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Alternative construction of groups of type E_6

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Vacation Research Scholarships are funded jointly by the Department of Education and the Australian Mathematical Sciences Institute.

Abstract

The groups of type E_6 are a family of finite simple groups, categorised as exceptional groups of Lie type in the Classification of Finite Simple Groups. These groups have elegant constructions using Lie theory or algebraic groups. In 2019 Bray, Stepanov and Wilson presented a new construction of the groups of type E_6 in 'Octonions, Albert vectors and the group $\mathrm{E}_6(F)$' which does not make use of Lie theory or algebraic groups. This report reviews this construction and proves it is equivalent to the original construction of the groups of type E_6 by Dickson 1901.

1 Introduction

The groups of type E_6 are a family of finite simple groups first discovered by Leonard Eugene Dickson in 1901 (Wilson 2009: 169). Within the Classification of Finite Simple Groups they are categorised as one of the families of exceptional groups of Lie type. Broadly, there are three ways of constructing the groups of type E_6: via Lie theory, via algebraic groups, and from the subgroup of automorphisms of a vector space preserving some cubic form (Gorenstein et al. 1997: 1). When using Lie theory (see Carter 1972) or algebraic groups (see Gorenstein et al. 1997: Chapter 1, Chapter 2) the groups of type E_6 share a similar construction to the other families of classical and exceptional groups of Lie type and as such these methods reveal a lot about the relationship between these families of groups. On the other hand, constructions of the exceptional groups from particular automorphisms of some vector space are less general and different families of exceptional groups require different treatment. Although less elegant, these methods reveal a lot of information about the structure of exceptional groups and 'gain markedly when it comes to performing concrete calculations' (Wilson 2009: 111).

In this report we review a recent refinement of the construction of the groups of type E_6 from the automorphisms of a vector space preserving a particular cubic form by Bray, Stepanov and Wilson (2019) in 'Octonions, Albert vectors and the group $\mathrm{E}_6(F)$' (the BSW construction). We also show that it is equivalent to Dickson's original construction in 1901.

Statement of Authorship

Rohan Hitchcock performed research, produced the proof in Section 4.1 and wrote this report. John Bamberg and Michael Giudici determined the direction of research, provided advice on research and edited this report.

2 Octonions

This section introduces octonion algebras – a type of composition algebra – which are a central component of the BSW construction of the groups of type E_6. To define the octonions we must first introduce quadratic and bilinear forms. We then define composition algebras and state some of the results used in the construction of the groups of type E_6.

2.1 Quadratic and Bilinear Forms

Quadratic and bilinear forms generalise many of the algebraic properties of the norm-squared and inner product respectively to vector spaces over arbitrary fields. In the following let F be a field.

Definition 2.1. A *bilinear form* on an F-vector space V is a map

$$\begin{aligned}\langle\ ,\ \rangle : V \times V &\longrightarrow F \\ (u, v) &\longmapsto \langle u, v\rangle\end{aligned}$$

such that for all $u, v, w \in V$ and $\lambda \in F$

(1) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$,

(2) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$,

(3) $\langle \lambda u, v\rangle = \lambda \langle u, v\rangle = \langle u, \lambda v\rangle$.

A bilinear form is said to be *symmetric* if it has the additional property that for all $u, v \in V$

(4) $\langle u, v\rangle = \langle v, u\rangle$.

A bilinear form on an F-vector space V is called *nondegenerate* if for all $v \in V$ it satisfies

$$(\forall x \in V\ \langle x, v\rangle = 0) \implies v = 0.$$

Definition 2.2. A *quadratic form* on an F-vector space V is a map

$$Q : V \longrightarrow F$$

which satisfies

(1) For all $u \in V$ and $\lambda \in F$ we have $Q(\lambda u) = \lambda^2 Q(u)$,

(2) The map

$$\begin{aligned}V \times V &\longrightarrow F \\ (u, v) &\longmapsto Q(u + v) - Q(u) - Q(v)\end{aligned}$$

defines a symmetric bilinear form.

A quadratic form is called *nondegenerate* if the bilinear form it induces is nondegenerate. Let Q be a quadratic form on an F-vector space V and $\langle\ ,\ \rangle$ the bilinear form induced by Q. Then for $u \in V$

$$\begin{aligned}\langle u, u\rangle &= Q(u + u) - Q(u) - Q(u) \\ &= Q(2u) - 2Q(u) \\ &= 2^2 Q(u) - 2Q(u) \\ &= 2Q(u)\end{aligned}$$

where we define $2 = 1 + 1$. Then whenever the characteristic of F is not equal to 2 (so that $2 \neq 0$) we have

$$Q(u) = \frac{1}{2} \langle u, u \rangle .$$

So whenever the characteristic of the field is not 2 the concepts of quadratic and symmetric bilinear forms coincide.

2.2 Defining composition algebras

Let F be a field. We define an *F-algebra* A as an F-vector space equipped with a multiplication which is compatible with the vector space structure in the sense that both left and right multiplication by a vector are linear maps. We also insist on the existence of a multiplicative identity. Explicitly, to be an F-algebra the multiplication on A must satisfy

(1) For all $x, y, z \in A$ we have $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$,

(2) For all $x, y \in A$ and $\lambda \in F$ we have $(\lambda x)y = \lambda(xy) = x(\lambda y)$,

(3) There exists an identity $1 \in A$ such that for all $x \in A$ we have $1 \cdot x = x \cdot 1 = x$.

Note that the multiplication on an F-algebra is not necessarily associative or commutative as we have defined it. For the purposes of this report a *subalgebra* of A is defined as a vector subspace which is closed under multiplication and contains the multiplicative identity from A.

Definition 2.3. A *composition algebra* C over F is an F-algebra equipped with a nondegenerate quadratic form $N : C \longrightarrow F$. We call N the *norm* of C.

An *octonion algebra* is simply an 8-dimensional composition algebra. Some composition algebras are quite familiar. The complex numbers $\mathbb{C}$ are a 2-dimensional composition algebra over the real numbers, where the norm of a complex number $z = x + iy \in \mathbb{C}$ is defined as

$$N(z) = |z|^2 = x^2 + y^2.$$

Another example having dimensions 2 over the real numbers is $\mathbb{R}^2$ when equipped with multiplication defined by

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1y_1 + x_2y_2, x_1y_2 + x_2y_1)$$

and norm defined by

$$N((x_1, x_2)) = x_1^2 - x_2^2.$$

Note that there are some non-zero elements of $\mathbb{R}^2$ with norm zero, but this is not the case in $\mathbb{C}$. This highlights an important distinction between composition algebras which is discussed further in Section 2.4.

Let C be a composition algebra over F with norm N and multiplicative identity $1 \in C$. Let $\langle\ ,\ \rangle$ denote the bilinear form induced by N. The *trace* $T : C \longrightarrow F$ of an element $x \in C$ is defined as

$$T(x) = \langle x, 1\rangle$$

and the *conjugate* of $x \in C$ is defined as

$$\overline{x} = T(x) \cdot 1 - x.$$

2.3 Some properties of composition algebras

Unfortunately, octonion algebras are never associative or commutative (Springer et al. 2000: 14) however it is possible to prove limited associativity and commutativity properties in some circumstances. This section states some of the properties used directly in subsequent sections. For a comprehensive summary of the properties of composition algebras, as well as proofs of the results stated here, see Section 1 of Springer et al. 2000. In the following let F be a field and C be a composition algebra over F.

Lemma 2.4. *For all* $x, y, z \in C$

(1) $x\overline{x} = \overline{x}x = N(x) \cdot 1$,

(2) $\overline{xy} = \overline{y}\overline{x}$,

(3) $\overline{x+y} = \overline{x} + \overline{y}$,

(4) $T(x(yz)) = T((xy)z)$.

Lemma 2.5. *The centre of a composition algebra is* $\langle 1 \rangle$.

Motivated by the previous lemma, we will consider the field F as being contained within C via the canonical isomorphism $F \cong \langle 1 \rangle$. From now on elements of the subalgebra $\langle 1 \rangle$ are considered to be elements of the field F and vice versa.

Lemma 2.6. *An element* $x \in C$ *is invertible if and only if* $N(x) \neq 0$. *If* x *is invertible then* $x^{-1} = N(x)^{-1}\overline{x}$.

An element of C which is not invertible is called *isotropic*.

2.4 Classification of composition algebras

It turns out that a composition algebra C over a field F can only have dimension 2, 4 or 8 or, if $\mathrm{char} F \neq 2$, 1 over F (Springer et al. 2000: 14). The dimension of a composition algebra determines many of the properties of the multiplication. The multiplication in composition algebras of dimension 2 is commutative and associative, in those of dimension 4 (quaternion algebras) it is associative but never commutative, and in those of dimension 8 it is never commutative or associative (Springer et al. 2000: 14).

There is an important distinction to be made between composition algebras in which every element is invertible, and those in which this is not the case. A composition algebra is called *split* if it contains non-zero isotropic (non-invertible) elements, or equivalently if it contains a non-zero element with zero norm (see Lemma 2.6). Otherwise a composition algebra is called *non-split*. The following important results are from page 19 and page 22 of Springer et al. 2000 respectively. Let F be a field.

Theorem 2.7. *There exist unique (up to isomorphism) split composition algebras over F for each dimension 2, 4, and 8.*

Theorem 2.8. *If F is finite then any composition algebra over F is necessarily split.*

Together Theorems 2.7 and 2.8 say that over a finite field there exists a unique octonion algebra, and furthermore it is always split. The uniqueness of split octonion algebras means we can fix a basis for the algebra independent of the field F, as is done in the BSW construction.

3 The BSW construction

This section outlines the BSW construction of the groups of type E_6 in Bray et al. 2019. For each field F and octonion algebra $\mathbb{O}$ over F we obtain a group of type E_6 which we will call $E_6(F)$. The group $E_6(F)$ will be finite whenever F is a finite field. Despite the notation, $E_6(F)$ does depend on the octonion algebra, however when F is finite (hence $E_6(F)$ is finite), $\mathbb{O}$ is unique and so the octonion algebra is not represented in the notation.

3.1 Constructing $E_6(F)$

Let F be a field and $\mathbb{O}$ an octonion algebra over F with multiplicative identity $1 \in \mathbb{O}$. The *Albert space* $\mathbb{J}$ is the F-vector space generated by matrices of the form

$$(a, b, c|A, B, C) = \begin{pmatrix} a & C & \overline{B} \\ \overline{C} & b & A \\ B & \overline{A} & c \end{pmatrix}$$

where $a, b, c \in \langle 1 \rangle \subset \mathbb{O}$ and $A, B, C \in \mathbb{O}$. The *Dickson-Freudenthal determinant* $\Delta : \mathbb{J} \longrightarrow \mathbb{O}$ of an Albert vector $X = (a, b, c|A, B, C) \in \mathbb{J}$ is defined as

$$\Delta(X) = abc - aA\overline{A} - bB\overline{B} + T(ABC).$$

Note that this expression is well defined even without associativity in $\mathbb{O}$ by the properties of composition algebras discussed in Section 2.3. The group $SE_6(F)$ is defined as the vector space automorphisms of $\mathbb{J}$ which preserve this determinant, and $E_6(F)$ is defined to be the quotient of $SE_6(F)$ by its centre. We are particularly interested in the case when F is a finite field, since this is when $E_6(F)$ is a finite group. The centre of the automorphism group of $\mathbb{J}$ is exactly the set of scalar maps, that is the maps of the form $f(X) = \lambda X$ for some non-zero $\lambda \in F$. By observing that the Dickson-Freudenthal determinant is a cubic form, we see that if f is to preserve this determinant then we must have $\lambda^3 = 1$. When F is a finite field with q elements, such a $\lambda \in F$ only exists when $q \equiv 1 \pmod 3$. Therefore when $F = \mathbb{F}_q$ we have $E_6(F) \cong SE_6(F)$ if and only if $q \not\equiv 1 \pmod 3$.

Suppose $\mathbb{O}$ is split, which is necessarily the case when F is finite by Theorem 2.8. Then $\mathbb{O}$ is unique by Theorem 2.7 so, without loss of generality we can choose a fixed basis for $\mathbb{O}$. In Bray et al. 2019 this is done in the following way. Let $I = \{\pm 0, \pm 1, \pm\omega, \pm\overline{\omega}\}$ and consider the set of symbols $\mathcal{B} = \{e_i\}_{i\in I} = \{e_{-1}, e_{\overline{\omega}}, e_\omega, e_0, e_{-0}, e_{-\omega}, e_{-\overline{\omega}}, e_1\}$. We define multiplication on the elements of $\mathcal{B}$ according to the following table in Figure 1 and consider the eight dimensional non-associative F-algebra generated by $\mathcal{B}$. Note that the

	e_{-1}	$e_{\overline{\omega}}$	e_ω	e_0	e_{-0}	$e_{-\omega}$	$e_{-\overline{\omega}}$	e_1
e_{-1}	0	0	0	0	e_{-1}	$e_{\overline{\omega}}$	$-e_\omega$	$-e_0$
$e_{\overline{\omega}}$	0	0	$-e_{-1}$	$e_{\overline{\omega}}$	0	0	$-e_{-0}$	$e_{-\omega}$
e_ω	0	e_{-1}	0	e_ω	0	$-e_{-0}$	0	$-e_{-\overline{\omega}}$
e_0	e_{-1}	0	0	e_0	0	$e_{-\omega}$	$e_{-\overline{\omega}}$	0
e_{-0}	0	$e_{\overline{\omega}}$	e_ω	0	e_{-0}	0	0	e_1
$e_{-\omega}$	$-e_{\overline{\omega}}$	0	$-e_0$	0	$e_{-\omega}$	0	e_1	0
$e_{-\overline{\omega}}$	e_ω	$-e_0$	0	0	$e_{-\overline{\omega}}$	e_1	0	0
e_1	$-e_{-0}$	$e_{-\omega}$	$e_{-\overline{\omega}}$	e_1	0	0	0	0

Figure 1: Multiplication table of the split octonion basis.

multiplicative identity is given by $e_0 + e_{-0} = 1$. For an element $x = \sum_{i\in I} \lambda_i e_i$ of this algebra we define a trace

$$T(x) = \lambda_0 + \lambda_{-0}$$

and norm

$$N(x) = \lambda_{-1}\lambda_1 + \lambda_{-\overline{\omega}}\lambda_{\overline{\omega}} + \lambda_{-\omega}\lambda_{-\omega} + \lambda_{-0}\lambda_0.$$

One can check that this defines a composition algebra by computing the norm of the product and the product of the norms of two arbitrary elements.

3.2 Generators for $SE_6(F)$

For certain 3×3 matrices M with entries in $\mathbb{O}$ we can define an endomorphism of the Albert space by $X \longmapsto \overline{M}^\top XM$ for $X \in \mathbb{J}$. Since octonions are not associative we cannot assume that $m_1(xm_2) = (m_1x)m_2$, so in general this endomorphism is not well defined. We therefore restrict the entries of M to be from a subalgebra S of $\mathbb{O}$ which is such that for all $x, y \in S$ and $z \in \mathbb{O}$ we have $(xz)y = x(zy)$. In Bray et al. 2019 these are called *sociable* subalgebras, and $S = \langle 1\rangle$ is one such example.

Let $\mathrm{Soc}(3,\mathbb{O})$ be the set of 3×3 matrices with entries in $\mathbb{O}$, such that for any $M \in \mathrm{Soc}(3,\mathbb{O})$ all entries of M come from the same sociable subalgebra. Note that for $M, N \in \mathrm{Soc}(3,\mathbb{O})$ the entries of N may come from a different sociable subalgebra than the entries of M. We define a map $\mu : \mathrm{Soc}(3,\mathbb{O}) \longrightarrow \mathrm{End}(\mathbb{J})$ by

$$\mu(M)(X) = \overline{M}^\top XM$$

for $M \in \mathrm{Soc}(3,\mathbb{O})$ and $X \in \mathbb{J}$. Suppose $M, N \in \mathrm{Soc}(3,\mathbb{O})$ are two matrices with entries taken from the same sociable subalgebra. Then for $X \in \mathbb{J}$ we have

$$(\overline{MN})^\top X(MN) = \overline{N}^\top(\overline{M}^\top XM)N.$$

So in this case we have $\mu(MN) = \mu(N) \circ \mu(M)$. For arbitrary $M, N \in \mathrm{Soc}(3, \mathbb{O})$ however, we cannot assume $MN \in \mathrm{Soc}(3, \mathbb{O})$.

For each $x \in \mathbb{O}$ consider the following matrices:

$$M_x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M'_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \qquad M''_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}$$

$$L_x = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad L'_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \qquad L''_x = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $m_x = \mu(M_x)$, $m'_x = \mu(M'_x)$, $m''_x = \mu(M''_x)$, $l_x = \mu(L_x)$, $l'_x = \mu(L'_x)$ and $l''_x = \mu(L''_x)$ denote the corresponding linear maps. One can check that these are elements of $SE_6(F)$. The remainder of this section sketches the proof of the following result from Bray et al. 2019.

Theorem 3.1. *When $\mathbb{O}$ is split $\{m_x, m'_x, m''_x, l_x, l'_x, l''_x\}_{x\in\mathbb{O}}$ generates $SE_6(F)$.*

For $X, Y \in \mathbb{J}$ where $X = (a, b, c|A, B, C)$ and $Y = (d, e, f|D, E, F)$ we define the map

$$\begin{aligned} M(Y, X) = {} & bcd + ace + abf - dA\overline{A} - eB\overline{B} - fC\overline{C} \\ & - a(D\overline{A} + A\overline{D}) - b(E\overline{B} + B\overline{E}) - c(F\overline{C} + C\overline{F}) + T(DBC + ECA + FAB). \end{aligned}$$

If $F \neq \mathbb{F}_2$, for any $\alpha \notin \{0, 1\}$ we can express this in terms of the determinant

$$M(X, Y) = \frac{1}{\alpha(\alpha-1)}\Delta(X + \alpha Y) - \frac{1}{\alpha - 1}\Delta(X + Y) + \frac{1}{\alpha}\Delta(X) - (\alpha + 1)\Delta(Y).$$

Using this map we categorise non-zero Albert vectors in the following way. For a non-zero Albert vector $X \in \mathbb{J}$

- X is called *white* if $M(Y, X) = 0$ for all $Y \in \mathbb{J}$ (it can be shown that if X is white then $\Delta(X) = 0$),
- X is called *grey* if $\Delta(X) = 0$ and X is not white,
- X is called *black* if $\Delta(X) \neq 0$ and X is not white.

The 1-space spanned by a white vector is called a *white point*, and similarly for the 1-spaces spanned by grey and black vectors. For example $X = (0, 0, 1|0, 0, 0)$ is a white vector, and $\langle X \rangle$ is a white point. The colouring of Albert vectors in this way is analogous to a notion of the 'rank' of an Albert vector: the colours white, grey and black correspond to the ranks 1, 2 and 3 respectively. This is discussed in more detail in Section 4.10.1 of Wilson 2009.

The next thing to note is that the colour of an Albert vector is preserved by elements of $SE_6(F)$. When $F \neq \mathbb{F}_2$ this is clear from the determinant form of $M(Y, X)$, but it can also be shown for $F = \mathbb{F}_2$. We will be concerned with the action of elements of $SE_6(F)$ on the white points and white vectors of $\mathbb{J}$.

From now on we assume $\mathbb{O}$ is split. The majority of the work done to prove Theorem 3.1 comes in the proof of the following three lemmas. Recall that a group action on a set S is called *primitive* if it is transitive, and for all $s \in S$ the stabiliser of s is a maximal subgroup of S.

Lemma 3.2. *The subgroup of $SE_6(F)$ generated by $\{m_x, m'_x, m''_x, l_x, l'_x, l''_x\}_{x\in\mathbb{O}}$ acts transitively on the white points of $\mathbb{J}$.*

Lemma 3.3. *The action of $SE_6(F)$ on the white points of $\mathbb{J}$ is primitive.*

Lemma 3.4. *The stabiliser of the white vector $(0,0,1|0,0,0)$ in $SE_6(F)$ is generated by $\{m_x, m'_x, l_x, l''_x\}_{x\in\mathbb{O}}$.*

Note that there are two different group actions being discussed here. Lemma 3.2 and Lemma 3.3 are results about the action of $SE_6(F)$ on the white *points*, while Lemma 3.4 is a result about the action of $SE_6(F)$ on the white *vectors*. Therefore Lemma 3.3 does not imply that the subgroup generated by $\{m_x, m'_x, l_x, l''_x\}_{x\in\mathbb{O}}$ is maximal in $SE_6(F)$.

Let $X = (0,0,1|0,0,0)$ and G be the group generated by $\{m_x, m'_x, m''_x, l_x, l'_x, l''_x\}_{x\in\mathbb{O}}$. Theorem 3.1 is proved by showing that the stabiliser of the white point $\langle X\rangle$ is strictly contained within the subgroup generated by G. By Lemma 3.3 this stabiliser is maximal in $SE_6(F)$, and so we must have $G = SE_6(F)$. From Lemma 3.2 is is clear that $\mathrm{stab}(\langle X\rangle) \neq G$ so it remains to show $\mathrm{stab}(\langle X\rangle) \subset G$.

The subgroup generated by $\{m_x, m'_x, l_x, l''_x\}_{x\in\mathbb{O}}$ stabilises the vector X, but the elements of the stabiliser of $\langle X\rangle$ can also map X to λX for $\lambda \in F^\times$. Let $\mathbb{O}^\times$ denote subset of invertible octonions. Then for each $u \in \mathbb{O}^\times$ consider the matrices

$$P_u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & u \end{pmatrix}.$$

Then we have

$$\mu(P_u)(X) = (0,0,N(u)|0,0,0)$$

Since $\mathbb{O}$ is split, it is clear from the formula for the norm that $N : \mathbb{O} \longrightarrow F$ is surjective, and so the actions of $\{\mu(P_u)\}_{u\in\mathbb{O}^\times}$ account for the remaining elements of the stabiliser of $\langle X\rangle$. Finally we note that

$$P_u = M'_{u^{-1}-1} L'_1 M'_{u-1} L'_{-u^{-1}}$$

and since the matrices in this product are all written over the same sociable subalgebra this implies

$$\mu(P_u) = l'_{-u^{-1}} \circ m'_{u-1} \circ l'_1 \circ m'_{u^{-1}-1} \in G.$$

Therefore $\mathrm{stab}(\langle X\rangle) \subset G$ as required.

4 Dickson's Construction

In 1901, with corrections and simplifications in 1908, Leonard Eugene Dickson produced the first construction of the groups of type $E_6(F)$ for an arbitrary field F (see Dickson 1901; Dickson 1908). As in Bray et al. 2019, $E_6(F)$ is constructed from the subgroup of the automorphisms of a 27-dimensional F-vector space D preserving a certain cubic form. Consider the variables x_i, y_i and z_{ij} for $i, j \in \{1, 2, 3, 4, 5, 6\}$ and $i \neq j$ taking values in F. The cubic form on this space is defined

$$C = \sum_{i \neq j} x_i y_j z_{ij} + \sum_{(*)} z_{ij} z_{k\ell} z_{mn}.$$

where in the sum labelled $(*)$ we have $(ij|k\ell|mn)$ taking the following values

12|34|56	12|35|64	12|36|45
13|24|65	13|25|46	13|26|54
14|23|56	14|25|63	14|26|35
15|23|64	15|24|36	15|26|43
16|23|45	16|24|53	16|25|34.

Dickson defines $E_6(F)$ as the group of invertible linear maps preserving C, where two linear maps are considered equal if they are scalar multiples of each other. The equality condition is equivalent to taking the quotient of the group of linear maps preserving C by its centre, since the centre of a linear automorphism group is exactly the maps which act as scalar multiplication. Hence, $SE_6(F)$ is the group of linear maps preserving C.

4.1 Equivalence to the BSW construction

Let D be the F-vector space defined above and $\mathbb{J}$ the Albert space constructed from the split octonion F-algebra. Let $SE_6(F)_D$ denote $SE_6(F)$ as defined above, and $SE_6(F)_{\mathbb{J}}$ denote $SE_6(F)$ as defined in Section 3. To show these are isomorphic it suffices to show that there exists a vector space isomorphism

$$\psi : D \longrightarrow \mathbb{J}$$

which preserves the cubic form. That is, for all $v \in D$ we have $\Delta(\psi(v)) = C(v)$. Given such a ψ, we can define

$$\begin{aligned} \Phi : SE_6(F)_D &\longrightarrow SE_6(F)_{\mathbb{J}} \\ f &\longmapsto \psi \circ f \circ \psi^{-1}. \end{aligned}$$

Clearly $\Phi(f)$ is linear for all $f \in SE_6(F)_D$ as it is the composition of linear maps. It also preserves the cubic form, since for $v \in \mathbb{J}$

$$\begin{aligned} \Delta(\Phi(f)(v)) &= \Delta(\psi(f \circ \psi^{-1}(v))) \\ &= C(f(\psi^{-1}(v))) \\ &= C(\psi^{-1}(v)) \\ &= \Delta(v). \end{aligned}$$

It is a group homomorphism since for $f, g \in SE_6(F)_D$

$$
\begin{aligned}
\Phi(f \circ g) &= \psi \circ (f \circ g) \circ \psi^{-1} \\
&= (\psi \circ f \circ \psi^{-1}) \circ (\psi \circ g \circ \psi^{-1}) \\
&= \Phi(f) \circ \Phi(g).
\end{aligned}
$$

Given an $f \in SE_6(F)_{\mathbb{J}}$ we can define $g = \psi^{-1} \circ f \circ \psi \in SE_6(F)_D$. Clearly $\Phi(g) = f$ and so Φ is surjective. Finally for injectivity suppose for $f, g \in SE_6(F)_D$ we have $\Phi(f) = \Phi(g)$. Then by injectivity and surjectivity of ψ we find $f = g$, and so Φ is injective, and hence an isomorphism.

Both D and $\mathbb{J}$ are 27-dimensional F-vector spaces, and so are isomorphic, so it remains to find an isomorphism which preserves the cubic forms. For an Albert vector $X = (a, b, c|A, B, C) \in \mathbb{J}$ we can write the Dickson-Freudenthal determinant as

$$\Delta(X) = abc - aN(A) - bN(b) - cN(C) + T(ABC)$$

by Lemma 2.4. Let $A = \sum_{i \in I} \alpha_i e_i$, $B = \sum_{i \in I} \beta_i e_i$ and $C = \sum_{i \in I} \gamma_i e_i$. Then using the expressions for the norm and trace of a split octonion algebra we have

$$
\begin{aligned}
\Delta(X) = abc &- a\alpha_{-1}\alpha_1 - a\alpha_{-\overline{\omega}}\alpha_{\overline{\omega}} - a\alpha_{-\omega}\alpha_\omega - a\alpha_{-0}\alpha_0 \\
&- b\beta_{-1}\beta_1 - b\beta_{-\overline{\omega}}\beta_{\overline{\omega}} - b\beta_{-\omega}\beta_\omega - b\beta_{-0}\beta_0 \\
&- c\gamma_{-1}\gamma_1 - c\gamma_{-\overline{\omega}}\gamma_{\overline{\omega}} - c\gamma_{-\omega}\gamma_\omega - c\gamma_{-0}\beta_0 \\
&- \alpha_{-1}\beta_{-0}\gamma_1 + \alpha_{\overline{\omega}}\beta_0\gamma_1 - \alpha_\omega\beta_{\overline{\omega}}\gamma_1 - \alpha_0\beta_{-1}\gamma_1 - \alpha_{-1}\beta_{-\omega}\gamma_{-\overline{\omega}} - \alpha_{\overline{\omega}}\beta_0\gamma_{-\overline{\omega}} - \alpha_{-0}\beta_{\overline{\omega}}\gamma_{-\overline{\omega}} + \alpha_{-\omega}\beta_{-1}\gamma_{-\overline{\omega}} \\
&+ \alpha_{-1}\beta_{-\overline{\omega}}\gamma_{-\omega} - \alpha_\omega\beta_0\gamma_{-\omega} - \alpha_{-0}\beta_\omega\gamma_{-\omega} - \alpha_{-\overline{\omega}}\beta_{-1}\gamma_{-\omega} - \alpha_{-1}\beta_1\gamma_0 + \alpha_0\beta_0\gamma_0 - \alpha_{-\overline{\omega}}\beta_{\overline{\omega}}\gamma_0 - \alpha_{-\omega}\beta_\omega\gamma_0 \\
&- \alpha_{\overline{\omega}}\beta_{-\overline{\omega}}\gamma_{-0} - \alpha_\omega\beta_{-\omega}\gamma_{-0} + \alpha_{-0}\beta_{-0}\gamma_{-0} - \alpha_1\beta_{-1}\gamma_{-0} - \alpha_{\overline{\omega}}\beta_1\gamma_\omega - \alpha_0\beta_{-\omega}\gamma_\omega - \alpha_{-\omega}\beta_{-0}\gamma_\omega + \alpha_1\beta_{\overline{\omega}}\gamma_\omega \\
&+ \alpha_\omega\beta_1\gamma_{\overline{\omega}} - \alpha_0\beta_{-\overline{\omega}}\gamma_{\overline{\omega}} - \alpha_{-\overline{\omega}}\beta_{-0}\gamma_{\overline{\omega}} - \alpha_1\beta_\omega\gamma_{\overline{\omega}} - \alpha_{-0}\beta_1\gamma_{-1} - \alpha_{-\omega}\beta_{-\overline{\omega}}\gamma_{-1} + \alpha_{-\overline{\omega}}\beta_{-\omega}\gamma_{-1} - \alpha_1\beta_0\gamma_{-1}.
\end{aligned}
$$

Writing the cubic form of a vector $v \in D$ gives

$$
\begin{aligned}
C(v) = x_1y_2z_{12} &+ x_1y_3z_{13} + x_1y_4z_{14} + x_1y_5z_{15} + x_1y_6z_{16} - x_2y_1z_{12} + x_2y_3z_{23} + x_2y_4z_{24} + x_2y_5z_{25} + x_2y_6z_{26} \\
&- x_3y_1z_{13} - x_3y_2z_{23} + x_3y_4z_{34} + x_3y_5z_{35} + x_3y_6z_{36} - x_4y_1z_{14} - x_4y_2z_{24} - x_4y_3z_{34} + x_4y_5z_{45} + x_4y_6z_{46} \\
&- x_5y_1z_{15} - x_5y_2z_{25} - x_5y_3z_{35} - x_5y_4z_{45} + x_5y_6z_{56} - x_6y_1z_{16} - x_6y_2z_{26} - x_6y_3z_{36} - x_6y_4z_{46} - x_6y_5z_{56} \\
&+ z_{12}z_{34}z_{56} - z_{12}z_{35}z_{46} + z_{12}z_{36}z_{45} - z_{13}z_{24}z_{56} + z_{13}z_{25}z_{46} - z_{13}z_{26}z_{45} \\
&+ z_{14}z_{23}z_{56} - z_{14}z_{25}z_{36} + z_{14}z_{26}z_{35} - z_{15}z_{23}z_{46} + z_{15}z_{24}z_{36} - z_{15}z_{26}z_{34} \\
&+ z_{16}z_{23}z_{45} - z_{16}z_{24}z_{35} + z_{16}z_{25}z_{34}.
\end{aligned}
$$

By comparing these two cubic forms it was possible to find the following isomorphism which preserves the cubic forms. Part of this work was done computationally using the satisfiability solver in Selinger 2016. We should not expect this isomorphism to be unique.

$$
\begin{array}{llllll}
x_1 \mapsto -\alpha_0 & x_2 \mapsto \beta_{-0} & x_3 \mapsto \gamma_{-1} & x_4 \mapsto \alpha_\omega & x_5 \mapsto -\beta_\omega & x_6 \mapsto -\gamma_{-\overline{\omega}} \\
y_1 \mapsto -\alpha_{-\omega} & y_2 \mapsto -\beta_{-\omega} & y_3 \mapsto -\gamma_{\overline{\omega}} & y_4 \mapsto -\alpha_{-0} & y_5 \mapsto -\beta_0 & y_6 \mapsto -\gamma_1 \\
z_{12} \mapsto -\gamma_\omega & & & & & \\
z_{13} \mapsto -\beta_{-\overline{\omega}} & z_{23} \mapsto \alpha_{-\overline{\omega}} & & & & \\
z_{14} \mapsto -a & z_{24} \mapsto -\gamma_{-0} & z_{34} \mapsto \beta_1 & & & \\
z_{15} \mapsto \gamma_0 & z_{25} \mapsto b & z_{35} \mapsto \alpha_1 & z_{45} \mapsto \gamma_{-\omega} & & \\
z_{16} \mapsto -\beta_{-1} & z_{26} \mapsto \alpha_{-1} & z_{36} \mapsto c & z_{46} \mapsto \beta_{\overline{\omega}} & z_{56} \mapsto \alpha_{\overline{\omega}}. &
\end{array}
$$

5 Conclusion

In this report we have summarised the construction of the groups of type E_6 by Bray et al. 2019. This was compared with the construction by Dickson 1901 and the two constructions were shown to be equivalent. We also presented the generators for $SE_6(F)$ found by Bray et al. 2019 and sketched their proof of this. Bray et al. 2019 were able to prove a number of other things about the groups of type E_6 using their construction, including the simplicity of $E_6(F)$, the order of $SE_6(F)$ and $E_6(F)$, and various facts about different subgroups of $SE_6(F)$ and $E_6(F)$. Most of these results rely on the octonion algebra used to construct the Albert space being split. As highlighted by Bray et al. 2019, an interesting direction for future research is to consider what happens in the case the octonion algebra is not split.

It may also be useful to find an isomorphism between the groups of type E_6 as constructed by Bray et al. 2019 and the constructions using Lie theory or algebraic groups. This may allow some of the insights of Bray et al. 2019 to be used when these groups arise more naturally in their Lie theory or algebraic group constructions.

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