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The Application of Lie Symmetry Methods to SDEs

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Abstract

This paper details the background theory of Lie symmetries of partial differential equations (PDE's). Then we present a method for applying this theory of Lie symmetries to stochastic differential equations (SDE's). Furthermore, we show that for geometric Brownian motion there exists a symmetry mapping the constant zero solution to the general solution. And, also discuss how uniqueness of the solution of SDE's could lead to symmetries corresponding to a transformation of the initial value of the solution.

Statement of Authorship

The first two sections are background material, well developed in their fields. In the third section an idea conceived by Dooley is presented. A special thanks to Alexander Hiller for being a sounding board for ideas.

1 Introduction

This paper presents the theory of Lie symmetries which were developed by Sophus Lie during the second half of the 19th century. They are valuable methods in the study of PDE's, however, there is far less study into how one may apply such methods to SDE's. The latter parts of the paper discuss an idea for this application to SDE's.

The first section details the symmetry methods of Sophus Lie. We closely follow the disposition given by Olver in [1] and refer the reader to this text for proofs. These symmetry methods describe precisely what a symmetry group of a system of PDE's is and how one may determine such symmetry group. The section following this reviews some background theory of stochastic calculus, primarily sourced from [4]. Finally the last section provides a bridge from Lie's methods to for PDE's to SDE's. This bridge is a system of PDE's associated with a SDE.

2 Lie Symmetry Methods

2.1 Manifolds and Lie Groups

Manifolds are the natural setting for the study of differential equations and symmetries. As we will see, they are also a key idea in the formulation of Lie groups. A manifold is a topological space that "locally resembles Euclidean space". What is meant by this is that there is an open covering $\{U_\alpha\}$ where each set U_α is homeomorphic to an open set of $\mathbb{R}^n$ [2].

Definition 2.1. An *m-dimensional manifold* is a set M, together with a countable collection of subsets $U_\alpha \subset M$, called *coordinate charts*, and one-to-one functions $\chi_\alpha : U_\alpha \to V_\alpha$ onto connected open subsets V_α of $\mathbb{R}^n$, called *local coordinate maps*, which satisfy:

1. The coordinate charts cover M:
$$\bigcup_\alpha U_\alpha = M$$
2. On the overlap of any pair of coordinate charts the composite map:
$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \to \chi_\beta(U_\alpha \cap U_\beta)$$
is a smooth function.
3. If $x \in U_\alpha, \tilde{x} \in U_\beta$ are distinct points of M, then there exist open subsets W of $\chi_\alpha(x)$ in V_α and $\tilde{W}$ of $\chi_\beta(\tilde{x})$ in V_β such that:
$$\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) = \varnothing$$

This definition allows for the study of topological properties (such as connectivity) but we shall require additional structure allowing for the notion of a differentiable function $f : M \to \mathbb{R}^n$. Unfortunately, it is not enough to stipulate that, for some chart $\chi_\alpha : U_\alpha \to \mathbb{R}^m$, $f \circ \chi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable. Since,
$$f \circ \chi_\alpha^{-1} = f \circ \chi_\beta^{-1} \circ (\chi_\beta \circ \chi_\alpha^{-1})$$
we can only expect $f \circ \chi_\alpha^{-1}$ to be differentiable, for all f that make $f \circ \chi_\beta^{-1}$ differentiable, if $\chi_\beta \circ \chi_\alpha^{-1}$ is also differentiable. So we define a *smooth manifold* to be one where for any two charts χ_α and χ_β, such that $U_\alpha \cap U_\beta \neq \varnothing$, their overlap function $\chi_\beta \circ \chi_\alpha^{-1}$ is a diffeomorphism [3]. Now, given two smooth manifolds M and N one might desire to define a smooth map between the two. Indeed, $f : M \to N$ is a smooth map if;
$$\tilde{\chi}_\beta \circ f \circ \chi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$
is itself a smooth map for every coordinate chart $\chi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^m$ on M and $\tilde{\chi}_\beta : \tilde{U}_\beta \to \tilde{V}_\beta \subset \mathbb{R}^n$ on N.

In concept a Lie group is a continuous group - which is to say the group elements may be continuously varied. However, for the study of differential equations we of course also require a notion differentiation. As we have seen, these two properties are contained nicely by manifolds and so we define a Lie group to be a group that possesses the structure of a manifold.

Definition 2.2. An *r-parameter Lie group* is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operation;

$$m : G \times G \to G, \quad m(g, h) = g \cdot h, \quad g, h \in G$$

and the inversion;

$$i : G \to G, \quad i(g) = g^{-1}, \quad g \in G$$

are smooth maps between manifolds.

Definition 2.3. Let M be a smooth manifold. A *local group of transformations* acting on M is given by a Lie group G, an open subset $\mathcal{U}$ with;

$$\{e\} \times M \subset \mathcal{U} \subset G \times M$$

which is the domain of definition of the group action, and a smooth map $\Psi : \mathcal{U} \to M$ with;

1. If $(h, x) \in \mathcal{U}$, $\big(g, \Psi(h, x)\big) \in \mathcal{U}$ and $(g \cdot h, x) \in \mathcal{U}$ then;
$$\Psi\big(g, \Psi(h, x)\big) = (g \cdot h, x) \tag{1}$$
2. $\forall x \in M$;
$$\Psi(e, x) = x \tag{2}$$
3. If $(g, x) \in \mathcal{U}$, then $\big(g^{-1}, \Psi(g, x)\big) \in \mathcal{U}$ and ;
$$\Psi\big(g^{-1}, \Psi(g, x)\big) = x$$

Returning to manifolds consider attaching a copy of $\mathbb{R}^m$ to each point $x \in M$, for a m-dimensional manifold M. These copies of $\mathbb{R}^m$ are the *tangent spaces* of M at x and are denoted T_xM. T_xM is a vector space of the tangent vectors to all the possible curves passing through x. Taking the disjoint union;

$$TM = \bigcup_{x \in M} T_xM$$

and defining the *projection map*;

$$\pi : TM \to M$$
$$v \mapsto x$$

one produces the *tangent bundle*. As we saw earlier one can map between manifolds but can one do the same but between the tangent bundles of manifolds? Given a smooth map $F : M \to N$, between to manifolds M and N, we have the map;

$$F_{*x} : T_xM \to T_{f(x)}N, \qquad x \in M$$

The union of all these F_{*x} produces a linear map known as the *differential* of F;

$$F_* : TM \to TN$$
$$v \mapsto w$$

for $v \in T_xM$ and $w \in T_{f(x)}N$. If given another map $G : N \to P$ to another manifold P, then;

$$(G \circ F)_* = G_* \circ F_*$$

which in essence is the chain rule [3]. A *section* of the tangent bundle is a continuous map;

$$s : M \to TM$$

such that;

$$\pi \circ s = id$$

A section lifts a slice out of the tangent bundle, specifying a vector field on the manifold where $s(x) \in T_xM$ is the vector assigned to the point $x \in M$. In local coordinates $(x^1, \cdots, x^m)$ such a vector field is of the form;

$$\mathbf{v}|_x = \xi(x) = \xi^1(x)\frac{\partial}{\partial x^1} + \cdots + \xi^m(x)\frac{\partial}{\partial x^m}$$

For the moment one can consider the $\frac{\partial}{\partial x^i}$ symbols to represent the unit vectors of the components of $\xi(x)$. One could have easily written the vector field as the m-tuple $\big(\xi^1(x), \cdots, \xi^m(x)\big)$ but we shall see shortly how these $\frac{\partial}{\partial x^i}$ act as partial differential operators. Also, on occasion $\frac{\partial}{\partial x^i}$ may be written as ∂_{x^i}.

Now, given a vector field $\mathbf{v}$ on M we might consider a curve $\phi : I \to M$ that with tangent vectors that coincide with $\mathbf{v}$. Such a curve is called an *integral curve* and is a curve $\phi(\varepsilon)$ such that;

$$\phi'(\varepsilon) = \mathbf{v}|_{\phi(\varepsilon)}, \qquad \phi(0) = x_0$$

If ϕ is not contained by any other integral curve then it is the *maximal integral curve* passing through the point x_0. That is to say there exists no curve $\tilde{\phi} : \tilde{I} \to M$, where $\tilde{I} \subset I$, passing through $\tilde{\phi}(0) = x_0$ such that $\tilde{\phi}(\varepsilon) = \phi(\varepsilon)$. The maximal integral curve passing through x is known as the *flow generated by* $\mathbf{v}$ and denoted $\Psi(\varepsilon, x)$. Rewriting the integral curve properties under this flow notation we have;

$$\Psi\big(\delta, \Psi(\varepsilon, x)\big) = \Psi(\delta + \varepsilon, x), \tag{3}$$

$$\Psi(0, x) = x, \tag{4}$$

$$\frac{d}{d\varepsilon}\Psi(\varepsilon, x) = \mathbf{v}|_{\psi(\varepsilon,x)} \tag{5}$$

When one compares (3)-(4) to (1)-(2) it is apparent that the flow coincides with a group action of $\mathbb{R}$ on M; this is called *one-parameter group of transformations* and will be prevalent later. The process of solving (4)-(5) for the flow/one-parameter group generated by $\mathbf{v}$ is referred to as *exponentiation*. The following notation is useful;

$$\Psi(\varepsilon, x) \equiv \exp(\varepsilon\mathbf{v})x$$

Under this notation properties (3)-(5) are;

$$\exp(\delta\mathbf{v})\exp(\varepsilon\mathbf{v})x = \exp\big((\delta+\varepsilon)\mathbf{v}\big)x,$$

$$\exp(0\mathbf{v})x = x,$$

$$\frac{d}{d\varepsilon}\big(\exp(\varepsilon\mathbf{v})x\big) = \mathbf{v}|_{\exp(\varepsilon\mathbf{v})x}$$

reminiscent of the properties of the usual exponential function.

Example 2.1. Vector Fields and Flows

(a) Take the real line $\mathbb{R}$ as the manifold on which we define the vector field $\mathbf{v} = x\partial_x$. The solution of (4)-(5) yields the usual exponential function;

$$\exp(\varepsilon\mathbf{v})x = e^{\varepsilon}x$$

(b) In the opposite direction consider $SO(2)$, the group of rotations on the plane. The one-parameter subgroup is given by;

$$\Psi\big(\varepsilon, (x, y)\big) = (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon)$$

This subgroup is of the form;

$$\mathbf{v} = \xi(x, y)\partial_x = \tau(x, y)\partial_y$$

So;

$$\xi(x, y) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}(x\cos\varepsilon - y\sin\varepsilon) = -y$$

$$\tau(x, y) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}(x\sin\varepsilon + y\cos\varepsilon) = \quad x$$

Therefore, the vector field $\mathbf{v} = -y\partial_x + x\partial_y$ generates the the group of rotations.

The notation we have used for vector fields suggests that they act as a differential operator - this is precisely what we shall find. Let us consider how a function $f : M \to \mathbb{R}$ changes under the flow generated by a vector field $\mathbf{v}$ on M, ie. the derivative of $f\big(\exp(\varepsilon\mathbf{v})x\big)$ with respect to ε;

$$\frac{d}{d\varepsilon} f\big(\exp(\varepsilon\mathbf{v})x\big) = \sum_{i=1}^{m} \xi^i\big(\exp(\varepsilon\mathbf{v})x\big)\frac{\partial f}{\partial x^i}\big(\exp(\varepsilon\mathbf{v})x\big) \equiv \mathbf{v}(f)\big(\exp(\varepsilon\mathbf{v})x\big) \tag{6}$$

In particular, when $\varepsilon = 0$ we have;

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f\big(\exp(\varepsilon\mathbf{v})x\big) = \mathbf{v}(f)(x)$$

Thus, $\mathbf{v}(f)$ is giving the infinitesimal change of f under the flow - perhaps more readily seen via the taylor expansion about $\varepsilon = 0$;

$$f\big(\exp(\varepsilon\mathbf{v})x\big) = f(x) + \varepsilon\mathbf{v}(f)(x) + \cdots$$

By this interpretation a vector field on M each $\mathbf{v}|_x$ provides a derivation on the space of smooth real-valued functions near x on M. So, the applicatio of $\mathbf{v}|_x$ to f produces a real number $\mathbf{v}(f)(x)$ amd this operation has the properties;

1. *Linearity*

$$\mathbf{v}(f+g) = \mathbf{v}(f) + \mathbf{v}(g)$$

2. *Leibniz' Rule (Product Rule)*

$$\mathbf{v}(f \cdot g) = \mathbf{v}(f)\cdot g + f \cdot \mathbf{v}(g)$$

We saw that the differential of a smooth map between manifolds $F : M \to N$ mapped tangent vectors on M to tangent vectors on N but if given a vector field $\mathbf{v}$ on M is $F_*(\mathbf{v})$ a well defined vector field on N? This certainly does not hold true in general so we define a class of vector fields for which it is true.

Definition 2.4. Given a smooth map $F : M \to N$ between the manifolds M and N, two vector fields $\mathbf{v}$ on M and $\mathbf{w}$ on N are F*-related* if;

$$F_*(\mathbf{v}|_x) = \mathbf{w}|_{F(x)}, \qquad \forall x \in M$$

When two vector fields are F-related, F maps the flow of one to the other;

$$F\big(\exp(\varepsilon\mathbf{v})x\big) = \exp\big(\varepsilon F_*(\mathbf{v})\big)F(x)$$

2.2 Lie Brackets and Algebras

The Lie bracket is a binary operator on vector fields. For two vector fields $\mathbf{v}, \mathbf{w}$ on M their Lie bracket $[\mathbf{v}, \mathbf{w}]$ is the unique vector field such that;

$$[\mathbf{v}, \mathbf{w}](f) = \mathbf{v}\big(\mathbf{w}(f)\big) - \mathbf{w}\big(\mathbf{v}(f)\big), \qquad \forall f : M \to \mathbb{R} \tag{7}$$

From (7) it is readily shown that the Lie bracket satisfies;

1. *Bilinearity*

$$[c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \mathbf{w}] = c_1[\mathbf{v}_1, \mathbf{w}] + c_2[\mathbf{v}_2, \mathbf{w}] \quad \text{and} \quad [\mathbf{v}, c_1\mathbf{w}_1 + c_2\mathbf{w}_2] = c_1[\mathbf{v}, \mathbf{w}_1] + c_2[\mathbf{v}, \mathbf{w}_2]$$

   for constants $c_1, c_2 \in \mathbb{R}$.

2. *Skew-Symmetry*

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$$

3. *Jacobi Identity*

$$\big[\mathbf{u}, [\mathbf{v}, \mathbf{w}]\big] + \big[\mathbf{w}, [\mathbf{u}, \mathbf{v}]\big] + \big[\mathbf{v}, [\mathbf{w}, \mathbf{u}]\big] = 0$$

Theorem 2.1. *If two vector fields* $\mathbf{v}$ *and* $\mathbf{w}$ *on a manifold* M *are* F*-related to well defined vector fields on another manifold* N*, then their Lie bracket is also* F*-related;*

$$F_*\big([\mathbf{v}, \mathbf{w}]\big) = \big[F_*(\mathbf{v}), F_*(\mathbf{w})\big]$$

Having defined vector fields on manifolds we, of course, immediately have a concept of vector fields on a Lie group G. This is of the utmost importance - indeed, there is a set of unique vector fields defined by there invariance under group multiplication. The vector space of such vector fields is known as the Lie algebra of G and, as we shall see, in some sense "generates" the group. So, if $g \in G$ we define *right multiplication* with the map;

$$\begin{aligned} R_g : G &\to G \\ h &\mapsto h \cdot g \end{aligned}$$

Then, a vector field $\mathbf{v}$ on G is *right-invariant* if;

$$R_{g*}(\mathbf{v}|_h) = \mathbf{v}|_{R_g(h)} = \mathbf{v}_{hg}, \qquad \forall g, h \in G$$

Also note that since the differential is linear, if $\mathbf{v}$ and $\mathbf{w}$ are both right-invariant vector fields then so is any linear combination of the two. This results in the set of right-invariant vector fields forming a vector space. A direct result of theorem (2.1) is that the Lie bracket of two right-invariant vector fields is also right-invariant;

$$R_{g*}\big([\mathbf{v},\mathbf{w}]\big) = \big[R_{g*}(\mathbf{v}), R_{g*}(\mathbf{w})\big] = [\mathbf{v},\mathbf{w}]$$

Definition 2.5. Lie Algebra The *Lie algebra* $\mathfrak{g}$ of a Lie group G is the vector space of all right-invariant vector fields on G together with the *Lie bracket.*

Any right-invariant vector field is uniquely determined by it's value at the identity. That is to say that since $R_g(e) = g$;

$$\mathbf{v}|_g = R_{g*}(\mathbf{v}|_e)$$

And conversely any $\mathbf{v}_e \in T_eG$ uniquely determines a right-invariant vector field on G;

$$R_{g*}(\mathbf{v}|_h) = R_{g*} \circ R_{h*}(\mathbf{v}|_e) = (R_g \circ R_h)_*(\mathbf{v}|_e) = \mathbf{v}|_{hg}$$

Thus, we may identify the Lie algebra of G with the tangent space of G at the identity;

$$\mathfrak{g} \simeq T_eG$$

If the algebra of a Lie group is of such importance and contains so much information about the group there must be something relating the two. This is the exponential map.

Definition 2.6. The *exponential map*;

$$\exp : \mathfrak{g} \to G$$

is obtained by setting $\varepsilon = 1$ in the flow generated by $\mathbf{v}$ through the identity;

$$\exp(\mathbf{v}) \equiv \exp(\mathbf{v})e$$

2.3 Symmetries of Differential Equations

Given a system $\mathcal{S}$ of differential equations a symmetry is some transformation that transforms a solution of $\mathcal{S}$ to another solution. In this section we seek to define a group of such symmetries and describe how to determine said group.

Groups and Differential Equations

Suppose the system of differential equations $\mathcal{S}$ is of p independent and q dependent variables; $x = (x^1, \cdots, x^p)$ and $u = (u^1, \cdots, u^q)$ respectively. Let $X \simeq \mathbb{R}^p$ be the space of independent variables and $U \simeq \mathbb{R}^q$ the space of dependent variables. Then the product space $X \times U$ represents the independent and dependent variables - this shall be referred to as the *base space*. To define a symmetry group we must first describe how a local group of transformations G, acting on $M \subset X \times U$, transforms a function $u = f(x)$ with a domain Ω. To do so, take the graph of f;

$$\Gamma_f = \{(x, f(x)) : x \in \Omega\}$$

Then G acts on this graph by;

$$g \cdot \Gamma_f = \{(\tilde{x}, \tilde{u}) = g \cdot (x, f(x)) : (x, f(x)) \in \Gamma_f\}$$

$g \cdot \Gamma_f$ will not always be the graph of a single-valued function but since G acts smoothly and the identity of G does not effect Γ_f, taking a sufficiently small subset of Ω ensures that $\Gamma_{\tilde{f}} = g \cdot \Gamma_f$ is the graph of a single-valued function $\tilde{f}$. This rigorous relationship is denoted $\tilde{f} = g \cdot f$ and the function $\tilde{f}$ is the *transform* of f by g. To explicitly determine this transformation suppose;

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u))$$

where Ξ_g and Φ_g are smooth functions. So;

$$\tilde{x} = \Xi_g(x, f(x)) = \Xi_g \circ (\mathbb{1} \times f)(x)$$
$$\tilde{u} = \Phi_g(x, f(x)) = \Phi_g \circ (\mathbb{1} \times f)(x)$$

To eliminate x from this system we take the inversion;

$$x = \Big(\Xi_g \circ (\mathbb{1} \times f)\Big)^{-1}(\tilde{x})$$

The inverse function theorem allows this to be done for g sufficiently close to the identity element; $\Xi_g \circ (\mathbb{1} \times f)$ collapses to the identity map at $g = e$, indicating the Jacobian is non-singular near the identity element. Of course one now substitutes this back into the system giving;

$$\tilde{u} = g \cdot f = \Big(\Phi_g \circ (\mathbb{1} \times f)\Big) \circ \Big(\Xi_g \circ (\mathbb{1} \times f)\Big)^{-1}$$

Now, understanding how a transformation group transforms a function on the space of independent and dependent variables one may define a symmetry group.

Definition 2.7. Let $\mathcal{S}$ be a system of differential equations. A *symmetry group* of the system $\mathcal{S}$ is a local group of transformations G acting on an open subset M of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of $\mathcal{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

Prolongation

A pivotal concept for the following is that of prolongation. Prolongation refers to the "prolonging" of the base space $X \times U$ to a space that also represents the derivatives of the dependent variables. First, note that for a function $u = f(x) = \big(f^1(x), \cdots, f^q(x)\big)$ of p independent and q dependent variables there are $q \cdot p_k$ kth order partial derivatives of the components f; where $p_k = \binom{p+k-1}{k}$. We shall denote these derivatives as;

$$u^\alpha_J = \frac{\partial^k}{\partial x^{j_1} \cdots \partial x^{j_k}} f^\alpha(x)$$

where $J = (j_1, \cdots, j_k)$ is an unordered k-tuple such that each $1 \leq j_i \leq p$ indicates the derivative being taken. So, let $U_k \simeq \mathbb{R}^{q \cdot p_k}$ be the space representative of u^α_J. Then, the product space $U^{(n)} = U \times U_1 \times \cdots \times U_n$ represents all derivatives up to order n. Finally, we have the *n-jet space* $X \times U^{(n)}$ representing the independent and dependent variables and the derivatives up to order n. Now, given a smooth function $f : X \to U$ there is a function $\mathrm{pr}^{(n)} f : X \to U^{(n)}$ such that for each $x \in X$ $u^{(n)} = \mathrm{pr}^{(n)} f(x)$ is a vector whose elements give the values of f and all the derivatives up to order n. $\mathrm{pr}^{(n)} f$ is called the *prolongation* of f.

We wish to use this concept of prolongation to re-frame differential equations in a more geometrical sense. Let us suppose that the system $\mathcal{S}$ of nth order differential equations in p independent and q dependent variables is given by;

$$\Delta_v(x, u^{(n)}) = 0, \qquad v = 1, \cdots, l$$

where $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$ and $\Delta(x, u^{(n)}) = \big(\Delta_1(x, u^{(n)}), \cdots, \Delta_l(x, u^{(n)})\big)$ is assumed to be smooth map from the nth jet space to Euclidean space;

$$\Delta : X \times U^{(n)} \to \mathbb{R}^l$$

Now, the subset on which Δ vanishes satisfies the system;

$$\mathcal{S}_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\} \subset X \times U^{(n)}$$

Within this framework a solution of the system $\mathcal{S}$ is a smooth function $u = f(x)$ such that the graph of it's nth prolongation is wholly contained by $\mathcal{S}_\Delta$;

$$\Gamma^{(n)}_f \equiv \{\big(x, \mathrm{pr}^{(n)} f(x)\big)\} \subset \mathcal{S}_\Delta$$

Prolongation of Group Actions

If a group of transformations G acts on the base space can one prolong the group action to act on the nth jet space? This nth prolongation of $\text{pr}^{(n)}G$ should be defined such that if G tranforms f into $\tilde{f}$ then $\text{pr}^{(n)}G$ will transform the nth order derivatives of f into those of $\tilde{f}$. To determine how $\text{pr}^{(n)}G$ acts on a point $(x_0, u_0^{(n)}) \in M^{(n)} = M \times U_1 \times \cdots \times U_n$, select a function $u = f(x)$ such that it's graph lies in M and it's derivatives agree with $(x_0, u_0^{(n)})$;

$$u_0^{(n)} = f(x_0)$$

. Then, the action of $\text{pr}^{(n)}G$ on $(x_0, u_0^{(n)})$ is given by the derivatives of $\tilde{f}$ at $\tilde{x}_0$;

$$\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)}) = (\tilde{x}_0, \tilde{u}_0^{(n)})$$

where;

$$\tilde{u}_0^{(n)} \equiv \text{pr}^{(n)}\tilde{f}(\tilde{x}_0)$$

With this prolongation of group action we define a criterion stipulating when a group of transformations is a symmetry group.

Theorem 2.2. *Let M be an open subset of $X \times U$ and suppose $\Delta(x, u^{(n)}) = 0$ is an nth order system of differential equations defined over M, with $\mathcal{S}_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\} \subset M^{(n)}$. Suppose G is a local group of transformations acting on M whose prolongation leaves $\mathcal{S}_\Delta$ invariant, meaning that whenever $(x, u^{(n)}) \in \mathcal{S}_\Delta$ we have $pr^{(n)}g \cdot (x, u^{(n)}) \in \mathcal{S}_\Delta$ for all $g \in G$. Then G is a symmetry group of the system of differential equations.*

Prolongation of Infinitesimal Generators and the Infinitesimal Criterion

The criterion of Theorem 2.2 is tedious to use as a method to determine a symmetry group - one would have to check that every point in M remains in $\mathcal{S}_\Delta$ for every element of G. To define a more workable criterion we first define the prolongation of an infinitesimal generator - this will simply be the infinitesimal generator of the prolonged group action.

Definition 2.8. Let $M \subset X \times U$ be open and suppose $\mathbf{v}$ is a vector field on M with the corresponding one-parameter subgroup $\exp(\varepsilon\mathbf{v})$. The nth prolongation of $\mathbf{v}$, denoted $\text{pr}^{(n)}\mathbf{v}$, is a vector field on $M^{(n)}$ and is defined to be the infinitesimal generator of the prolonged one-parameter subgroup $\text{pr}^{(n)}\big(\exp(\varepsilon\mathbf{v})\big)$;

$$\text{pr}^{(n)}\mathbf{v}|_{(x,u^{(n)})} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \text{pr}^{(n)}\big(\exp(\varepsilon\mathbf{v})\big)(x, u^{(n)})$$

Definition 2.9. Let;

$$\Delta_v(x, u^{(n)}) = 0, \qquad v = 1, \cdots, l$$

be a system of differential equations. The system is said to be of *maximal rank* if the Jacobian matrix;

$$J_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta_v}{\partial x^i}, \frac{\partial \Delta_v}{\partial u_J^\alpha}\right)$$

is of rank l whenever $\Delta_v(x, u^{(n)}) = 0$.

Theorem 2.2 together with Definition 2.8 allows one to describe a criterion that stipulates when a vector field on the base space generates a symmetry group.

Theorem 2.3 (Infinitesimal Criterion)**.** *Suppose;*

$$\Delta_v(x, u^{(n)}) = 0, \qquad v = 1, \cdots, l$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and;

$$pr^{(n)}\mathbf{v}[\Delta_v(x, u^{(n)})] = 0, \qquad \textit{whenever } \Delta_v(x, u^{(n)}) = 0 \tag{8}$$

for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system.

Prolongation Formula

Definition 2.10 (Total Derivative)**.** Let $P(x, u^{(n)})$ be a smooth function of x, u and derivatives of u up to order n, defined on an open subset $M^{(n)} \subset X \times U^{(n)}$. The *total derivative* of P with respect to x^i is the unique smooth function $D_iP(x, u^{(n+1)})$ defined on $M^{(n+1)}$ with the property that if $u = f(x)$ is any smooth function;

$$D_iP\big(x, \mathrm{pr}^{(n+1)}f(x)\big) = \frac{\partial}{\partial x^i}\Big(P\big(x, \mathrm{pr}^{(n+1)}f(x)\big)\Big)$$

Proposition 2.1. *Given $P(x, u^{(n)})$, the ith total derivative of P has the general form;*

$$D_iP = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_J u_{J,i}^\alpha \frac{\partial P}{\partial u_J^\alpha} \tag{9}$$

where, for $J = (j_1, \cdots, j_k)$;

$$u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i} = \frac{\partial^{k+1}u^\alpha}{\partial x^i \partial x^{j_1} \cdots \partial x^{j_k}}$$

Also note that the inner sum in (9) is over all J such that $0 \leq k \leq n$, where n is the highest order derivative appearing in P.

Theorem 2.4. *Let;*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be a vector field on an open subset $M \subset X \times U$. The nth prolongation of $\mathbf{v}$ *is the vector field;*

$$pr^{(n)}\mathbf{v} = \mathbf{v}\sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}$$

defined on the jet space $M^{(n)} \subset X \times U^{(n)}$. The second summation being over all $J = (j_1, \cdots, j_k)$ such that $1 \leq j_i \leq p$ and $1 \leq k \leq n$. The coefficient functions ϕ_α^J are given by;

$$\phi_\alpha^J(x,u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha\right) + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$.

Symmetry Group Calculation

Finally, we reach the goal of the theory developed in this section. Taking the infinitesimal criterion together with the prolongation formula produces a tool to compute the symmetry group of a system of differential equations. We shall allow the coefficients ξ^i and ϕ_α, of the infinitesimal generator of the supposed symmetry group, to be unknown functions. The functions ϕ_α^J of the prolongation will then be explicit expressions of the derivatives of ξ^i and ϕ_α. So, the infinitesimal criterion will be in terms of x, u, the derivatives of u and ξ^i, ϕ_α and their derivatives. Then equating the coefficients of the partial derivatives to zero yields the *defining equations* - which shall provide explicit forms to ξ^i and ϕ_α. Lastly, one of course exponentiates the infinitesimal generators to recover the symmetry groups.

3 Stochastic Calculus

The theory of stochastic calculus is built upon the stochastic integral. This paper requires understanding of stochastic integrals of the form $\int_0^t XdM$ where M is a martingale and X is a stochastic process satisfying particular assumptions. For this we refer the reader to [4] for a rigorous development of such an integral.

This paper studies Lie symmetries of SDE's of the form;

$$X_t = X_0 + \int_0^t \mu(s,X_s)dt + \int_0^t \sigma(s,X_s)dW_s \tag{10}$$

where X_0 is the initial value of the process X_t and W_t is brownian motion. $\mu(t,x)$ and $\sigma(t,x)$ are the *drift* and *diffusion*, respectively. The integral form of (10) is rather cumbersome so we use the standard notation;

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{11}$$

Note that (11) is a notational convenience with the rigorous interpretation given by (10). Now, a particularly useful and well known result in stochastic calculus us Itô's Lemma. One can think of Itô's Lemma as a chain rule for functions of stochastic processes.

Theorem 3.1 (Itô's Lemma)**.** *Let M be a continuous local martingale and V be a continuous process which is locally of bounded variation. Let f be a continuous real-valued function on $\mathbb{R}^2$ such that the partial derivatives $f_x(t,x)$, $f_{xx}(t,x)$ and $f_t(t,x)$ exist ans are continuous for all $(t,x) \in \mathbb{R}^2$. Then;*

$$f(V_t, M_t) = f(V_0, M_0) + \int_0^t f_x(V_s, M_s)dM_s + \int_0^t f_t(V_s, M_s)dV_s + \frac{1}{2}\int_0^t f_{xx}(V_s, M_s)d[M]_s$$

where $[M_s]$ is the quadratic variation process of M.

For further explanation and a proof of Theorem 3.1 we again refer the reader to [4]. With Itô's Lemma we may derive a system of PDE's associated with a SDE of the form (11) which, as we shall see in the next section, is central to the application of symmetry methods to SDE's. Consider the case of Itô's Lemma when $V_t = t$ and $M_t = W_t$. Since $[W]_t = t$, if a process is given by $X_t = f(t, W_t)$ then;

$$dX_t = df(t, W_t) = \Big(f_t(t, W_t) + \frac{1}{2}f_{xx}(t, W_t)\Big)dt + f_x(t, W_t)dW_t \tag{12}$$

Now, if one assumes that a solution to (11) is of the form $X_t = f(t, W_t)$ then by (12) we have;

$$\mu\big(t, f(t, W_t)\big)dt + \sigma\big(t, f(t, W_t)\big)dW_t = \Big(f_t(t, W_t) + \frac{1}{2}f_{xx}(t, W_t)\Big)dt + f_x(t, W_t)dW_t$$

Equating the coefficients of dt and dW_t, respectively, produces the the Itô system.

Definition 3.1 (Ito System)**.** Given a SDE of the form (11) it's associated *Itô system* is the following system of PDE's;

$$\mu\big(t, f(t, W_t)\big) = f_t(t, W_t) + \frac{1}{2}f_{xx}(t, W_t) \tag{13}$$

$$\sigma\big(t, f(t, W_t)\big) = f_x(t, W_t) \tag{14}$$

4 Symmetry Methods for Stochastic Differential Equations

The path to applying the symmetry methods detailed in section 1 to SDE's lies with the Itô system - instead of reworking Lie's theory we simply ask how does the application of a symmetry to a SDE's

Itô system effect the SDE itself? So, supposing that the symmetries of (13)-(14) are generated by a vector field;

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u$$

Then, we have the nth prolongation of $\mathbf{v}$;

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \phi^t\partial_{u_t} + \phi^x\partial_{u_x} + \phi^{tt}\partial_{u_{tt}} + \phi^{xx}\partial_{u_{xx}}$$

Now, by (2.3) the infinitesimal criterion for (13)-(14), in a coordinate system $u = f(t,x)$, is;

$$\tau \cdot \mu_t + \phi \cdot \mu_u = \phi^t + \frac{1}{2}\phi^{xx}$$
$$\tau \cdot \sigma_t + \phi \cdot \sigma_u = \phi^x$$

Example 4.1 (Geometric Brownian Motion)**.** Consider the SDE;

$$dX_t = \mu_0 X_t dt + \sigma_0 X_t dW_t$$

where μ_0 and σ_0 are constants. One of the infinitesimal generators of the symmetry group is;

$$\mathbf{v} = \exp\left(\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t + \sigma_0 x\right)\partial_u$$

After exponentiation the symmetry that this generator produces is;

$$\tilde{X_t} = X_t + \varepsilon e^{\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t + \sigma_0 W_t}$$

where ε is a group parameter of the one-parameter subgroup generated by $\mathbf{v}$. Interestingly, this symmetry maps the constant zero solution to the general solution. Indeed, if ones sets $X_t \equiv 0$ and lets $\varepsilon = X_0$ then;

$$\tilde{X_t} = X_0 e^{\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t + \sigma_0 W_t}$$

. We also see that of one set the given solution to the general one $X_t \equiv X_0 e^{\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t + \sigma_0 W_t}$ then

$$\tilde{X_t} = (X_0 + \varepsilon)e^{\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t + \sigma_0 W_t}$$

Which appears to indicate that under the conditions of strong uniqueness this map may correspond to a change in the initial value of the solution process.

5 Conclusion

We have seen the theory that develops Lie symmetries and how they may be used to find symmetry groups of PDE's. We, also had a short exploration into are these methods can be applied to SDE's via their associated Itô system. Lastly we consider a symmetry of geometric Brownian motion that maps the constant zero solution to the general solution. One might wish to further this paper by determining exactly when a symmetry can map the zero solution to the general one. Also, in future work one could further examine the apparent correspondence between symmetries of SDE's and a change in initial condition under assumptions of strong uniqueness.

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