AMSI VACATIONRESEARCH SCHOLARSHIPS 2019–20

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Conway's Tiling Groups

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1. Abstract

Tiling problems are as follows; a region is defined, and a set of tiles. Then we must show that the region can or cannot be filled by the tiles without overlap. A traditional approach to solving them is by using colouring arguments. This paper explores the application of group theory to tiling problems. It focuses mainly on a theorem by Conway, which turns the geometric problem into a new one involving group words. We apply Conway's theorem to regions similar to chessboards but with one set of diagonally opposite corners removed, starting from regions of 3 units by 3 units, noting that while the theorem has an implication that holds when a tiling exists, it can hold even if there is no tiling for a region.

2. Introduction

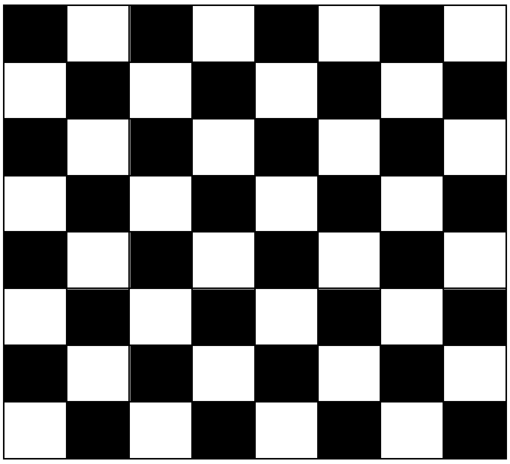
The subject of tiling problems goes quite a way back – people have been filling spaces with objects pretty much since objects and spaces (and people) have existed. The main difference is that now we use maths to show whether a particular region can or cannot be tiled. Showing whether a region can or cannot be tiled is generally a difficult problem. A traditional method of doing this is with a colouring argument. In this report we will explore Conway's group theoretic methods. This report explores some of the ways that Conway's theorem can be used, focusing mainly on regions similar to chessboards, with dominoes as tiles. The main thing to note here is that the implication in the theorem, which states that if a region can be tiled by certain tiles then we can build the word for the region out of the words for the tiles, does not go both ways. That is, even if we can build the word for the region out of the words for the tiles, a tiling does not necessarily exist.

All the working shown is my own; the actual theorem used is from Conway's paper on group theory and tiling problems.

3. Tiling Problems

A Tiling Problem is as follows – given a region R and a set of tiles T, can R be tiled using the tiles from T? An example is a chessboard as R and some dominoes as T. There are a few ways to solve a tiling problem, one of which is a colouring argument. If we colour the 8 by 8 grid of the chessboard in alternating black and white cells, we can see upon counting the cells that there are 32 black cells and 32 white cells. A domino, under the same colouring, has one black and one white cell. To fill the region, there must be an equal amount of each colour. Essentially, If the chessboard can be tiled by dominoes, then there must be an equal number of black cells and white cells. It is easy to see that the chessboard can be tiled by dominoes – try it!

Figure 1, a chess board and dominoes.

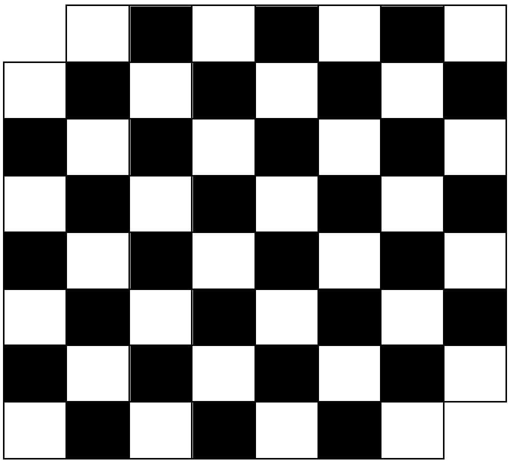


One black cell and one white cell.

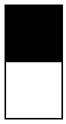
32 black cells and 32 white cells.

Now take, for example, a chessboard with one set of diagonally opposite corners removed. There is no tiling for this region using dominoes. Using a colouring argument, the number of black cells will be 2 less than the number of white cells.

Figure 2, a chessboard with one set of diagonally opposite corners removed



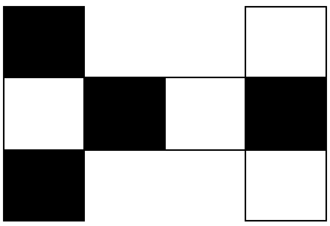
32 white cells and 30 black cells



One black cell and one white cell

Another useful example to mention regarding colouring arguments is a region shaped a bit like a capital H. The colouring argument here suggests that the region could be tiled by dominoes – there are equal amounts of black and white cells. However, looking at the region, there is no tiling with dominoes. A colouring argument can show that a region cannot be tiled, but it doesn't always apply.

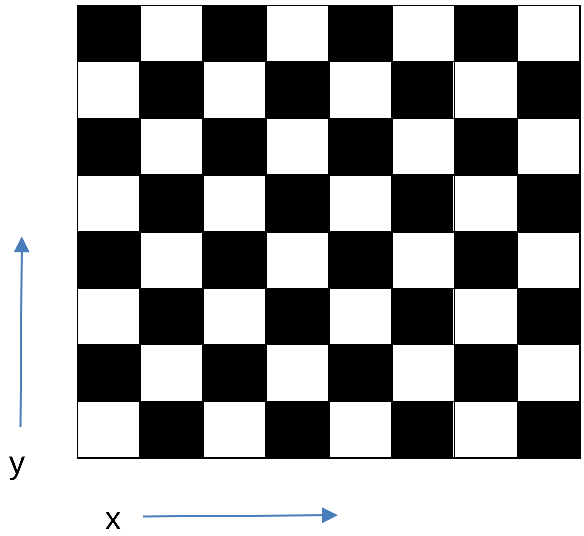
Figure 3, a H shaped region.



4. “Group word” technique

These regions can also be described using a word. Looking back at the chessboard example, if one were to walk around the outside of the chessboard with one step in the positive x direction being ‘x’, and one step in the positive y direction being ‘y’, and moving back in the opposite directions of each being their inverses, a word for the chessboard could be $x^8y^8x^{-8}y^{-8}$, starting from the bottom left corner. Since this sequence of steps comes back to where the ‘path’ starts, this requires that the word for the region is equal to e (where e is the empty word). This means that the words for any possible regions are group words.

Figure 4, another chessboard.



The domino tiles can then be described using a similar process, resulting in the words $xy^2x^{-1}y^{-2}$, and $x^2yx^{-2}y^{-1}$.

Figure 5, more dominoes.

This can be made more precise using a Cayley graphs of the groups involved, but I will not go into details in this report.

5. Conway's Theorem

Conway's Tiling Theorem

For a region R with associated word w_R and set of tiles T_1, T_2,…,T_n, with associated words w_2, w_2,…,w_n, if R can be tiled by those tiles, then the following implication holds in all groups:

$$(w_1 = e, w_2 = e, \dots, w_n = e) \Rightarrow w_r = e$$

This theorem can be proven with the following inductive step.

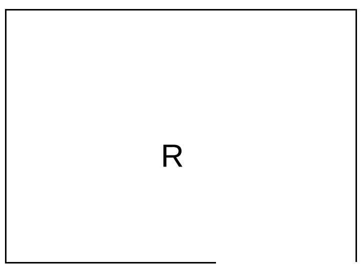
Assume all regions S smaller than R have the following property

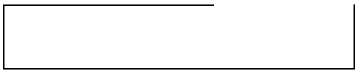
$$\text{If S can be tiled, then } w_s = e \text{ holds in all groups}$$

Where w_s is the word describing S.

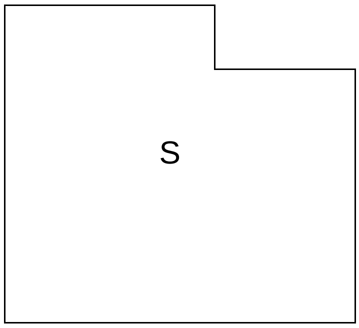
Now assume R can be tiled by dominoes and then remove one domino from the tiling. We then have a smaller region S, and by inductive hypothesis $w_s = e$.

Figure 6, A region before and after removing a tile.





An observation here: a 'walk' around R



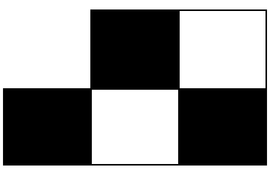
can be obtained from a 'walk' around S by walking around the removed tile in one direction, and then back the other way. This walk, w_T, by the inductive hypothesis has w_T=e. Then, since w_R=$w_S w_T$, we obtain w_R=0, as claimed.

6. Examples

Most of the following examples are not able to be tiled, and it can be shown using a colouring argument or Conway's theorem.

The first example is similar to the previous chessboard regions, only much smaller. To show that this region cannot be tiled with dominoes (which we can see by a colouring argument) we must find a group G such that for some x and y in G, $x^2y=yx^2$, $xy^2=y^2x$, and the word for the region $x^2yxy^2 \neq y^2xyx^2$. Note that $x^2yx^{-2}y^{-1} \Leftrightarrow x^2y = yx^2$, and similarly for other words. So we can write the group words as equations, $x^2y=yx^2$, $xy^2=y^2x$, and $x^2yxy^2=y^2xyx^2$.

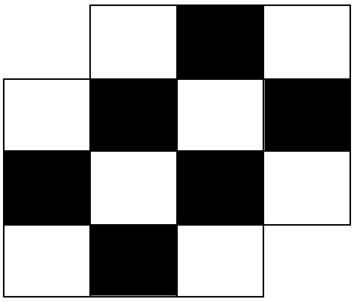
Figure 7, a smaller region.



Here, the group of symmetries of a triangle is sufficient to show that the region cannot be tiled, with x=(12) and y=(23). This makes checking the relations for the dominoes redundant, as x and y are of order 2 and we would just be checking x=x and y=y. Then checking the relation for the region, which simplifies to $xy \neq yx$, indeed xy=(12)(23) and yx=(23)(12).

The next region is slightly larger, and to show that it cannot be tiled we need to find a group G such that for some x and y in G, $x^2y=yx^2$, $xy^2=y^2x$, and $x^3yxy^3 \neq y^3xyx^3$.

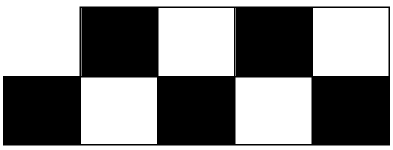
Figure 8, a larger region.

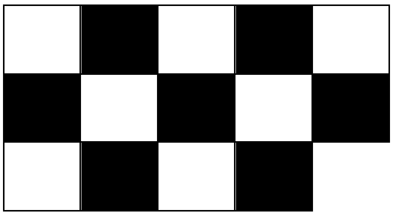


Here, using the group of symmetries of a triangle with x=(13) and y=(12) works. Once again, using elements of order 2 is helpful.

The next region is bigger again, and to show that this region cannot be tiled with dominoes, we must find a group G such that for some x and y in G, $x^2y=yx^2$, $xy^2=y^2x$, and $x^4yxy^4 \neq y^4xyx^4$.

Figure 9, another larger region.

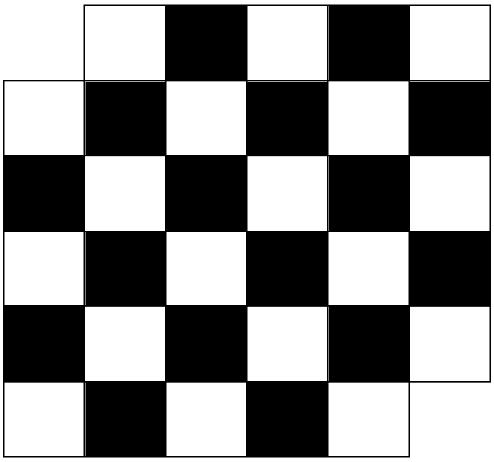




Here, using the group of symmetries of a triangle with x=(13) and y=(12)(34), we can show that the region cannot be tiled.

The next region is larger again, and to show that it cannot be tiled we must find a group G such that for some x and y in G, $xy^2=y^2x$, $x^2y=yx^2$, and $x^5yxy^5 \neq y^5xyx^5$.

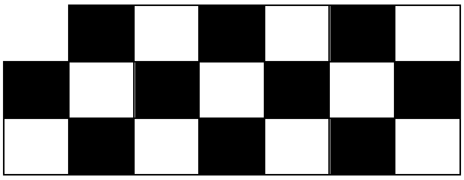
Figure 10, yet another larger region.

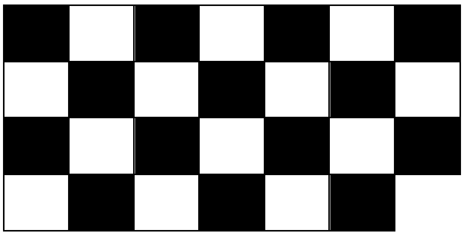


Here, if we let G be the group of symmetries of a pentagon with x=(13)(45) and y=(25)(34), we can show that the region cannot be tiled.

To show that the next region cannot be tiled, we must again find a group G such that for some x and y in G, $x^2y=yx^2$, $xy^2=y^2x$, and $x^6yxy^6 \neq y^6xyx^6$.

Figure 11, another region.

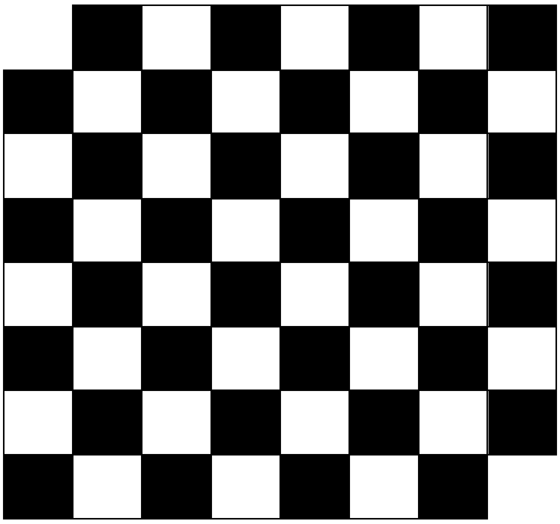




Here, using the symmetries of a square as G with x=(14)(23) and y=(24), we can see that the region cannot be tiled.

Next up is the last example of a chessboard region, and to show that it cannot be tiled we must find a group G such that for some x and y in G, $xy^2=y^2x$, $x^2y=yx^2$, and $x^7yxy^7 \neq y^7xyx^7$.

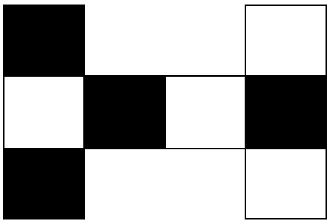
Figure 12, the last chessboard.



For this region, if we let G be the group of symmetries of a heptagon, with x=(14)(23)(57) and y=(27)(36)(45) we can see that the region cannot be tiled.

The last example is that of the H shaped region mentioned previously, described by $xyx^2y^{-1}xy^3=y^3xy^{-1}x^2yx$. The implication $x^2y = yx^2$ & $xy^2 = y^2x \Rightarrow xyx^2y^{-1}xy^3 = y^3xy^{-1}x^2yx$ Holds in all groups so there should be a tiling! What this suggests is that Conway's theorem only goes one way; if there is a tiling the implication holds, but if the implication holds there is not always a tiling.

Figure 13, H region revisited



7. Conclusions.

What can be seen from this report is that while useful, much like colouring arguments Conway's theorem here has it's flaws. The implication only goes one way. If the relations in the theorem can be satisfied, there is not necessarily a tiling. However as the theorem states, if a region can be tiled, then those relations are satisfied.

Another interesting thing to note is that for the 'chessboard-like' regions of the form $x^n yxy^n = y^n xyx^n$, to show that they do not have a tiling (with dominoes) by using Conway's theorem, the smallest non-abelian group is sufficient here.

References

- Conway, J.H & Lagarias, J.C 1988, 'Tiling with Polyominoes and Combinatorial Group Theory', *Journal of Combinatorial Theory*, Series A, no. 53, pp.183-208.
- Thurston, W.P 1990, 'Conway's Tiling Groups', *The American Mathematical Monthly*, vol. 97, no. 8, pp.757-773.