group theory, measure theory (mainly the Haar measure) and representation theory. We then present some facts about harmonics analysis on locally compact Abelian groups and then move to the results of harmonic analysis on compact groups (finite, $\mathcal{SU}(2)$, and $\mathcal{SO}(3)$).

After this, we propose some next steps in terms of theory and application.

1.3 Statement of Authorship

We have sourced the information present from many texts, though the notation and structure defintely vary from any of those listed in the references.

The theoretical knowledge here is well established and no new theorems are presented, but this project has been a chance to understand the process of learning graduate-level mathematics, especially for someone in the engineering discipline.

Auxiliary software is original and was written to produce representations and to build a base that may be used to perform transforms on finite groups. This can be found in full in the appendix.

2 Introduction

Joseph Fourier in his attempt to understand heat conduction discovered the phenomenon that periodic functions can be represented by an infinite sum of weighted sinusoids

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Of course, $\sin(x)$ and $\cos(x)$ have periodicity of 2π, but we can rewrite the function with an arbitrary (but *finite*) period $(1/\xi_0)$ by some algebraic massaging

$$\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad \longrightarrow \quad \sum_{n=0}^{\infty} a_n \cos(2\pi\xi_0 nx) + b_n \sin(2\pi\xi_0 nx).$$

Fourier series can also be expressed using the orthogonality of $\sqrt{-1}$ and the real numbers, exploiting Euler's identity[1]

$$\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad \cong \quad \sum_{n=-\infty}^{\infty} G_n e^{2\pi i n \xi_0 x}.$$

The coefficients (G_n) of the Fourier series are also calculable, namely by the formula

$$G_n = \frac{1}{\xi_0} \int_{-1/(2\xi_0)}^{1/(2\xi_0)} f(x) e^{-2\pi i n \xi_0 x} \, dx$$

for some function $f : \mathbb{R} \longrightarrow \mathbb{C}$ with period $1/\xi_0$. These facts have been well known for hundreds of years and are well exposited in any good education in the quantitative sciences [McLean, 2015].

The insight of Fourier series has been generalised to non-periodic functions, namely through the Fourier transform, which we will derive in the next section. This has allowed mathematicians, engineers and scientists to both break-down complex functions (Fig 1, right) into simpler constituents as well as use Fourier series to approximate functions that are not realisable with real-world components (Fig 1, left).

But the functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ are quite a small amount of those that exist. What if we could take functions from something more general? Like that of a *group* from abstract algebra? This immediately raises questions about what functions and groups this could be done for and whether the properties of these transformations still exist or have some analogue.

We hope to answer some of these questions and illuminate a field of study rich with intersections of different fields of mathematics.

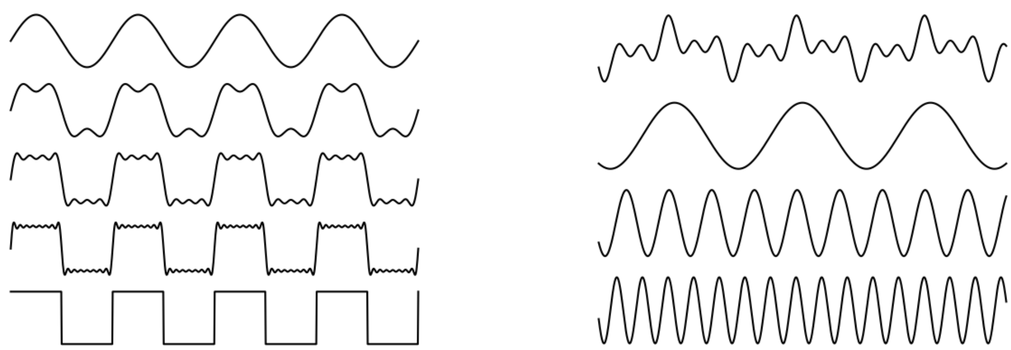


Figure 1: (Left) Using Fourier series, we can generate the square-wave function. (Right) Decomposition of functions into constituents using the Fourier transform.

[1]Euler's identity is $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

3 Group Theory

All subsections in the Group Theory section take a lot from Artin's Algebra. Most of what is here is exposited by Artin in a much more gentle and intuitive fashion. We point those with a desire for greater understanding to his text [Artin, 2010].

3.1 Definition of a Group

A group, G requires a set of elements (X), an associative law $(\circ)$ and some structure amongst the set. By structure what is meant is that there must be an identity element

$$\exists\ e \in G \mid g \circ e = e \circ g = g,\ \forall g \in G,$$

inverse elements

$$\exists\ g^{-1} \in G \mid g^{-1} \circ g = e, \forall g \in G,$$

and closure

$$(g \circ h) \in G \mid \forall g, h \in G.$$

3.2 Finite Groups

A finite group G is one who has a finite number of elements. We say $|G| < \infty$ if G is a finite group.

3.3 Subgroups

H is a subgroup of G if it shares the group action and some subset of elements from G. H also has to satisfy the structural attributes of a group (identity, inverses and closure) to be a subgroup.

If H is a subgroup of G we write:

$$H < G$$

3.4 Cosets

A coset of a subgroup H is a set of elements formed by taking some $g \in G$ and computing

$$g \circ H = \{g \circ h \mid h \in H\}$$

where $\circ$ is the group action of H and G.

3.5 Normal Subgroups

A normal subgroup is one that forms identical left and right coset spaces. Explicitly we can state this as

$$g \circ H = H \circ g \mid \forall g \in G$$

or more commonly

$$g \circ H \circ g^{-1} = H.$$

If H is a normal subgroup of G we write

$$H \triangleleft G.$$

It is helpful to note all subgroups with a commutative associative law (termed Abelian groups) will be normal. It's easy to see why with a one-line proof

$$g \circ H \circ g^{-1} = H \circ (g \circ g^{-1}) = H \circ e = H.$$

3.6 A Brief Taxonomy of Groups

3.6.1 Abelian Groups

A group with a commutative law, i.e. $a \circ b = b \circ a$. e.g. $(\mathbb{R}, +)$, $(\mathbb{R}\backslash 0, \times)$ and $(\mathbb{C}\backslash 0, \times)$ [Artin, 2010].

3.6.2 Compact Groups

Will be thought of as *closed and bounded* to save ourselves an exposition in topology [Munkres, 2000]. Colloquially, this is the topological way of saying finite. All finite groups, and the Lie groups $\mathcal{SU}(2)$ and $\mathcal{SO}(3)$ are compact.

For example, through the method of Möbius transformations $\mathcal{SU}(2)$ can be shown to be isomorphic to a four-dimensional unit-sphere S^3 [Chirikjian and Kyatkin, 2001]. This sphere is finite in volume, meaning it does not cover all of four-dimensional space. Because of this (and other proofs) we can classify $\mathcal{SU}(2)$ as compact.

3.6.3 Locally Compact Groups

We can think of closed and bounded in a neighbourhood of a point, e.g. $(\mathbb{R}, +)$ is locally compact but not compact [Munkres, 2000].

For example, the real line is an infinitely long, and so is not compact. However, any proper subset of the real line is locally compact, as it would have to be finite.

3.6.4 Special Matrices

Special groups of matrices have determinant = +1. Some of the groups (Lie groups, in fact) under examination for harmonic analysis are: $\mathcal{SU}(2)$, the *special* group of unitary matrices and $\mathcal{SO}(3)$, the *special* group of orthogonal matrices.

3.6.5 $\mathcal{SU}(2)$ in General Form

For the uninitiated, the general form of $\mathcal{SU}(2)$ is stated as:

$$\mathcal{SU}(2) := \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C},\ \alpha\overline{\alpha} + \beta\overline{\beta} = 1 \right\}$$

And indeed it is unitary

$$\begin{aligned}\begin{bmatrix}\alpha & \beta \\ -\overline{\beta} & \overline{\alpha}\end{bmatrix}\begin{bmatrix}\alpha & \beta \\ -\overline{\beta} & \overline{\alpha}\end{bmatrix}^* &= \begin{bmatrix}\alpha & \beta \\ -\overline{\beta} & \overline{\alpha}\end{bmatrix}\begin{bmatrix}\overline{\alpha} & -\beta \\ \overline{\beta} & \alpha\end{bmatrix} \\ &= \begin{bmatrix}\alpha\overline{\alpha}+\beta\overline{\beta} & 0 \\ 0 & \overline{\beta}\beta+\overline{\alpha}\alpha\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\end{aligned}$$

3.6.6 $\mathcal{SO}(3)$ in General Form

Every matrix in $\mathcal{SO}(3)$ can be written as a multiplication of three matrices

$$\begin{bmatrix}\cos(\varphi_1) & -\sin(\varphi_1) & 0 \\ \sin(\varphi_1) & \cos(\varphi_1) & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\cos(\varphi_2) & -\sin(\varphi_2) & 0 \\ \sin(\varphi_2) & \cos(\varphi_2) & 0 \\ 0 & 0 & 1\end{bmatrix}$$

where φ_1, φ_2 and θ are Euler angles and have range of $\varphi_{1,2} \in [0, 2\pi)$, and $\theta \in [0, \pi]$.

3.7 Matrix Lie Groups

First explored by Sophus Lie[2], all matrix Lie groups are subgroups of $\mathcal{GL}(n;\mathbb{C})$. They share the property that for a sequence of matrices A_m that converge to a limit matrix A, the limit is also in the group. Hall discusses this property, examples and counter examples at length in his first chapter [Hall, 2015].

The unitary group of matrices $\mathcal{U}(n;\mathbb{C})$ and the orthogonal group of matrices $\mathcal{O}(n;\mathbb{C})$ are subgroups of $\mathcal{GL}(n;\mathbb{C})$, contain their limit points and the identity matrix, making them Lie groups as well.

Lie groups also have manifold properties, though they won't be discussed as these properties are not the focus of this report.

3.7.1 Relationship Between $\mathcal{SU}(2)$ and $\mathcal{SO}(3)$

$\mathcal{SU}(2)$ is related to $\mathcal{SO}(3)$ by a surjective mapping, that is $\mathcal{SU}(2)$ can be mapped onto $\mathcal{SO}(3)$.

$$\tau : \mathcal{SU}(2) \longrightarrow \mathcal{SO}(3)$$

Chirikjian proves this using Möbius transformations to show that $\mathcal{SU}(2)$ is isomorphic to the 3-sphere (a four-dimensional sphere) S^3 and that $\mathcal{SO}(3)$ is isomorphic to the three-sphere's upper half [Chirikjian and Kyatkin, 2001], but we state the results here without proof.

$$\mathcal{SU}(2) \cong S^3$$

$$\mathcal{SO}(3) \cong S^3_{\geq 0}$$

So we assert that $\exists\ \tau'$ such that

$$\tau' : S^3 \longrightarrow S^3_{\geq 0}$$

[2]Pronounced *Lee*.

implying that $\exists\ \tau$ such that

$$\tau : \mathcal{SU}(2) \longrightarrow \mathcal{SO}(3).$$

In fact, we can also express $\mathcal{SO}(3)$ as a quotient group [Hewitt and Ross, 1965]

$$\mathcal{SO}(3) = \mathcal{SU}(2)/\{I, -I\}.$$

4 Measure Theory

A measure is a function on a set that maps it to the real line

$$\mu : S \longrightarrow \mathbb{R}.$$

This allows for a scalar value to be placed on sets. This body of theory is paramount in formulating the mathematical study of analysis.

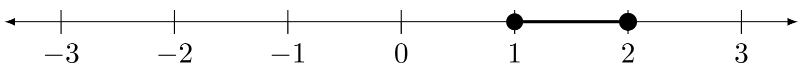
Say our set S was all points $x \in \mathbb{R}^3$ with $|x| \leq 1$. Then a measure $\mu(S)$ could be interpretted as a volume of a sphere. Similarly, if our points lied in $\mathbb{R}^2$ and again satisfied $|x| \leq 1$ then we would have a measure $\mu(S)$ that could be interpretted as an area of a circle.

4.1 Lebesgue Measure

For the sake of brevity, we can say that a Lebesgue measure is one that obtains the length of a given interval.

Example:

$$\mu([1, 2]) = (2 - 1) = 1$$



There is a lot to say on the Lebesgue measure, though that is not the focus of this report, this topic is well discussed by Tao [Tao, 2016]. The very brief introduction here is only in service to the measure we will be using for harmonic analysis, the Haar measure.

4.2 Haar Measure

The Haar measure is invariant under *translation* or better said: invariant under *group action*

$$\mu_L(S) = \mu_L(g \circ S) \quad | \quad G := (X, \circ),\ g \in G,\ S \subset G.$$

Allows us to *measure* the size/length/area/volume of a subset of a group. This then allows us to integrate over a group, i.e.

$$\mu_L(S) = \int_S d_{\mu_L}(g)$$

where $S \subset G$ and μ_L is the *left* Haar measure [Folland, 2015].

The Haar measure is often referred to as the *left* Haar measure because many of the groups that it is used on are non-Abelian. We must specify whether the measure is invariant to group action on the left (left Haar measure) or invariant to group action on the right (right Haar measure). Groups who have identical left and right Haar measures are called *unimodular*.

Let us now detail some simple derivations of Haar measures on $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash 0, \times)$.

4.2.1 Proof of Haar Measure on $(\mathbb{R}, +)$

Assertion: The Haar measure on the group $(\mathbb{R}, +)$ is simply the Lebesgue measure.

Proof:

$$\begin{aligned}
&S \subset \mathbb{R} \\
&S' := \{c \circ s \mid s \in S, c = \text{constant in } \mathbb{R}\} \\
&S = [a, b] \\
&S' = [c + a, c + b]
\end{aligned}$$

Assume $d_{\mu_L}(x) = dx$, i.e. that the left Haar measure is the Lebesgue measure (for those unfamiliar with the Lebesgue measure, this can be simply thought of as the dx we all know and love).

$$\begin{aligned}
\mu_L(S) &= \int_a^b d_{\mu_L}(x) = \int_a^b \ dx = (b - a) \\
\mu_L(S') &= \int_{c+a}^{c+b} d_{\mu_L}(x) = \int_{c+a}^{c+b} \ dx = (c + b) - (c + a) = (b - a)
\end{aligned}$$

$\mu_L(S) = \mu_L(S')$ and so the Haar Measure on $(\mathbb{R}, +)$ is the Lebesgue Measure

$$d_{\mu_L}(x) = d(x) = dx.$$

4.2.2 Proof of Haar Measure on $(\mathbb{R}\backslash 0, \times)$

Assertion: The Haar Measure on $(\mathbb{R}\backslash 0, \times)$ is $\frac{dx}{|x|}$

Proof:

$$\begin{aligned}
&S \subset \mathbb{R} \\
&S' := \{c \circ s \mid s \in S, c = \text{constant in } \mathbb{R}\backslash 0\} \\
&S = [a, b] \\
&S' = [ca, cb]
\end{aligned}$$

Assuming $d_{\mu_L}(x) = dx$ doesn't work!

$$\begin{aligned}
\mu_L(S) &= \int_a^b d_{\mu_L}(x) = \int_a^b \ dx = (b - a) \\
\mu_L(S') &= \int_{ca}^{cb} d_{\mu_L}(x) = \int_{ca}^{cb} \ dx = (cb) - (ca) = c(b - a)
\end{aligned}$$

and so

$$\mu_L(S) \neq \mu_L(S').$$

We have to do a normalisation, meaning we assume $d_{\mu_L}(x) = dx/|x|$.

$$\mu(S) = \int_a^b d_{\mu_L}(x) = \int_a^b \frac{dx}{|x|} = \ln(|b|) - \ln(|a|) = \ln\left(\left|\frac{b}{a}\right|\right)$$

$$\mu(S') = \int_{ca}^{cb} d_{\mu_L}(x) = \int_{ca}^{cb} \frac{dx}{|x|} = \ln(cb) - \ln(ca) = \ln\left(\left|\frac{cb}{ca}\right|\right)$$
$$= \ln\left(\left|\frac{b}{a}\right|\right)$$

Therefore $\mu_L(S) = \mu_L(S')$.

4.3 Other Haar Measures

Tabulations of these measures can be found in most books on Abstract Harmonic Analysis, as in Folland [Folland, 2015].

Group $(X, \circ)$	**Left Haar Measure** (μ_L)
$(\mathbb{R}, +)$	dx
$(\mathbb{R}\backslash 0, \times)$	$dx/\|x\|$
$(\mathbb{C}\backslash 0, \times)$	$dx\ dy/(x^2 + y^2)$
$(\mathcal{GL}_n, \times)$	$dX/\|\det(X)\|^n$
$(\mathbb{H}\backslash 0, \times)$	$dx\ dy\ dz\ dw/(x^2 + y^2 + z^2 + w^2)^2$

4.3.1 Haar's Theorem

There exists μ_L that is unique up to a multiplicative constant for any locally compact group. It is also:

1. Countably additive.
2. Non-trivial on the Borel (measurable) sets of G.
3. Left-translation invariant $\mu(g \circ S) = \mu(S) \mid S \subset G$.
4. Finite on compact sets[3].
5. Algebraic groups for whom the left and right Haar measure are the same are called *unimodular* [Bump, 2013].

The measure was formulated by Haar in 1935, then Cartan and Weil proved the uniqueness and existence theorems in 1940 [Folland, 2015]. So what does this mean? We can now integrate over

[3]Here we are translating the idea of finite from topology (compactness) to finite values of the real line. This is a beautiful connection between topology and analysis that is provided by the Haar measure.

groups

$$\mu_L(G) = \int_G d_{\mu_L}(g) \qquad | \ g \in G$$
$$\mu_L(\mathcal{SO}(3)) = \int_{\mathcal{SO}(3)} d_{\mu_L}(g) \qquad | \ g \in \mathcal{SO}(3)$$
$$\mu_L(\mathcal{SU}(2)) = \int_{\mathcal{SU}(2)} d_{\mu_L}(g) \qquad | \ g \in \mathcal{SU}(2).$$

Note that often the group's measure is normalised, meaning the Haar measure of the entire group is +1, e.g.

$$\mu_L(G) = 1$$
$$\mu_L(\mathcal{SO}(3)) = 1$$
$$\mu_L(\mathcal{SU}(2)) = 1.$$

5 Representation Theory

5.1 What is Representation Theory?

Representation Theory is a way of mapping groups to vector spaces. Once there, we can use the tools of linear algebra to examine algebraic groups [Artin, 2010]

$$\rho : G \longrightarrow V.$$

Note that ρ is used to indicate a representation and σ to indicate an *irreducible* representation.

Caution:
Two representations can be equivalent but vary in their basis, or be composed of several smaller representations (more on that in section 5.3) [Artin, 2010].

Example:
We can take some elements of the symmetry group S_3 and write them in permutation notation

$$e = (\mathbf{1}) \qquad x = (\mathbf{1\ 2\ 3}) \qquad y = (\mathbf{1\ 2}).$$

We can also express this group as

$$S_3 = \{e,\ x,\ x^2,\ y,\ xy,\ x^2y\}.$$

If we consider ρ to be the map from the group to a permutation matrix,

$$\rho : S_3 \longrightarrow V$$

then we can write this *representation* as

$$\rho(S_3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

We now have a set of matrices that obey the multiplication table of the group and each *represent* one group element.

5.2 Characters

A character is a function from a group to a field (commonly $\mathbb{C}$). For our purposes, all characters mentioned will take the trace of a representation

$$\chi : \rho(g) \longmapsto \text{trace}(\rho(g)).$$

Characters simplify representation theory because of some properties of the trace operator:

1. trace(A) is independent of the basis of A.
2. The trace of a representation of a group element g is constant over a conjugacy class of G.

From now on, we will write characters as trace($\rho(g)$) or $\chi(g)$. To elaborate on conjugacy classes, it's best to give an example.

Example:
If we define the group elements in permutation notation, $e = (\mathbf{1})$, $x = (\mathbf{1\ 2\ 3})$ and $y = (\mathbf{1\ 2})$. Then $S_3 = \{e, x, x^2, y, xy, x^2y\}$ and its conjugacy classes are:

$$\text{conj}(S_3) = \{\{e\}, \{x, x^2\}, \{y, xy, x^2y\}\}$$

so we would have the equalities

$$\chi(x) = \chi(x^2)$$
$$\chi(y) = \chi(xy) = \chi(x^2y).$$

Note: An element of a group can belong to one and only one conjugacy class.

With groups being partitioned by their conjugacy classes, characters give us a tool to use on representations that removes distractions generated by the machinery of the vector space (namely, the basis).

5.3 Irreducible Representations

For finite groups, Maschke's Theorem tells us that every representation of a finite group G is a *direct sum* of irreducible representations.

$$\rho \approx n_1\rho_1 \oplus \ldots \oplus n_r\rho_r$$

$\rho_1, \ldots, \rho_r$ are the irreducible representations of G, where

$$n_i\rho_i = \underbrace{\rho_i \oplus \ldots \oplus \rho_i}_{n_i \text{ copies of } \rho_i}.$$

An irreducible representation is one that *cannot* be made up of smaller representations. We will denote σ_j to be the j-th irreducible representation[4] of G [Artin, 2010].

Example:
The irreducible unitary representations of the S_3 group (the symmetries of a triangle) are σ_A, σ_Σ and σ_T.

[4] As opposed to ρ_j just being *some* representation of G, that may or may not be irreducible.

σ_A gives a two dimensional representation of the triangle, mimicking its $2\pi/3$ roations with the transformations provided by using $\sigma_A(x)$, similarly flipping along one of its axes can be done by using $\sigma_A(y)$. σ_Σ is the sign representation, it can be calculated by taking the determinant of the permutation that represents it, and σ_T is the trivial representation that takes all elements of the group to the trivial (1×1) vector of $+1$.

Putting S_3 into each irreducible representation we get:

$$\sigma_A(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_A(x) = \begin{bmatrix} \cos\left(\frac{2\sigma}{3}\right) & -\sin\left(\frac{2\sigma}{3}\right) \\ \sin\left(\frac{2\sigma}{3}\right) & \cos\left(\frac{2\sigma}{3}\right) \end{bmatrix} \qquad \sigma_A(y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_\Sigma(e) = [1] \qquad \sigma_\Sigma(x) = [1] \qquad \sigma_\Sigma(y) = [-1]$$
$$\sigma_T(e) = [1] \qquad \sigma_T(x) = [1] \qquad \sigma_T(y) = [1]$$

We have given the representations only for e, x and y because the other three elements of S_3 can be determined from the values given.

5.4 The Peter-Weyl Theorem

5.4.1 Prelude

The Peter-Weyl theorem makes use of the notion of an L^p space, particularly the L^2 space. An L^2 space is one that consists of all square integrable functions over a domain D. We can state this more explicitly as

$$\int_D |f(x)|^2 \, dx < \infty$$

In terms of locally compact groups, we will be more closely examining the space of $L^2(G)$, which is the space of square integrable functions over the group G.

5.4.2 The Actual Theorem

We define $\mathcal{E}_A$ to generally be the span of the columns of matrix A, $[\sigma] \in \hat{G}$ to be the irreducible unitary representations of $\hat{G}$, the dual object of G, and $\mathcal{C}(G)$ to be the continuous functions on the group G. However we define $\mathcal{E}$ to be

$$\mathcal{E} := \text{the linear span of} \bigcup_{[\sigma]\in\hat{G}} \mathcal{E}_\sigma$$

i.e. the combined span of all the $[\sigma] \in \hat{G}$. Let G be a compact group. Then

- $\mathcal{E}$ is uniformly dense in $\mathcal{C}(G)$
- $L^2(G) = \bigoplus_{[\sigma]\in\hat{G}} \mathcal{E}_\sigma$
- and if σ_{ij} is given by

  $$\{\sqrt{d_\sigma}\sigma_{ij} : i, j = 1, \ldots, d_\sigma, [\sigma] \in \hat{G}\}$$

  then $[\sigma] \in \hat{G}$ is an orthonormal basis for $L^2(G)$.

Consequently, each $[\sigma] \in \hat{G}$ occurs with multiplicity d_σ [Folland, 2015].

This establishes an L^2 isomorphism between compact groups ($\mathcal{SU}(2)$ and $\mathcal{SO}(3)$) and their unitary irreducible representations. This is comparable to the isomorphism that exists between the regular Fourier transform stated via Plancherel's theorem in section 11.3.3.

We use the Peter-Weyl theorem as a cornerstone for establishing an L^2 theory for compact Lie groups which leads to establishing their equations of harmonic analysis.

5.5 Representation of $\mathcal{SU}(2)$

For $\mathcal{SU}(2)$, a compact group, we have the Peter-Weyl theorem as an analogue for the Maschke's Theorem. Note that because $\mathcal{SU}(2)$ is non-Abelian, its representations are in dimensions greater than one (a consequence of Schur's lemma) [Folland, 2015].

In fact, from the mid-20th century two-volume tome of Hewitt and Ross, we can source the following formula for generating the matrix coefficients for representations of $\mathcal{SU}(2)$

$$\sigma_{j,k}^{(l)}(u) = (-1)^{j-k}\left(\frac{(l-j)!(l+j)!}{(l-k)!(l+k)!}\right)^{1/2} \times$$
$$\sum_{s=\max(0,k-l)}^{\min(l+k,l-j)} (-1)^s \binom{l+k}{s}\binom{l-k}{l-j-s} \alpha^{l-j-s}\,\overline{\alpha}^{\,l+k-s}\,\beta^s\,\overline{\beta}^{\,j-k+s}.$$

Regarding notation, here j and k are indexing variables, l is a half integer increment (i.e. $l = \{0, \frac{1}{2}, 1, \frac{3}{2} \ldots\}$), and this produces an entry of an $(m \times m)$ matrix, where $m = 2l + 1$ [Hewitt and Ross, 1965] [Gou, 2019]. The parantheses around vertical entries indicate the binomical coefficient[5], e.g.

$$\binom{l+k}{s} = \frac{(l+k)!}{s!(l+k-s)!}.$$

To calculate the representations by hand is a laborious process, so we have used a computer to do them using original code provided in appendix section 10.2.

$$l = 0 \quad \Rightarrow \quad m = 2(0) + 1 = 1 \quad \Rightarrow \quad (1 \times 1)$$
$$[1]$$

$$l = \frac{1}{2} \quad \Rightarrow \quad m = 2\left(\frac{1}{2}\right) + 1 = 2 \quad \Rightarrow \quad (2 \times 2)$$
$$\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$

$$l = 1 \quad \Rightarrow \quad m = 2(1) + 1 = 3 \quad \Rightarrow \quad (3 \times 3)$$

[5] Sometimes called n-choose-k.

$$\begin{bmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\overline{\beta} & \alpha\overline{\alpha}-\beta\overline{\beta} & \sqrt{2}\beta\overline{\alpha} \\ \overline{\beta}^2 & -\sqrt{2}\overline{\alpha}\overline{\beta} & \overline{\alpha}^2 \end{bmatrix}$$

$$l = \frac{3}{2} \quad \Rightarrow \quad m = 2\left(\frac{3}{2}\right)+1 = 4 \quad \Rightarrow \quad (4\times 4)$$

$$\begin{bmatrix} \alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^3 \\ -\sqrt{3}\alpha^2\overline{\beta} & -2\alpha\beta\overline{\beta}+\alpha^2\overline{\alpha} & 2\alpha\beta\overline{\alpha}-\beta^2\overline{\beta} & \sqrt{3}\beta^2\overline{\alpha} \\ \sqrt{3}\alpha\overline{\beta}^2 & -2\alpha\overline{\alpha}\overline{\beta}+\beta\overline{\beta}^2 & \alpha\overline{\alpha}^2-2\beta\overline{\alpha}\overline{\beta} & \sqrt{3}\beta\overline{\alpha}^2 \\ -\overline{\beta}^3 & \sqrt{3}\overline{\alpha}\overline{\beta}^2 & -\sqrt{3}\overline{\alpha}^2\overline{\beta} & \overline{\alpha}^3 \end{bmatrix}$$

$$l = 2 \quad \Rightarrow \quad m = 2(2)+1 = 5 \quad \Rightarrow \quad (5\times 5)$$

$$\begin{bmatrix} \alpha^4 & 2\alpha^3\beta & \sqrt{6}\alpha^2\beta^2 & 2\alpha\beta^3 & \beta^4 \\ -2\alpha^3\overline{\beta} & \alpha^3\overline{\alpha}-3\alpha^2\beta\overline{\beta} & -0.5\sqrt{6}\left(-2\alpha^2\beta\overline{\alpha}+2\alpha\beta^2\overline{\beta}\right) & 3\alpha\beta^2\overline{\alpha}-\beta^3\overline{\beta} & 2\beta^3\overline{\alpha} \\ \sqrt{6}\alpha^2\overline{\beta}^2 & -\frac{\sqrt{6}\left(3\alpha^2\overline{\alpha}\overline{\beta}-3\alpha\beta\overline{\beta}^2\right)}{3} & \alpha^2\overline{\alpha}^2-4\alpha\beta\overline{\alpha}\overline{\beta}+\beta^2\overline{\beta}^2 & -\frac{1}{3}\sqrt{6}\left(-3\alpha\beta\overline{\alpha}^2+3\beta^2\overline{\alpha}\overline{\beta}\right) & \sqrt{6}\beta^2\overline{\alpha}^2 \\ -2\alpha\overline{\beta}^3 & 3\alpha\overline{\alpha}\overline{\beta}^2-\beta\overline{\beta}^3 & -\frac{\sqrt{6}\left(2\alpha\overline{\alpha}^2\overline{\beta}-2\beta\overline{\alpha}\overline{\beta}^2\right)}{2} & \alpha\overline{\alpha}^3-3\beta\overline{\alpha}^2\overline{\beta} & 2\beta\overline{\alpha}^3 \\ \overline{\beta}^4 & -2\overline{\alpha}\overline{\beta}^3 & \sqrt{6}\overline{\alpha}^2\overline{\beta}^2 & -2\overline{\alpha}^3\overline{\beta} & \overline{\alpha}^4 \end{bmatrix}$$

$$l = \frac{5}{2} \quad \Rightarrow \quad m = 2\left(\frac{5}{2}\right)+1 = 6 \quad \Rightarrow \quad (6\times 6)$$

$$\begin{bmatrix} \alpha^5 & \sqrt{5}\alpha^4\beta & \sqrt{10}\alpha^3\beta^2 & \sqrt{10}\alpha^2\beta^3 & \sqrt{5}\alpha\beta^4 & \beta^5 \\ -\sqrt{5}\alpha^4\overline{\beta} & -4\alpha^3\beta\overline{\beta}+\alpha^4\overline{\alpha} & -3\sqrt{2}\alpha^2\beta^2\overline{\beta}+2\sqrt{2}\alpha^3\beta\overline{\alpha} & -2\sqrt{2}\alpha\beta^3\overline{\beta}+3\sqrt{2}\alpha^2\beta^2\overline{\alpha} & 4\alpha\beta^3\overline{\alpha}-\beta^4\overline{\beta} & \sqrt{5}\beta^4\overline{\alpha} \\ \sqrt{10}\alpha^3\overline{\beta}^2 & 3\sqrt{2}\alpha^2\beta\overline{\beta}^2-2\sqrt{2}\alpha^3\overline{\alpha}\overline{\beta} & 3\alpha\beta^2\overline{\beta}^2-6\alpha^2\beta\overline{\alpha}\overline{\beta}+\alpha^3\overline{\alpha}^2 & -6\alpha\beta^2\overline{\alpha}\overline{\beta}+3\alpha^2\beta\overline{\alpha}^2+\beta^3\overline{\beta}^2 & 3\sqrt{2}\alpha\beta^2\overline{\alpha}^2-2\sqrt{2}\beta^3\overline{\alpha}\overline{\beta} & \sqrt{10}\beta^3\overline{\alpha}^2 \\ -\sqrt{10}\alpha^2\overline{\beta}^3 & -2\sqrt{2}\alpha\beta\overline{\beta}^3+3\sqrt{2}\alpha^2\overline{\alpha}\overline{\beta}^2 & 6\alpha\beta\overline{\alpha}\overline{\beta}^2-3\alpha^2\overline{\alpha}^2\overline{\beta}-\beta^2\overline{\beta}^3 & -6\alpha\beta\overline{\alpha}^2\overline{\beta}+\alpha^2\overline{\alpha}^3+3\beta^2\overline{\alpha}\overline{\beta}^2 & 2\sqrt{2}\alpha\beta\overline{\alpha}^3-3\sqrt{2}\beta^2\overline{\alpha}^2\overline{\beta} & \sqrt{10}\beta^2\overline{\alpha}^3 \\ \sqrt{5}\alpha\overline{\beta}^4 & -4\alpha\overline{\alpha}\overline{\beta}^3+\beta\overline{\beta}^4 & 3\sqrt{2}\alpha\overline{\alpha}^2\overline{\beta}^2-2\sqrt{2}\beta\overline{\alpha}\overline{\beta}^3 & -2\sqrt{2}\alpha\overline{\alpha}^3\overline{\beta}+3\sqrt{2}\beta\overline{\alpha}^2\overline{\beta}^2 & \alpha\overline{\alpha}^4-4\beta\overline{\alpha}^3\overline{\beta} & \sqrt{5}\beta\overline{\alpha}^4 \\ -\overline{\beta}^5 & \sqrt{5}\overline{\alpha}\overline{\beta}^4 & -\sqrt{10}\overline{\alpha}^2\overline{\beta}^3 & \sqrt{10}\overline{\alpha}^3\overline{\beta}^2 & -\sqrt{5}\overline{\alpha}^4\overline{\beta} & \overline{\alpha}^5 \end{bmatrix}$$

5.6 Representation of $\mathcal{SO}(3)$

Because of the two-to-one surjective homomorphism from $\mathcal{SU}(2)$ to $\mathcal{SO}(3)$ mentioned in section 3.7.1, the representations of $\mathcal{SO}(3)$ correspond to the non-negative integer values of l mentioned in the representation of $\mathcal{SU}(2)$ [Hewitt and Ross, 1965].

6 Harmonic Analysis on Locally Compact Abelian Groups

This section on locally compact Abelian groups is brief and only to elucidate the importance of duality in harmonic analysis and provide parallels when we examing the non-Abelian case.

6.1 Schur's Lemma

Schur's lemma is two-parted and we quote it here from Folland [Folland, 2015] without proof.

1. A unitary representation σ of G is irreducible if and only if $\mathcal{C}(\sigma)$ contains only scalar multiples of the identity.

2. Suppose σ_1 and σ_2 are irreducible unitary representations of G. If σ_1 and σ_2 are equivalent then $\mathcal{C}(\pi_1, \pi_2)$ is one dimensional; otherwise, $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.

A result of Schur's lemma is that if G is Abelian, then every irreducible representation of G is one-dimensional.

6.2 Dual Groups

Dual groups ($\hat{G}$) are the set of irreducible unitary representations (IURs) of a group, G. As a consequence of Schur's lemma, all Abelian groups have IURs that are one dimensional, meaning they lie in Hilbert space we can represent with $\mathbb{C}$ (recall that because of the unitary property of these representations, each dimension is complex).

For locally compact Abelian groups, these IURs are often termed the *characters* of the group.

We now list some dual groups.

$$\hat{\mathbb{R}} \cong \mathbb{R}$$
$$\hat{\mathbb{T}} \cong \mathbb{Z}$$
$$\hat{\mathbb{Z}} \cong \mathbb{T}$$

Theorem: If G is discrete, then $\hat{G}$ is compact. If G is compact then $\hat{G}$ is discrete [Folland, 2015].

6.3 Pontryagin Duality

Pontryagin[6] duality theorem states that the characters of the characters is isomorphic to the group itself!

$$\hat{\hat{G}} \cong G$$

[Folland, 2015].

Caution: The Pontryagin duality theorem states that $G \longrightarrow \hat{\hat{G}}$ is an isomorphism between topological groups, not that they are equivalent. This fact is often overlooked and isomorphisms are often stated as equivalences.

7 Harmonic Analysis on Compact Groups

We now crescendo and state the results of harmonic analysis on both finite groups and compact Lie groups.

[6]Sometimes spelt *Pontrjagin*.

7.1 Harmonic Analysis on Finite Groups

$$\hat{f}(\sigma_j) = \sum_{g \in G} f(g)\sigma_j(g^{-1})$$

Where $\hat{f}(\sigma_j)$ is the harmonic analysis of $f(g)$ at σ_j.

If we want the entire spectrum, we will have to do it for each irreducible unitary representation.

$$\mathcal{F}^{-1}(f) = f(g) = \frac{1}{|G|} \sum_{j=1}^{\alpha} d_j \ \mathrm{trace}(\hat{f}(\sigma_j)\sigma_j(g))$$

As shown by the formula, we will need all transforms of all irreducible unitary representations (i.e. the whole *spectrum* of f) $\{\sigma_1, \sigma_2, \dots, \sigma_\alpha\}$ to recover the function [Chirikjian and Kyatkin, 2001].

7.2 Harmonic Analysis on Compact Lie Groups

$$\hat{f}(\sigma_i) = \int_G f(g)\sigma_i(g^{-1}) \ d_{\mu_L}(g)$$

$$\mathcal{F}^{-1}\left(\hat{f}\right) = f(g) = \sum_{\sigma_i \in \hat{G}} d_{\sigma_i} \ \mathrm{trace}\left(\hat{f}(\sigma_i)\sigma_i(g)\right)$$

$\hat{G}$ is the collection of all possible IURs, σ, and is countably infinite. The set of all transforms

$$\left\{\hat{f}(\sigma_i)\right\}_{i=1}^{\infty}$$

is termed the *spectrum* of f [Chirikjian and Kyatkin, 2001].

7.3 Convolution

Citing the results provided by Chirikjian which *holds for both* finite groups and compact Lie groups, we have

$$\mathcal{F}(f_1 * f_2)(\sigma_j) = \hat{f}_2(\sigma_j)\hat{f}_1(\sigma_j)$$

where the result is not commutative [Chirikjian and Kyatkin, 2001].

7.4 Plancherel Theorem

The main differences in the Plancherel theorem for finite and compact groups is that compact Lie groups have a countably-infinite number of representations. Constrastingly, finite groups will have only have a finite number, here denoted by α.

7.4.1 Finite Groups

$$\frac{1}{|G|} \sum_{g \in G} |f(g)|^2 = \sum_{j=1}^{\alpha} d_{\sigma_j} \ \mathrm{trace}\left(\hat{f}(\sigma_j)^* \hat{f}(\sigma_j)\right)$$

[Terras, 2001]

7.4.2 Compact Lie Groups

$$\int_G |f(g)|^2 \, d\mu_L(g) = \sum_{i=1}^{\infty} d_{\sigma_i} \text{trace}\left(\hat{f}(\sigma_i)^* \ \hat{f}(\sigma_i)\right)$$

[Folland, 2015]

7.5 Orthogonality Relations for Compact Groups

Recall that for the harmonics of a complex-exponential Fourier series, each is orthogonal from one another by the inner product

$$\int_0^{2\pi} e^{inx}\overline{e^{imx}} \, dx \ = \begin{cases} 2\pi & n = m \\ 0 & n \neq m \end{cases}.$$

We now have the IURs filling the place of e^{inx} for a Fourier series and so now, our IURs should be orthogonal.

In fact, this is the case, as shown by the Schur orthogonality relations, a result that follows from Schur's lemma.

7.5.1 Schur's Orthogonality Relations

Let σ and σ' be IURs of G and consider $\mathcal{E}_\sigma$ and $\mathcal{E}_{\sigma'}$ to be subspaces of $L^2(G)$, then

1. Then if $\sigma \neq \sigma'$ then $\mathcal{E}_\sigma \perp \mathcal{E}_{\sigma'}$
2. If $\{e_j\}$ is any orthonormal basis for $\mathcal{H}_\sigma$ and σ_{ij} is given by

$$\sigma_{ij}(g) = \langle \sigma(g)e_j, e_i \rangle$$

then $\{\sqrt{d_\sigma} \ \sigma_{ij} : i, j = 1, \ldots, d_\sigma\}$ is an orthonormal basis for $\mathcal{E}_\sigma$ [Folland, 2015].

We can also state this in a manner similar to our original harmonics

$$\int_G \langle \sigma(g)e_j, e_i \rangle \left\langle \sigma'(g)e_{j'}, e_{i'} \right\rangle \ d_{\mu_L}(g) = \begin{cases} \delta_{ij}\delta_{i'j'}\frac{1}{d_\sigma} & \sigma = \sigma' \\ 0 & \sigma \neq \sigma' \end{cases}$$

where δ_{ij} is the Kronecker-delta function defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

8 Next Steps

We can split the next steps into theory and application.

8.1 Theory

Next steps would be to understand the techniques of harmonic analysis of non-compact groups, like $\mathcal{SE}(3)$, the special Euclidean three-dimensional group[7]. $\mathcal{SE}(3)$ is the collection of all rotations and translations in $\mathbb{R}^3$ that do not cause reflections.

These kinds of movements are the ones we experience in the mechanics of everyday life and so $\mathcal{SE}(3)$ has myriads of uses in application. The complexity lies in the non-compactness of the group, for which the theory is spotty in places and considerably more complex than for compact groups.

8.2 Application

The next steps in terms of application is two-parted:

1. Development of a library for the finite group Fourier transform. Given input of only a list of elements, a group action and function to be analysed, it would validate these inputs and return a Fourier transform. The beginnings of this can be seen in appendix section 10.3.

2. Development of a fast Fourier transform (FFT) library to perform harmonic analysis on functions for $\mathcal{SE}(3)$. This may then be used to analyse orbital mechanics, robot kinematics and noise-reduction in sensors. This software would build on the work detailed by Chirikjian and Kyatkin [Chirikjian and Kyatkin, 2001]. The finished library should function on embedded systems, whilst being time and resource sensitive. The candidate language is currently `C++` 11.

9 Conclusion

This work has been heavily focused on building a bridge between traditional Fourier analysis, through the finite group Fourier transform and to the harmonic analysis of compact Lie groups.

Current research involves the theory of non-compact groups, though understanding the current state-of-the-art involves first learning the theory of compact groups, which is one of the outcomes this project intended to accomplish.

[7] Some authors refer to this group as $\mathcal{M}(3)$.

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10 Appendix

We have notation, background theory and code in the appendix.

Software is kept in a private github repository and can be made available for anyone wishing to fork the repository, though there will likely have been changes after the publication of this report.

The author can be found for software requests at `github.com/xandhiller`.

10.1 Notation

Notation	Meaning
$\Rightarrow$	Logically leads to
$\therefore$	Therefore
$\because$	Because
$\subset$	Subset, i.e. if $U \subset X$ then all elements of U can be found in X.
$\subseteq$	Proper subset, i.e. $U \subset X$ and $U \neq X$.
$\sim$	Similar *or* related by an equivalence relation
$\simeq$	Homomorphic
$\cong$	Isomorphic
$\in$	Contained in
$\exists$	Exists
$\mathbb{N}$	Set of natural numbers *excluding* zero, e.g. $1, 2, 3, \ldots$
$\mathbb{Z}$	Set of integers, e.g. $-2,\ -1,\ 0,\ 1,\ 2,\ \ldots$
$\mathbb{Q}$	Set of rational numbers, i.e. those that can be expressed as a *ratio* of two integers.
$\mathbb{R}$	Set of real numbers, e.g. $0,\ 1,\ 2,\ e,\ 3.01,\ \pi,\ 4,\ \ldots$
$\mathbb{C}$	Set of complex numbers, e.g. $a + ib$ where $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$.
$X \longrightarrow Y$	Set X is mapped to the set Y.
$x \longmapsto x^2$	The point x is mapped to the point x^2.
$\text{conj}(G)$	The conjugacy classes of the group G.
$\|X\|$	The cardinality of the set X, i.e. how many elements X has.
$\|G\|$	The cardinality of the group G, i.e. how many elements G has.
ρ, ρ_g or $\rho(g)$	Some representation of the group element g. Not necessarily irreducible.
σ, σ_g or $\sigma(g)$	Some *irreducible* representation of the group element g.

Caution:
Caligraphic text such as $\mathcal{SO}(n)$, $\mathcal{SU}(n)$ or $\mathcal{GL}(n)$ is used to denote Lie groups. Should the need arise to denote *collections* (a set of sets) then we will also use caligraphic text, though the difference between a Lie group and a collection should be clear from context.

10.2 Code for Producing Matrix Elements of $\mathcal{SU}(2)$ Representation

```python
#! /usr/bin/python3.7 --

from sympy import Symbol, Matrix, init_printing, factorial, binomial, sqrt
from sympy import conjugate as conj
from numpy import arange, power
from sympy.matrices import zeros
from sympy import latex
init_printing()

```

```python
# Written with bad variable names to match notation in Hewitt and Ross
def repres_elem(j,k,l,
                alpha=Symbol('alpha', complex=True),
                beta=Symbol('beta', complex=True)):
    # Have to do this because numpy isn't playing nicely with negative exponents.
    prefix_exp_power = int(j-k)
    prefix = (-1)**(prefix_exp_power) * sqrt((factorial(l+j)*factorial(l-j)/(factorial(l+k)*factorial(l-k))))
    # Setup vars for iteration/sum iteration
    upper=min(l-j,l+k)
    lower=max(0,k-j)
    # Run iteration (summation)
    summand=0
    for s in arange(lower,upper+1,1):
        summand += (-1)**s * binomial(l+k,s) * binomial(l-k, l-j-s) *  alpha**(l-j-s) * conj(alpha)**(l+k-s) * beta**(s) * conj(beta)**(j-k+s)
    # Done
    return prefix*summand

##########################
l=5/2
##########################
R = zeros(int(2*l+1))
elems=[]
for j in arange(-l,l+1,1):
    row=[]
    for k in arange(-l,l+1,1):
        row.append(repres_elem(j,k,l))
    elems.append(row)

R = Matrix(elems)
display(R)
print(latex(R))
```

10.3 Base Code for Finite Group Fourier Transform

```python
#! /usr/bin/python3.7 --

import numpy as np

'''
Custom matrix class that has the attributes of equality and not-equality that
I want, which is element-wise.
'''
class matrix(np.matrix):
    def __eq__(self, other):
        if type(other) == type(None):
            return False
        A = self
        B = other
        if A.shape == B.shape:
            for i in range(A.shape[0]):
                for j in range(A.shape[1]):
                    if A[i,j] != B[i,j]:
                        return False
        else:
            raise 'Matrix Shape Error'
        return True

    def __ne__(self, other):
        if type(other) == type(None):
            return True
        A = self
        B = other
        if A.shape == B.shape:
            for i in range(A.shape[0]):
                for j in range(A.shape[1]):
                    if A[i,j] != B[i,j]:
                        return True
        else:
            raise 'Matrix Shape Error'
        return False


############################################################################
class finite_group():

    def __init__(self, elems, action):
        self.elems = elems
        self.action = action
        self.cardinality = len(self.elems)
        if self.is_matrix_group():
            self.matrix_group = True
        else:
            self.matrix_group = False
        if self.is_group() is False:
            self.elems = None
            self.action = None
            raise 'IsNotGroup'

    '''
    True: All elements are matrices
```

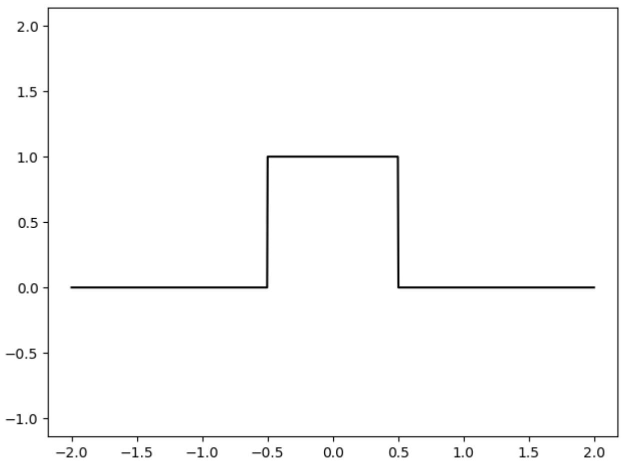
```
    False: At least one element is not a matrix in the set of elements
    '''
    def is_matrix_group(self):
        for g in self.elems:
            if type(g) == type(np.matrix([0])):
                continue
            else:
                return False
        return True

    '''
    Satisfies all of the group structure properties, namely the existence of an
    identity, existence of inverses and closure.
    '''
    def is_group(self):
        if self.has_identity() and self.has_closure() and self.has_inverses():
           return True
        else:
           return False

    '''
    If action(g,h) == action(h,g) for all g and h, then the group is abelian,
    meaning that the group has a commutative action.
    '''
    def is_commutative(self):
        X = self.elems
        for g in X:
            for h in X:
                if self.action(g,h) != self.action(h,g):
                    return False
                else:
                    continue
        return True

    def has_identity(self):
        X = self.elems
        for g in X:
            for h in X:
                # Found identity candidate
                if self.action(g,h) == h:
                    # Confirming identity candidate
                    for j in X:
                        if self.action(g,j) == j:
                            continue
                        else:
                            break
                    self.identity = g
                    return True
        return False

    '''
    Description:
        For all g,h in the group action(g,h) should be in the group as well.
        Otherwise, the set and action do not satisfy the requirements of a group
        and instead are a groupoid.
    Return Values:
        True: if it does have closure
        False: if it does not have closure
    '''
    def has_closure(self):
        X = self.elems
        for g in X:
            for h in X:
                if self.action(g,h) not in X:
                    return False
        return True

    def has_inverses(self):
        X = self.elems
        for g in X:
            result = self.get_inverse(g)
            if result == None:
                return False
        return True

    '''
    Get inverse of a certain element
    '''
    def get_inverse(self, el):
        X = self.elems
        for g in X:
            if self.action(g,el) == self.identity:
                return g
        # Inverse not found
        return None

    def center(self):
        pass

    def get_identity(self):
        return self.identity

################################################################################
```

```python

def main():
    x = matrix([
            [0,0,1],
            [1,0,0],
            [0,1,0]])
    y = matrix([
            [0,1,0],
            [1,0,0],
            [0,0,1]])
    I = matrix([
            [1,0,0],
            [0,1,0],
            [0,0,1]])

    print(type(x))
    my_set = [I, x, y, x*y, x*x*y, x*x]
    def my_action(a,b):
        return a*b
    G = finite_group(my_set, my_action)
    print(G.is_group())
    print(G.get_identity())
    print(G.get_inverse(x)*x)
    print(I.trace())

if __name__ == '__main__':
    main()
```

11 Background

11.1 Arriving at the Fourier Transform

Fourier series are useful for periodic functions, but what about transforming those that are aperiodic? For example, if we look at the classic example of the rectangle function



it takes a value of $+1$ over $-1/2$ to $1/2$ and then never repeats. The period of such a function must be the whole real line. We then have to extend our fundamental period to be infinitely long, i.e. $1/\xi_0 \longrightarrow \infty$. But first we need to slightly rearrange the coefficient equation

$$G_n = \frac{1}{\xi_0}\int_{-1/(2\xi_0)}^{1/(2\xi_0)} f(x)e^{-2\pi n\xi_0 x}\,dx \qquad \longrightarrow \qquad G_n\xi_0 = \int_{-1/(2\xi_0)}^{1/(2\xi_0)} f(x)e^{-2\pi n\xi_0 x}\,dx$$

and we now can extend the period by taking the limit of both sides

$$\lim_{1/\xi_0\to\infty}\big(G_n\xi_0\big) = \lim_{1/\xi_0\to\infty}\left(\int_{-1/(2\xi_0)}^{1/(2\xi_0)} f(x)e^{-2\pi n\xi_0 x}\,dx\right)$$

$$n\xi_0 \longrightarrow \xi \qquad \frac{1}{2\xi_0}\longrightarrow\infty \qquad \frac{-1}{2\xi_0}\longrightarrow -\infty \qquad G_n\xi_0 \longrightarrow \hat{f}(\xi)$$

then substituting our new values, we have arrived at the definition of the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\ dx.$$

11.2 The Fourier Transform and its Properties

We first turn our attention to the original definition of the transformation of functions over the real line.

$$\mathcal{F}(f) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\ dx$$

There are a few components to this transformation that we can examine, first the function f is one that maps from the real-line to the complex numbers

$$f : \mathbb{R} \longrightarrow \mathbb{C}.$$

$e^{-2\pi i\xi x}$ is the kernel[8] of the transform and dx is the Lebesgue measure [Tao, 2016].

The results below can be derived using single variable calculus. The consequences in engineering of these results can be discussed at great length, though we leave that to other authors such as Osgood [Osgood, 2019].

11.3 General Properties

11.3.1 Convolution

Convolution is an operation on two functions f and g where we calculate

$$\int_{-\infty}^{\infty} f(x-\tau)g(\tau)\ d\tau.$$

Convolution in one domain leads to multiplication in the codomain, explicitly this means

$$\mathcal{F}(f * g) = \mathcal{F}\left(\int_{-\infty}^{\infty} f(x-\tau)g(\tau)\ d\tau\right) = \hat{f}\ \hat{g}$$

and

$$\mathcal{F}^{-1}(\hat{f} * \hat{g}) = \mathcal{F}^{-1}\left(\int_{-\infty}^{\infty} \hat{f}(\xi-\tau)\hat{g}(\tau)\ d\tau\right) = f\ g.$$

11.3.2 Translation Effects

A delay in the domain (here, the real line $\mathbb{R}$) leads to a multiplication of a function in the codomain.

$$\mathcal{F}(f(x-a)) = \int_{\mathbb{R}} f(x-a)e^{-2\pi i\xi x}\ dx = e^{2\pi i\xi(-a)}\hat{f}(\xi)$$

[8]Note that when we use *kernel* what is meant is the kernel of the integral transformation and not an algebraic homomorphism mapping into the identity.

11.3.3 Parseval-Plancherel Theorem

For periodic functions with a period P where

$$P := \left[\frac{-1}{2\xi_0}, \frac{1}{2\xi_0}\right] \subseteq \mathbb{R}$$

$$\left\|f\right\|^2_{L^2(P)} = \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |G_n|^2$$

For aperiodic functions:

$$\left\|f\right\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi$$

Proof and exposition of these theorems can be found in Folland [Folland, 2015].

11.3.4 Orthogonality Relations

For a Fourier series of a function, we get something that looks like

$$f(x) = \ldots + G_0 + G_1 e^{ix} + G_2 e^{i2x} + G_3 e^{i3x} + \ldots$$

we often refer to the e^{inx} term as a harmonic. In later parts of this report the orthogonality of coefficients of a Fourier series will be important to link our idea of harmonic analysis to Fourier analysis and so we shall demonstrate the orthogonality of harmonics here.

We always test for orthogonality with the inner product:

$$\int f \, \overline{g} \, dx.$$

If the integral is zero, then f and g are orthogonal. Now let us suppose we have two different harmonics, e^{inx} and e^{imx}, where $n - m \neq 0$ and we are testing for orthogonality over $[0, 2\pi]$, then

$$\int_0^{2\pi} e^{inx}\overline{e^{imx}} \, dx = \int_0^{2\pi} e^{i(n-m)x} \, dx = \frac{e^{i(n-m)2\pi} - 1}{i(n-m)} = 0 \qquad \forall \, n, m \in \mathbb{Z} \mid n - m \neq 0$$

and if $n - m = 0$? Then $n = m$ and so we have

$$\int_0^{2\pi} e^{i(m-m)x} \, dx = \int_0^{2\pi} \, dx = 2\pi \qquad \forall \, n, m \in \mathbb{Z} \mid n - m = 0$$

in summary, for integers n and m:

$$\int_0^{2\pi} e^{inx}\overline{e^{imx}} \, dx \;=\; \begin{cases} 2\pi & n = m \\ 0 & n \neq m \end{cases}$$

11.4 Some Linear Algebra

11.4.1 Basis

A basis of a vector space V is a set of vectors that are linearly independent and span the space. One of the simplest examples is e_1, e_2 and e_3 that we use in a 3-dimensional vector space. This basis (and every basis) is simply a set of vectors we can use to express all other vectors in the form of linear combinations. i.e.

$$\forall\ v \in V \quad v = a_1 e_1 + \ldots + a_n e_n \quad | \quad a_1, \ldots, a_n \in \mathbb{R}.$$

Notation can vary, however. For example, if we denote our basis $\mathbf{B}$ and denote each column by $\mathbf{b_1}, \ldots, \mathbf{b_n}$ then we get:

$$\mathbf{B} = \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n_1} & \ldots & b_{nn} \end{bmatrix} = [\mathbf{b_1} \ldots \mathbf{b_n}]$$

where our example of a three dimensional vector space with the standard basis would take the form

$$\mathbf{B}_3 = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[Strang, 2006].

11.4.2 Orthonormal Basis

An orthonormal basis is one that is both *orthogonal* and *normalised*. Orthogonality is where each $\mathbf{b_i} \perp \mathbf{b_j}$ for all $i \neq j$, i.e. $\mathbf{b_i} \cdot \mathbf{b_j} = 0$ for all vectors not the same. Normalisation means that the length of each basis vector is one, $|b| = 1$ [Strang, 2006].

11.4.3 Trace Operator

Given a matrix A of size $n \times n$, we can take the diagonal entries and sum them, this is called *taking the trace of* A. More explicitly, if we denote A_{ij} as the entry of A on its i-th row and j-th column, then the trace is

$$\text{trace}(A) = \sum_{i=1}^{n} A_{ii}.$$

Trace is invariant of basis [Strang, 2006] and should it be performed on a matrix group then the operation is constant over the group's conjugacy classes [Artin, 2010].

11.4.4 Groups of Matrices

$\mathcal{GL}(n)$ is the Lie group of all $n \times n$ invertible matrices[9].

$$\forall A \in \mathcal{GL}(n)\ \exists\ A^{-1} \in G \mid A^{-1}A = AA^{-1} = I$$

[9] Recall that not all matrices have inverses and so $\mathcal{GL}(n)$ is a subset of all the matrices that exist.

$\mathcal{U}(n)$ denotes the Lie group of unitary matrices of size $n \times n$, who have the property that $U^* = U^{-1}$ where the * operation indicates Hermitian conjugate,

$$UU^* = U^*U = I.$$

$\mathcal{O}(n)$ is the Lie group of orthogonal matrices of size $n \times n$ who have the property $A^{-1} = A^T$,

$$AA^T = A^TA = I.$$

11.5 Duality

11.5.1 Duality in Sets

The complement of a set A is all the elements not in A, and is usually denoted by $\overline{A}$.

Observe that the operation is an *involution*, i.e. it is its own inverse (like the function $1/x$ or $-x$). Because of this, the complement of A is $\overline{A}$ but the complement of $\overline{A}$ is A, more explicitly

$$\overline{\overline{A}} = A.$$

This inherent relationship between the two is one of the simplest and illustrative examples of the principle of duality.

11.5.2 Duality in Vector Spaces

Consider the vector space V of length n column vectors. What other space could act on this space and send it to the scalars? Only the space of all length n row vectors, which we will call V^*. Then for $v \in V$ and $v^* \in V^*$

$$v^*v = \begin{bmatrix} v_1^* & \dots & v_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1v_1^* + \dots + v_nv_n^* = \sum_i^n v_iv_i^*.$$

This gives us a way to produce a scalar from any v and v^* from their respective spaces.

We say that V^* is the space of linear mappings to $\mathbb{R}$ on V, and that V is the space of linear mappings to $\mathbb{R}$ on V^*. V and V^* are termed *dual vector spaces* of each other.

11.6 Miscellaneous

Below are some other frequently used pieces of mathematics that don't easily fit into any other section.

11.6.1 Direct Sum ($\oplus$)

Say we have two matrices A and B defined by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then their direct sum $A \oplus B$ is

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix}.$$

At times, direct sums involve many matrices and take a form similar to the Σ notation often used for arithmetic summation, for example

$$\bigoplus_{i=1}^{n} A_i = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix}.$$

Group representations are often made up of smaller representations composed with a direct sum.

11.6.2 Tensor Product ($\otimes$)

For a vector $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ the tensor product between them is

$$v \otimes w = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1w_1 & v_1w_2 & \dots & v_1w_m \\ v_2w_1 & v_2w_2 & \dots & v_2w_m \\ \vdots & \vdots & \ddots & \vdots \\ v_nw_1 & v_nw_2 & \dots & v_nw_m \end{bmatrix}$$

[Neuenschwander, 2015].

11.6.3 Permutation Notation

A permutation is a bijective function from a set onto itself. Consider a sequence of values $\{x_1, x_2, x_3, x_4\}$, we can rearrange these in the order $\{x_4, x_1, x_2, x_3\}$ but how do we state this mathematically? Perhaps as a mapping

$$\{x_1, x_2, x_3, x\} \quad \longmapsto \quad \{x_4, x_1, x_2, x_3\}$$

although the convention is often to use permutation matrices or permutation notation. A permutation matrix for our rearrangement would look like

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and permutation notation would take the form (**1 2 3 4**), [Artin, 2010]. The meaning of this bracketed notation is that element 1 goes to place 2, element 2 goes to place 3, element 3 goes to place 4 and element 4 goes to place 1. It is easy to see why this notation is used when we examine how brief it is to state (**3 32**) instead of writing a 32×32 permutation matrix (assuming 32 elements in the set).

11.6.4 Dense Sets

For a topological space X, a subset $A \subset X$ is called *dense* if every point in X belongs to A or is a limit point of A [Munkres, 2000].