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Model Categories

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Abstract

The goal of this project is to understand Quillen's notion of model categories, which allows us to define a general notion of homotopy outside the realm of algebraic topology. Whilst there are many examples of these, we primarily concern ourselves with understanding the structure of model categories and the accompanying homotopy category, ultimately giving us our homotopy theory. We also look at recent developments which allow us to define model categories in terms of weak factorisation systems, which is equivalent to Quillen's original definition and gives a characterisation of cofibrantly generated model categories - those which are most common in practice.

Acknowledgements

I would first and foremost like to thank my supervisor Vigleik Angeltveit for not only his guidance and enthusiasm throughout this project, but for also telling me that the 'Aussie' way of talking about adjoints is to call them 'mates'. I would also like to thank Ivo Vekemans for providing additional guidance throughout the project and thank the other students at the Australian National University who came to the meetings.

I would like to thank the Mathematical Sciences Institute at the Australian National University for providing me with a workspace for this project.

Finally, I would like to thank the Australian Mathematical Sciences Institute for providing me with the opportunity to undertake this projects and extend this thanks to the Department of Education for helping to fund the whole program.

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1. Introduction

Homotopy theory in algebraic topology gives us a way of saying two things are equivalent, whether this be two continuous maps, two spaces, or two other topological objects. The primary goal of this report is to gain an understanding of the construction of model categories presented in Quillen (1967), allowing us to define "homotopy theory" in generality outside of algebraic topology. On the outside, model categories are fairly simple - a category with three distinguished classes of morphisms called weak equivalences, fibrations, and cofibrations, which satisfy some axioms. Despite this seemingly simple definition, there is a wealth of structure to be found within model categories. One of the main structures we will consider is the homotopy category of a model category, obtained by formally inverting the weak equivalences. The main difficulty encountered when attempting to show that a category admits this model structure is in determining what each class of morphisms should be. In this report, we will avoid proving that a particular category is a model category but will present some examples.

We will assume of the reader a basic familiarity with algebraic topology equivalent to chapter one of Hatcher (2002), enough to understand the basics of homotopy theory. The essentials of this have been included in Appendix II. Throughout we may give examples from other areas of algebraic topology such as higher homotopy theory and homological algebra. We also require some category theory, which has been included in Appendix I. We will loosely be following Hirschhorn (2003) and Hovey (1999), often referring to other papers as necessary.

We first introduce model categories as in Quillen (1967), initially called closed model categories, and proceed to explore some of the fundamental properties of the distinguished classes of morphisms (Section 2.1). Recent developments have shown that there is an equivalent way to define a model category in terms of a weak factorisation system, which appears to be a more natural definition when we examine cofibrantly generated model categories (Section 2.2). The key result we will look at here will tell us precisely how to construct a cofibrantly generated model category. In order to put a weak factorisation system on these model categories, however, we need to apply the small object argument which we discuss in detail in Appendix IV. Once we have an understanding of what a model category is, we will define the homotopy category of a model category (Section 3). To do this we briefly describe the category formed by formally inverting a class of morphisms in some category as put forward by Gabriel and Zisman (1967). Once we have the general notion of a homotopy we will explore some of the key properties that come with definition. Our efforts will culminate as we prove what Hovey (1999) calls the 'fundamental theorem about model categories'.

Statement of Authorship

All of the results presented in this report are not original. Most can be found in Hovey (1999), Hirschhorn (2003), and Quillen (1967). A lot of the proofs throughout this report were outlined by my supervisor, Vigleik Angeltveit, in meetings and then then details were fleshed out by myself, often through discussions with other students attending the meetings and with various texts, referenced as appropriate throughout this report.

2. Model Categories

2.1. Defining Model Categories

Model categories were first defined by Quillen (1967). This definition was originally called a 'closed model category' but is now called a model category. We present the definition which can be found in Dwyer and Spaliński (1995) and Hirschhorn (2003), which can be viewed as a strengthening of Quillen's original definition.

The key differences with Quillen are that we require all small limits and colimits rather than just finite ones and we also require the factorisations to be functorial (Definition 4.11). In practice we can typically shift between these varying restrictions without too much difficulty.

Definition 2.1. A *model category* $\mathcal{M}$ is a category with three distinguished classes of morphisms:

- Weak equivalences ($\xrightarrow{\sim}$), denoted $\mathcal{W}$;
- Fibrations ($\twoheadrightarrow$), denoted $\mathcal{F}ib$;
- Cofibrations ($\rightarrowtail$), denoted $\mathcal{C}of$.

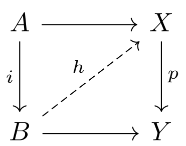
Each of these classes is closed under composition and contains all identity morphisms. An *acyclic (co)fibration* is a (co)fibration which is also a weak equivalence. The classes of morphisms and $\mathcal{M}$ must also satisfy the following axioms:

(M1) (Limit axiom) $\mathcal{M}$ is bicomplete (Definition 4.16).

(M2) (Two out of three axiom) If f and g are composable morphisms in $\mathcal{M}$ and two of f, g, and gf are weak equivalences then so is the third.

(M3) (Retract axiom) Each class of morphisms ($\mathcal{W}$, $\mathcal{C}of$, $\mathcal{F}ib$) is closed under retracts (Definition 4.20).

(M4) (Lifting Axiom) Acyclic cofibrations have the left lifting property with respect to fibrations and acyclic fibrations have the right lifting property with respect to cofibrations (Definition 6.1). That is, for any commutative square in $\mathcal{M}$



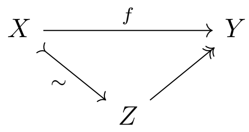
a lift h exists if either i is an acyclic cofibration and p is a fibration or i is a cofibration and p is an acyclic fibration.

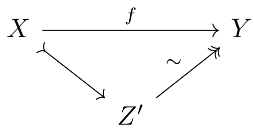
(M5) (Factorisation axiom) Every morphism $f : X \to Y$ in $\mathcal{M}$ can be factored in two ways:

(i) As $f = pi$ where i is an acyclic cofibration and p is a fibration.

(ii) As $f = pi$ where i is a cofibration and p is an acyclic fibration.

That is, for any morphism $f : X \to Y$ in $\mathcal{M}$ the following two diagrams exist and commute.

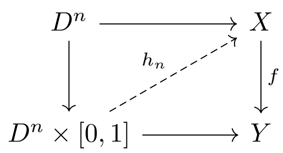




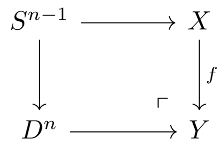
Note. We will frequently denote the classes of acyclic fibrations and acyclic cofibrations as $\mathcal{W}\cap\mathcal{F}ib$ and $\mathcal{W}\cap\mathcal{C}of$ respectively. We also use the notation $i \boxslash p$ to say that i has the left lifting property with respect to p and p has the right lifting property with respect to i. From M4 we can then write that $(\mathcal{W} \cap \mathcal{C}of) \boxslash \mathcal{F}ib$ to say that every map in $\mathcal{W} \cap \mathcal{C}of$ has the left lifting property with respect to every map in $\mathcal{F}ib$.

The above definition illustrates Quillen originally called these 'closed' model categories. The lifting axiom (M4) tells us that the class of acyclic cofibrations has the left lifting property with respect to the class of fibrations. It turns out that knowing any two of the classes of morphisms is sufficient for determining what the third should be. This in itself is a nontrivial result and details can be found Hirschhorn (2003) (Proposition 7.2.7, page 112).

Example 2.2. **Top** can be given a model structure. The weak equivalences are weak homotopy equivalences, the fibrations are Serre fibrations, and the cofibrations are relative cell complexes.
A map $f : X \to Y$ is a *weak homotopy equivalence* if the induced map on homotopy groups $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all n and all choices of basepoint $x \in X$.
A map $p : X \to Y$ is a *Serre fibration* if for all $n \geq 0$ there exists a map $h_n : D^n \times [0, 1] \to X$ such that the following diagram commutes.



A map $f : X \to Y$ is a *relative cell complex* if X is a subspace of Y and f can be constructed as a transfinite composition (Definition 7.6) of pushouts (Definition 4.19) of the following form.



A detailed proof that this defines a model structure on **Top** can be found in Hirschhorn (2019).

Example 2.3. A *non-negatively graded chain complex* C over an R-module, where R is an associative ring with a unit, is a sequence of R-modules $\{C_n\}_{n\geq 0}$ together with boundary maps $d_n : C_{n+1} \to C_n$ such that $d_n d_{n+1} = 0$ for all n. The *n-th homology group* of C is defined to be the quotient $H_n C = \ker(d_n)/\mathrm{im}(d_{n+1})$. The category $\mathbf{Ch}_R$ has non-negatively graded chain complexes over R as its objects and morphisms $f : C \to D$ consisting of components $f_n : C_n \to D_n$ satisfying $f_n d_n = d_n f_{n+1}$.
Following Dwyer and Spaliński (1995), we can make $\mathbf{Ch}_R$ into a model category by setting a morphism $f : C \to D$ to be:

- A weak equivalence if it induces isomorphisms $f_n^* : H_n C \to H_n D$ for all $n \geq 0$.
- A fibration if for any two morphisms $g_1, g_2 : D_n \to E$, where E is an R-module, it holds that if $g_1 f_n = g_2 f_n$ then $g_1 = g_2$. That is, f_n is an epimorphism.
- A cofibration if for any two morphisms $g_1, g_2 : E \to C_n$, where E is an R-module, it holds that if $f_n g_1 = f_n g_2$ then $g_1 = g_2$. Moreover the cokernel of f_n is a projective R-module. That is, f_n is a monomorphism.

There are three important subcategories of a model category determined by how objects interact with the initial and terminal objects (Definition 4.12) of the model category.

Definition 2.4. Let $\mathcal{M}$ be a model category. We can an objects $X \in \mathcal{M}$ *cofibrant* if every morphism $0 \to X$ is a cofibration. Similarly, we call an object $X \in \mathcal{M}$ *fibrant* if every morphism $X \to 1$ is a fibration. An objects $X \in \mathcal{M}$ is bifibrant if it is both cofibrant and fibrant.

Definition 2.5. For a model category $\mathcal{M}$ there are the following full subcategories (Definition 4.6):

- $\mathcal{M}_c$ - the full subcategory of cofibrant objects.
- $\mathcal{M}_f$ - the full subcategory of fibrant objects.
- $\mathcal{M}_{cf}$ - the full subcategory of bifibrant objects.

Definition 2.6. Let $\mathcal{M}$ be a model category. For any object $X \in \mathcal{M}$ there is a *cofibrant replacement functor* $Q : \mathcal{M} \to \mathcal{M}$ such that QX is cofibrant and there is a natural acyclic fibration $q_X : QX \to X$. Similarly, there is a *fibrant replacement functor* $R : \mathcal{M} \to \mathcal{M}$ such that RX is fibrant and there is a natural acyclic cofibration $r_X : X \to RX$.

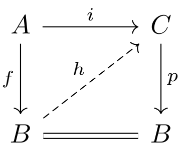
Our next step is to determine some of the additional properties of the three classes of morphisms. Before we do this we are going to need the retract argument.

Proposition 2.7 (The Retract Argument)**.** *Let $f : A \to B$ be a morphism in a category $\mathcal{M}$ which can be factored as $f = pi$ where $i : A \to C$ and $p : C \to B$.*

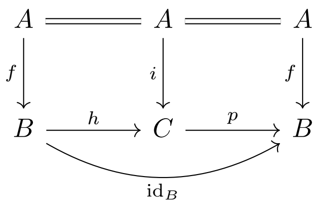
(i) *If $f \boxslash p$ then f is a retract of i.*

(ii) *If $i \boxslash f$ then f is a retract of p.*

Proof. Suppose that $f \boxslash p$. Then the following diagram commutes.



We can extend this diagram to the following, which also commutes.



This diagram gives us that f is a retract of i (Definition 4.20). The proof when $i \boxslash f$ is similar. □

We can now show that axiom M4 actually gives us the necessary and sufficient condition for a morphism to be a (co)fibration or an acyclic (co)fibration. We hinted at this before as knowing two classes of morphisms allows you know the third in Definition 2.1.

Proposition 2.8. *Let $i : A \to B$ and $p : X \to Y$ be morphisms in a model category $\mathcal{M}$. Then*

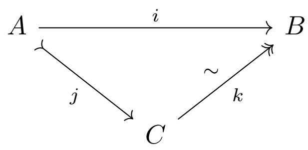
(i) *i is a cofibration if and only if $i \boxslash (\mathcal{W} \cap \mathcal{F}ib)$.*

(ii) *i is an acyclic cofibration if and only if $i \boxslash \mathcal{F}ib$.*

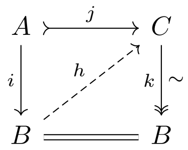
(iii) *p is a fibration if and only if $(\mathcal{W} \cap \mathcal{C}of) \boxslash p$.*

(iv) *p is an acyclic fibration if and only if $\mathcal{C}of \boxslash p$.*

Proof. (i) First, suppose that i is a cofibration. By axiom M4 we know that $i \boxslash (\mathcal{F}ib \cap \mathcal{W})$.
Conversely, suppose that $i \boxslash (\mathcal{F}ib \cap \mathcal{W})$ and we want to show that i is a cofibration. Using M5 we can factor i as



where $j \in \mathcal{C}of$ and $k \in \mathcal{W} \cap \mathcal{F}ib$. By assumption, there is a lift $h : B \to C$ so that the following diagram commutes.



By the retract argument it follows that i is a retract of j. Since j is a cofibration, axiom M3 gives us that i must also be a cofibration.

(ii) The proof is similar to (i) but in the second part we factor i as an acyclic cofibration followed by a fibration. The proofs of parts (iii) and (iv) are similar. □

Lemma 2.9. *Let $\mathcal{M}$ be a model category.*

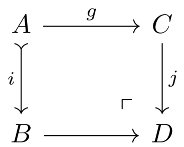
(i) *Cofibrations are closed under pushouts (Definition 4.19).*

(ii) *Acyclic cofibrations are closed under pushouts.*

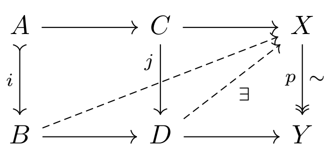
(iii) *Fibrations are closed under pullbacks (Definition 4.18).*

(iv) *Acyclic fibrations are closed under pullbacks.*

Proof. We will only prove (i) as (ii) is similar and the remaining cases are dual.
Let $i : A \to B$ be a cofibration in $\mathcal{M}$ and suppose we have the following pushout square.



We want to show that j, which is the pushout of i along g, is also a cofibration. By Proposition 2.8 we only need to show that $j \boxslash (\mathcal{W} \cap \mathcal{F}ib)$. Suppose $p : X \to Y$ is an acyclic fibration. If the following solid arrow diagram commutes



then since $i \in \mathcal{C}of$ we know that $i \boxslash (\mathcal{W} \cap \mathcal{F}ib)$ which gives us a lift $B \to X$. By the universal property of pushout (Definition 4.19) this gives us the lift $D \to X$ and hence $j \in \mathcal{C}of$. □

It is natural to consider functors between model categories and more importantly if they preserve the classes of morphisms. In the case of an adjunction it turns out that they do under certain conditions.

Proposition 2.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ be adjoint functors. F preserves acyclic cofibrations if and only if G preserves fibrations, and G preserves acyclic fibrations if and only if F preserves cofibrations.*

Proof. Since F and G are adjoints we have that $\mathcal{N}(FB, X) \cong \mathcal{M}(B, GX)$ (Definition 4.21). This means that we have a one-to-one correspondence between the following solid arrow squares.

$$\begin{array}{ccc} A & \longrightarrow & GX \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle Gp} \\ B & \longrightarrow & GY \end{array} \qquad \begin{array}{ccc} FA & \longrightarrow & X \\ {\scriptstyle Fi}\downarrow & \overset{h'}{\nearrow} & \downarrow{\scriptstyle p} \\ FB & \longrightarrow & Y \end{array}$$

This correspondence implies that the lift h exists if and only if the lift h' exists. If $i \in \mathcal{C}of_{\mathcal{M}}$ and $p \in (\mathcal{W}_{\mathcal{N}} \cap \mathcal{F}ib_{\mathcal{N}})$ then M4 completes the proof. Similarly if $i \in (\mathcal{W}_{\mathcal{M}} \cap \mathcal{C}of_{\mathcal{M}})$ and $p \in \mathcal{F}ib_{\mathcal{N}}$. □

2.2. Cofibrantly Generated Model Categories

In this section we present a definition of a model category in terms of weak factorisation systems (see Appendix III) which is equivalent to Quillen's original definition. This leads us to define a cofibrantly generated model category, which is a very common form of model category encountered in practice. We then present the theorem which gives us the construction of a cofibrantly generated model category.

Definition 2.11. A *model category* $\mathcal{M}$ is a bicomplete (Definition 4.16) category with three distinguished classes of morphisms - weak equivalences $\mathcal{W}$, fibrations $\mathcal{F}ib$, and cofibrations $\mathcal{C}of$ - such that:

(i) (2-out-of-three) If f and g are composable morphisms in $\mathcal{M}$ and two of f, g, and gf are weak equivalences then so is the third.

(ii) $\mathcal{W}$ is closed under retracts.

(iii) The pairs $(\mathcal{W} \cap \mathcal{C}of, \mathcal{F}ib)$ and $(\mathcal{C}of, \mathcal{W} \cap \mathcal{F}ib)$ are both weak factorisation systems on M.

Following what we have done in Appendix III, for a set of maps I in a category $\mathcal{C}$ we use I-cell to denote the subcategory of relative I-cell complexes. We also use the following notation.

$$\mathcal{R} = I^{\boxslash} := \{f \in \operatorname{mor} \mathcal{C} : i \boxslash f \text{ for all } i \in I\},$$
$$\mathcal{L} = {}^{\boxslash}\mathcal{R} := \{f \in \operatorname{mor} \mathcal{C} : f \boxslash r \text{ for all } r \in \mathcal{R}\}.$$

The main importance of our new definition comes from cofibrantly generated model categories. In practice, most model categories we encounter are cofibrantly generated or can be modified in some way to make them cofibrantly generated (Hirschhorn, 2003).

Definition 2.12. A model category $\mathcal{M}$ is called *cofibrantly generated* if there are sets of maps I (called the generating cofibrations) and J (called the generating acyclic cofibrations) such that:

(i) The domains of maps in I and J are small relative to I-cell and J-cell respectively (Definition 7.13).

(ii) $\mathcal{F}ib = J^{\boxslash}$.

(iii) $\mathcal{W} \cap \mathcal{F}ib = I^{\boxslash}$.

By comparing the previous two definitions it it clear how the weak factorisation systems in a cofibrantly generated model category are constructed. We call such a weak factorisation cofibrantly generated as well. Before progressing we give a key property of cofibrantly generated weak factorisation systems.

Proposition 2.13. *Let $(\mathcal{L}, \mathcal{R})$ be a cofibrantly generated weak factorisation system in a category $\mathcal{C}$ with generating cofibrations I. Then every map in $\mathcal{L}$ is a retract of a relative I-cell complex.*

Proof. Suppose that $i : A \to B$ is in $\mathcal{L}$. Then we have that i factors as

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & C & \end{array}$$

where $\beta \in \mathcal{R}$ and α is a relative I-cell complex. Since $i \in \mathcal{L}$ it has the left lifting property with respect to β giving us the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ {\scriptstyle i}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle\beta} \\ B & = & B \end{array}$$

By the retract argument (Proposition 2.7) we obtain the following commutative diagram.

$$\begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle i}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle i}\downarrow \\ B & \xrightarrow{s} & C & \xrightarrow{\beta} & B \end{array}$$

Hence i is a retract of the map $\alpha \in I - cell$. □

The following theorem will tell us how to construct cofibrantly generated model categories and is the major result for this section. There are many advantages to knowing this (see Hovey (1999) chapter 2, page 36 for details).

Theorem 2.14. *Suppose $\mathcal{M}$ is a bicomplete category, $\mathcal{W} \subset \mathcal{C}$ is a subcategory, and I and J are sets of maps. Then these produce a cofibrantly generated model structure with I and J as the sets of generating cofibrations and acyclic cofibrations respectively and $\mathcal{W}$ the subcategory of weak equivalences if and only if the following hold:*

(i) *$\mathcal{W}$ satisfies the two-out-of-three axiom and is closed under retracts.*

(ii) *Domains of I are small relative to I-cell.*

(iii) *Domains of J are small relative to J-cell.*

(iv) *$J - cell \subset \mathcal{W} \cap {}^{\boxslash}(I^{\boxslash})$.*

(v) *$I^{\boxslash} \subset \mathcal{W} \cap J^{\boxslash}$.*

(vi) *Either $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash}) \subset {}^{\boxslash}(J^{\boxslash})$ or $\mathcal{W} \cap J^{\boxslash} \subset I^{\boxslash}$.*

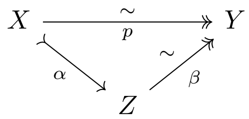
Conditions (ii) and (iii) are required for the small object arguments on I and J (see Appendix III), (iv) says that trivial relative cell complexes are weak equivalences and cofibrations, (v) says that one definition of acyclic fibrations is contained in the other, and (vi) says that the reverse inclusion for (iv) or (v) must hold (respectively). This last statement may not seem obvious at first, but by Remark 7.16 we know that J-cell is contained in $\mathcal{L}_J = {}^{\boxslash}(J^{\boxslash})$.

Proof. Suppose $\mathcal{M}$ is a cofibrantly generated model category. Then all the conditions hold by definition.
Conversely, suppose that the conditions hold. It remains to show that the lifting axiom is satisfied. We first assume that $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash}) \subset {}^{\boxslash}(J^{\boxslash})$ in (vi). Then every acyclic cofibration has the left lifting property with respect to the fibrations $J^{\boxslash}$. This means that the following diagram commutes.

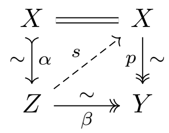
$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle\sim}\downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

The lift is immediate since $A \to B$ is in ${}^{\boxslash}(J^{\boxslash})$ which is the class of retracts of relative J-cell complexes and lifts against fibrations by the small object argument.

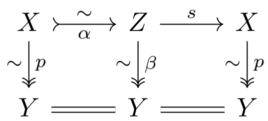
For the other lift, suppose that $X \to Y$ is an acyclic fibration, giving the following factorisation.



Since $\mathcal{W}$ satisfies the two-out-of-three axiom we have that α is also an acyclic cofibration. From the part of the lifting axiom we have already proven, we have the following commutative diagram.



By the retract argument (Proposition 2.7) this then gives us the following commutative diagram.



Hence p is a retract of β and thus $p \in I^{\boxslash}$.

The case where we assume $\mathcal{W} \cap J^{\boxslash} \subset I^{\boxslash}$ is similar. □

3. The Homotopy Category of a Model Category

The idea of the homotopy category of a model category is to find a way to formally invert the weak equivalences. In general, the main problem this could pose is that we may end up with too many morphisms. As it turns out, a model category permits this quite nicely. More formally, if $\mathcal{M}$ is a model category we want to define $\mathcal{M}[\mathcal{W}^{-1}]$.

3.1. The Category of Fractions

Gabriel and Zisman first put forward the notion of formally inverting a class of morphisms by defining a category of fractions along with a 'calculi of fractions' (Gabriel and Zisman, 1967). The process described utilises when a functor makes a morphism invertible: if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $f \in \operatorname{mor} \mathcal{C}$ then F makes f invertible if Ff is invertible. Whilst we won't focus on the details on the category of fractions we will give some description to give the reader a better idea of the structure of the homotopy category. First, we need to know what a free category is (see Riehl (2017), Example 4.1.13 on page 120 for additional details).

Definition 3.1. A *directed graph* G consists of a collection of vertices V, a collection of edges E between the vertices, and two functions $E \to V$ determining the start and end of each edge. The category of directed graphs is denoted **DirGraph**, with morphisms as directed graph morphisms (taking vertices to vertices, edges to edges, preserving incidence relations, and preserving the start and end of each edge).

Definition 3.2. Let F : **Cat** $\to$ **DirGraph** be the forgetful functor (forgetting the categorical structure on a category $\mathcal{C}$) which admits a left adjoint H. The *free category* on a directed graph G has the vertices of G as its objects, an identity morphisms for each vertex, and a morphism for each (finite) composition of edges.

Composition in the free category is defined as edge concatenation. The adjunction gives a natural bijection between functors $HG \to \mathcal{C}$ and morphisms $G \to F\mathcal{C}$.

Suppose that we have a class of morphisms $\mathcal{W}$ in a category $\mathcal{C}$. The *category of fractions* $\mathcal{C}[\mathcal{W}^{-1}]$ is formed by by taking the the free category on the directed graph of $\mathcal{C}$ with a 'backwards' arrow for each morphism in $\mathcal{W}$ modulo some relations (see Riehl (2019) for details). The resulting category $\mathcal{C}[\mathcal{W}^{-1}]$ can be described as follows:

- Objects: same as in $\mathcal{C}$.
- Morphisms: composition of edges in directed graph,

$$X \overset{f_1}{\text{———}} Z_1 \overset{f_2}{\text{———}} Z_2 \text{———} \cdots \text{———} Z_{n-1} \overset{f_n}{\text{———}} Y$$

- Identity: empty string (viewing the morphisms as strings).
- Composition: concatenation of edge paths.
- $\sim$: replace two morphisms by their composite.

The final property above allows us to characterise the following zig-zags:

$$(A \xrightarrow{w\in\mathcal{W}} B \xleftarrow{w\in\mathcal{W}} A) \sim \mathrm{id}_A \qquad\qquad (B \xleftarrow{w\in\mathcal{W}} A \xrightarrow{w\in\mathcal{W}} B) \sim \mathrm{id}_B$$

Remark 3.3. $\mathcal{C}[\mathcal{W}^{-1}]$ has a universal property:

$$\left\{ \begin{matrix} \text{Functors } F : \mathcal{C} \to \mathcal{D} \text{ which have} \\ F(\mathcal{W}) \subset \text{isomorphisms of } \mathcal{D} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Functors} \\ G : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D} \end{matrix} \right\}.$$

3.2. The Homotopy Category

Definition 3.4. Let $\mathcal{M}$ be a model category. The *homotopy category of a model category* $\mathrm{Ho}\,\mathcal{M}$ is the category of fractions $\mathcal{M}[\mathcal{W}^{-1}]$ as described in Section 3.1 formed by formally inverting the class of weak equivalences $\mathcal{W}$.

The homotopy category is the cornerstone for defining a general homotopy theory. Before we do this, however, we show that the homotopy category of the full subcategories of cofibrant, fibrant, and bifibrant objects (Definition 2.5) a nice equivalence with $\mathrm{Ho}\,\mathcal{M}$.

Proposition 3.5. *The inclusion functors induce equivalences of categories.*

$$\begin{array}{ccc} Ho\,\mathcal{M}_{cf} & \xrightarrow{\simeq} & Ho\,\mathcal{M}_c \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ Ho\,\mathcal{M}_f & \xrightarrow{\simeq} & Ho\,\mathcal{M} \end{array}$$

Proof. We first show the equivalence $\mathrm{Ho}\,\mathcal{M}_c \xrightarrow{\simeq} \mathrm{Ho}\,\mathcal{M}$. Consider the inclusion functor $i : \mathcal{M}_c \to \mathcal{M}$. This functor preserves weak equivalences. Importantly, this induces a functor $\mathrm{Ho}\,i : \mathrm{Ho}\,\mathcal{M}_c \to \mathrm{Ho}\,\mathcal{M}$. We can find an inverse for i by using the cofibrant replacement functor (Definition 2.6) $Q : \mathcal{M} \to \mathcal{M}_c$. This means we have a natural acyclic fibration $q_X : QX \to X$. We want to show that this functor preserves $\mathcal{W}$ too. To do this, suppose we have a weak equivalence $f : X \to Y$. Using the cofibrant replacement of X and Y we obtain the following commutative square.

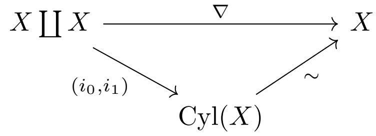
$$\begin{array}{ccc} QX & \overset{\sim}{\underset{q_X}{\twoheadrightarrow}} & X \\ \downarrow{\scriptstyle g} & & {\scriptstyle f}\downarrow{\scriptstyle\sim} \\ QY & \overset{\sim}{\underset{q_Y}{\twoheadrightarrow}} & Y \end{array}$$

Using the two-out-of-three axiom M2 and the fact that $\mathcal{W}$ is closed under composition (Definition 2.1) we have that $fq_X = q_Y g$ and that $fq_X \in \mathcal{W}$. Since $q_Y \in \mathcal{W}$ we must have that g is also a weak equivalence. Hence we have $\mathrm{Ho}\,(i)\mathrm{Ho}\,(Q) \cong \mathrm{id}_{\mathrm{Ho}\,\mathcal{M}}$ and $\mathrm{Ho}\,(Q)\mathrm{Ho}\,(i) \cong \mathrm{id}_{\mathrm{Ho}\,\mathcal{M}_c}$. Since q is natural, we have that both of these isomorphisms are natural giving us the desired equivalence.
The remaining equivalences follow a very similar proof, using a dual argument with the fibrant replacement for $\mathrm{Ho}\,\mathcal{M}_f$. □

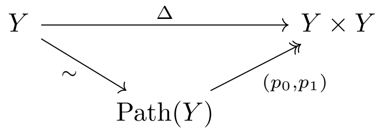
We are now ready to define the general notion of a homotopy.

Definition 3.6. Let $f, g : X \to Y$ be morphisms in a model category $\mathcal{M}$.

(i) A *cylinder object* for X is a factorisation the fold morphism $\nabla : X \coprod X \to X$ into a cofibration followed by a weak equivalence as follows.



(ii) A *path object* for Y is a factorisation of the diagonal morphism $\Delta : Y \to Y \times Y$ into a weak equivalence followed by a fibration as follows.



(iii) A *left homotopy* $f \overset{\ell}{\sim} g$ is a morphism $F : \mathrm{Cyl}(X) \to Y$, for some cylinder object $\mathrm{Cyl}(X)$ for X, with $Fi_0 = f$ and $Fi_1 = g$.

(iv) A *right homotopy* $f \overset{r}{\sim} g$ is a morphism $\tilde{F} : X \to \mathrm{Path}(Y)$, for some path object $\mathrm{Path}(Y)$ for Y, with $p_0\tilde{F} = f$ and $p_1\tilde{F} = g$.

(v) We say that f and g are *homotopic* $f \sim g$ if $f \overset{\ell}{\sim} g$ and $f \overset{r}{\sim} g$.

(vi) A morphism f is a *homotopy equivalence* if there is some morphism $h : Y \to X$ satisfying $hf \sim \mathrm{id}_X$ and $fh \sim \mathrm{id}_Y$.

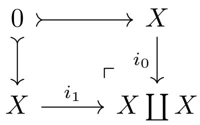
The above definition is completely analogous to Definition 5.1. The cylinder object plays the role of $X \times [0,1]$ and the left homotopy plays the role of the associated map $H : X \times [0,1] \to Y$. The path object plays the role of the space of continuous functions $[0,1] \to Y$ and the right homotopy plays the role of the family of maps $h_t : X \to Y$.

Remark 3.7. The factorisation axiom M5 of model category ensures that we can always find a cylinder object and a path object for any object X in a model category $\mathcal{M}$ as the diagonal and fold morphisms are always defined. These objects are not necessarily the 'best' choices in all situations.

The natural step from here is to examine the properties of these cylinder and path objects along with the morphisms associated with them. The first result we look at tells us when the morphisms $i_0, i_1 : X \to \mathrm{Cyl}(X)$ and $p_0, p_1 : \mathrm{Path}(X) \to X$ in Definition 3.6 are cofibrations and fibrations, respectively. To do this we first need to know when the morphisms $X \to X \coprod X$ and $X \times X \to X$ are cofibrations and fibrations. The following lemmas are presented in Hirschhorn (2003).

Lemma 3.8. *Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$. If X is cofibrant then the injections $i_0, i_1 : X \to X \coprod X$ are cofibrations. Dually, if X is fibrant then the projections $p_0, p_1 : X \times X \to X$ are fibrations.*

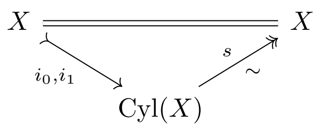
Proof. For the first statement, let X be cofibrant. We have the following pushout square.



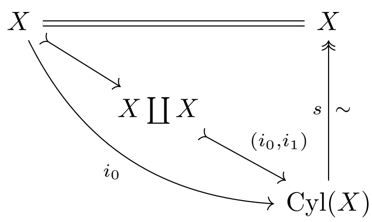
Since cofibrations are closed under pushout (Lemma 2.9) it follows that i_0 and i_1 are cofibrations. The case where X is fibrant is a dual argument using the pullback. □

Lemma 3.9. *Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$. If $Cyl(X)$ is a cylinder object for X then the morphisms $i_0, i_1 : X \to Cyl(X)$ are weak equivalences. If X is cofibrant then they are acyclic cofibrations. Dually, if $Path(X)$ is a path object for X then the morphisms $p_0, p_1 : Path(X) \to X$ are weak equivalences. If X is fibrant then they are acyclic fibrations.*

Proof. We prove the case for cylinder objects and the case for path objects is a dual argument. Using the factorisation axiom M5 have the following factorisation of the identity morphism on X.



Since the identity morphism is a weak equivalence the two-out-of-three axiom M2 gives us that i_0 and i_1 are also weak equivalences. If X is cofibrant then the morphisms $X \to X \coprod X$ are cofibrations so our factorisation is as follows.



□

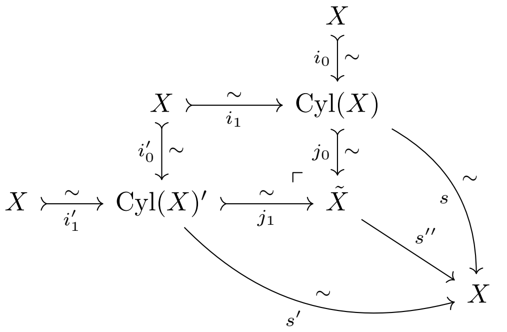
The above lemma tells us when the maps for cylinder and path objects are cofibrations and fibrations respectively. The reason for asking this question is to determine when the composition of left (dually right) homotopies is again a left (right) homotopy.

Lemma 3.10. *Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}$ with X cofibrant and Y fibrant.*

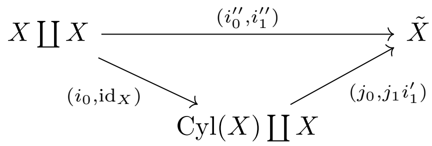
(i) *If $Cyl(X)$ and $Cyl(X)'$ are cylinder objects for X with morphisms $i_0, i_1 : X \to Cyl(X)$, $s : Cyl(X)$, $i'_0, i'_1 : X \to Cyl(X)'$, and $s' : Cyl(X)' \to X$, then there is a third cylinder object $Cyl(X)''$ which is the pushout of the diagram $Cyl(X) \leftarrow X \to Cyl(X)'$.*

(ii) *If $Path(Y)$ and $Path(Y)'$ are path objects for Y with morphisms $p_0, p_1 : Path(Y) \to Y$, $r : Y \to Path(Y)$, $p'_0, p'_1 : Path(Y)' \to Y$, and $r' : Y \to Path(Y)'$, there there is a third path object $Path(Y)''$ which is the pullback of the diagram $Path(Y) \to Y \leftarrow Path(Y)'$.*

Proof. We will only prove the first statement as the second follows by a dual argument. Since X is cofibrant Lemma 3.9 tells us that i_0, i_1, i'_0, and i'_1 are acyclic cofibrations. Since acyclic cofibrations are closed under pushouts we obtain the following pushout diagram.



We want to show that $\tilde{X}$ is indeed a cylinder object for X. First, by the two-out-of-three axiom M2 we have that $s'' : \tilde{X} \to X$ is a weak equivalence. We now need to show that the morphism $(i_0'', i_1'') : X \coprod X \to \tilde{X}$ is a cofibration. Using the above diagram we can factor this morphism as in the following diagram.

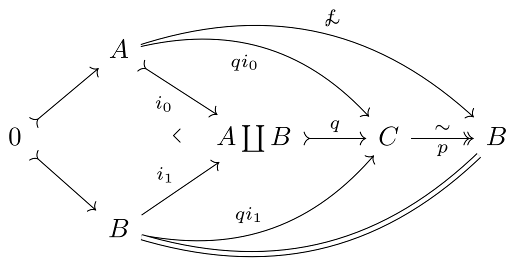


The first morphism $(i_0, \mathrm{id}_X) : X \coprod X \to \mathrm{Cyl}(X) \coprod X$ is the pushout of the diagram $X \coprod X \leftarrow X \to \mathrm{Cyl}(X)$. Since acyclic cofibrations are closed under pushout we have that (i_0, id_X) is an acyclic cofibration. Similarly, the second morphism $(j_0, j_1 i_1')$ is the pushout of the diagram $\mathrm{Cyl}(X) \leftarrow X \coprod X \to \mathrm{Cyl}(X) \coprod X$ and is thus a cofibration. Since cofibrations are closed under composition it follows that (i_0'', i_1'') is a cofibration and $\tilde{X}$ is indeed a cylinder object for X. □

We are nearly ready to examine the key properties of left and right homotopies. We currently know when various maps are (acyclic) (co)fibrations and when the composition of left (dually right) homotopies is again a left (right) homotopy. Before we do this we are going to need Ken Brown's lemma which will give us a key property of functors between model categories.

Lemma 3.11. *Let $F : \mathcal{M} \to \mathcal{N}$ be a functor where $\mathcal{M}$ is a model category and $\mathcal{N}$ has a subcategory of weak equivalences which satisfy the two-out-of-three axiom. Suppose that F takes acyclic cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually for acyclic fibrations and fibrant objects.*

Proof. Let A and B be cofibrant objects in $\mathcal{M}$, $f : A \to B$ be a weak equivalence between cofibrant objects, and $C \in \mathcal{M}$. We want to show that $F(f)$ is a weak equivalence. Consider the following commutative diagram.



First, the pushout square on the left and closure of cofibrations under pushout tell us that the inclusions $i_0 : A \to A \coprod B$ and $i_1 : B \to A \coprod B$ are cofibrations. We then factor the morphism $A \coprod B \to B$ into a cofibration $q : A \coprod B \to C$ followed by an acyclic fibration $p : C \to B$ using the factorisation axiom M5. By the two-out-of-three axiom M2 we must have that qi_0 and qi_1 are weak equivalences as the outer arrows and p are weak equivalences. Moreover, since the class of cofibrations is closed under composition, they are also

cofibrations and hence are acyclic cofibrations of cofibrant objects. Then, by our assumption, $F(qi_0)$ and $F(qi_1)$ are weak equivalences.
Since $F(\mathrm{id}_B) = F(pqi_1) = F(p)F(qi_1)$ is a weak equivalence the two-out-of-three axiom gives us that $F(p)$ is also a weak equivalence. Then $F(f) = F(pqi_0) = F(p)F(qi_0)$ is also weak equivalence.
The dual argument is much the same, using a pullback and the terminal object instead. □

The following proposition gives us the main properties of left and right homotopies.

Proposition 3.12. *Let $\mathcal{M}$ be a model category and $f, g : B \to X$, $e : A \to B$, and $h : X \to Y$ be morphisms in $\mathcal{M}$.*

(i) *$f \overset{\ell}{\sim} g$ gives $hf \overset{\ell}{\sim} hg$. Dually, $f \overset{r}{\sim} g$ gives $fe \overset{r}{\sim} ge$.*

(ii) *If X is fibrant then $f \overset{\ell}{\sim} g$ gives $fe \overset{\ell}{\sim} ge$. Dually, if B is cofibrant then $f \overset{r}{\sim} g$ gives $hf \overset{r}{\sim} hg$.*

(iii) *If B is cofibrant then $\overset{\ell}{\sim}$ is an equivalence relation on $\mathcal{M}(B, X)$. Dually, if X is fibrant then $\overset{r}{\sim}$ is an equivalence relation on $\mathcal{M}(B, X)$.*

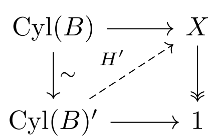
(iv) *If B is cofibrant and h is an acyclic fibration or weak equivalence of fibrant objects then the induced map $h_* : \mathcal{M}(B, X)/\overset{\ell}{\sim} \to \mathcal{M}(B, Y)/\overset{\ell}{\sim}$ is an isomorphism. Dually, if X is fibrant and e is an acyclic cofibration or weak equivalence of cofibrant objects then the induced map $e_* : \mathcal{M}(B, X)/\overset{r}{\sim} \to \mathcal{M}(A, X)/\overset{r}{\sim}$ is an isomorphism.*

(v) *If B is cofibrant then $f \overset{\ell}{\sim} g$ gives $f \overset{r}{\sim} g$. Dually, if X is fibrant then $f \overset{r}{\sim} g$ gives $f \overset{\ell}{\sim} g$.*

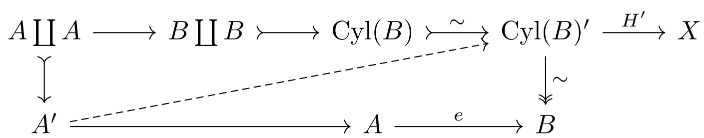
Proof. We only prove the statements for left homotopies as the rest follow by a dual argument.

(i) Suppose that $f \overset{\ell}{\sim} g$. Then there is a cylinder object $\mathrm{Cyl}(B)$ of B. If $F : \mathrm{Cyl}(B) \to X$ is the left homotopy then $hF : \mathrm{Cyl}(B) \to Y$ is the left homotopy giving $hf \overset{\ell}{\sim} hg$.

(ii) Suppose that $f \overset{\ell}{\sim} g$ and X is fibrant. Then there is a cylinder object $\mathrm{Cyl}(B)$ such that the map $s : \mathrm{Cyl}(B) \to B$ is an acyclic fibration. This is reasonable since we can factor the map $s : \mathrm{Cyl}(B) \to B$ as a cofibration followed by an acyclic fibration generating another cylinder object for B and extending the left homotopy $F : \mathrm{Cyl}(B) \to X$. Our goal here is to make a 'better' cylinder object for B. To go this we use the factorisation axiom M5 to factor s as an acyclic cofibration $\mathrm{Cyl}(B) \to \mathrm{Cyl}(B)'$ followed by an acyclic fibration $\mathrm{Cyl}(B)' \to B$. The first morphism here is an acyclic cofibration by the two-out-of-three axiom M2. Consider the following commutative solid arrow diagram.



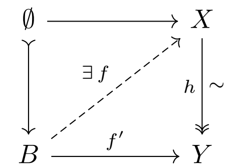
Since X is fibrant we have a left homotopy $H' : \mathrm{Cyl}(B)' \to X$. Putting all of this together we get the following commutative solid arrow diagram.



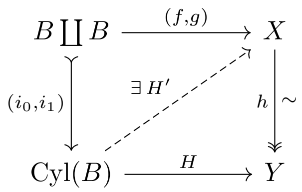
The lift exists since $\mathcal{C}of \boxslash (\mathcal{W} \cap \mathcal{F}ib)$. The desired left homotopy is the composition of this lift and H'.

(iii) Symmetry is immediate. If $f = g$ then we can take $B = \mathrm{Cyl}(B)$ and the left homotopy $F = f$, giving us reflexivity. Transitivity is achieved by Lemma 3.10 and property (ii) above.

(iv) First suppose that h is an acyclic fibration and consider the functor $F : \mathcal{M}(B,X)/\overset{\ell}{\sim} \to \mathcal{M}(B,Y)/\overset{\ell}{\sim}$. We want to show that F is an isomorphism. For surjectivity, consider the morphism $f' : B \to Y$. Since B is cofibrant we have the following diagram.

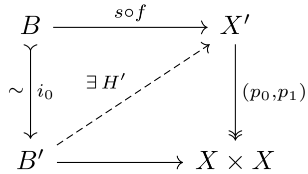


Hence for any $f' : B \to Y$ we have a morphism $f : B \to X$, since $\mathcal{C}of \boxslash (\mathcal{W} \cap \mathcal{F}ib)$, such that $hf = f'$ and this gives us surjectivity. For injectivity suppose we have $hf \overset{\ell}{\sim} hg$ with the left homotopy $H : \mathrm{Cyl}(B) \to Y$. This gives us the following diagram



The lift $H' : \mathrm{Cyl}(B) \to X$ is a left homotopy giving $f \overset{\ell}{\sim} g$ and hence F is injective as well.
Now suppose that h is a weak equivalence of fibrant objects. Then h can be factored into an acyclic cofibration followed by an acyclic fibration. The result then follows by applying Ken Brown's lemma and the case for when h is an acyclic fibration.

(v) Suppose that $f \overset{\ell}{\sim} g$ by the left homotopy $H : \mathrm{Cyl}(B) \to X$ where $\mathrm{Cyl}(B)$ is a cylinder object for B. By lemma 3.9 we have that $i_0 : B \to \mathrm{Cyl}(B)$ is an acyclic cofibration. Let $\mathrm{Path}(X)$ be a path object for X giving us a weak equivalence $s : X \to \mathrm{Path}(X)$ followed by a fibration $(p_0, p_1) : \mathrm{Path}(X) \to X \times X$. This gives us the following diagram.



The lift, which exists since $(\mathcal{W} \cap \mathcal{C}of) \boxslash \mathcal{F}ib$, $H' : \mathrm{Cyl}(B) \to \mathrm{Path}(X)$ gives us the right homotopy $\tilde{H} = H' i_1$.

□

These properties give rise to some immediate corollaries.

Corollary 3.13. *If $f, g : B \to X$ are morphisms in a model category $\mathcal{M}$ with B cofibrant and X fibrant, then $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ give the same equivalence relation on $\mathcal{M}(B,X)$.*

Corollary 3.14. *The homotopy relation $\sim$ on morphisms in $\mathcal{M}_{cf}$ (Definition 2.5) defines an equivalence relation.*

The previous corollary allows us to define the category $\mathcal{M}_{cf}/\sim$. The functor $G : \mathcal{M}_{cf} \to \mathcal{M}_{cf}/\sim$ will invert the homotopy equivalences in $\mathcal{M}_{cf}$, but we want to invert the weak equivalences. As it turns out, the weak equivalences in $\mathcal{M}_{cf}$ are precisely the homotopy equivalences (see Hovey (1999) Proposition 1.2.8 on pages 11 and 12 for a proof).

Corollary 3.15. *Let $\mathcal{M}$ be a model category. Given functors $F : \mathcal{M}_{cf} \to Ho\,\mathcal{M}_{cf}$ and $G : \mathcal{M}_{cf} \to \mathcal{M}_{cf}/\sim$ there is an isomorphism of categories $j : \mathcal{M}_{cf}/\sim \to Ho\,\mathcal{M}_{cf}$ with $jG = F$ and j is the identity on objects.*

We finally present the result which ties all of work together (Hovey (1999) calls this the 'fundamental theorem about model categories'). The best part is that we have done most of the work for proving it!

Theorem 3.16. *Let $\mathcal{M}$ be a model category, $F : \mathcal{M} \to Ho\,\mathcal{M}$ be the canonical functor, Q the cofibrant replacement functor (Definition 2.6), and R the fibrant replacement functor. Let $X, Y \in \mathcal{M}$.*

(i) *The inclusion $\mathcal{M}_{cf} \to \mathcal{M}$ induces $\mathcal{M}_{cf}/\sim \xrightarrow{\cong} Ho\,\mathcal{M}_{cf} \xrightarrow{\cong} Ho\,\mathcal{M}$.*

(ii) *We have the natural isomorphisms*

$$\mathcal{M}(QRX, QRY)/\sim\, \cong Ho\,\mathcal{M}(FX, FY) \cong \mathcal{M}(RQX, RQY)/\sim .$$

(iii) *There is a natural isomorphism $Ho\,\mathcal{M}(FX, FY) \cong \mathcal{M}(QX, RY)/\sim$. If X and Y are cofibrant and fibrant, respectively, then we have the natural isomorphism $Ho\,\mathcal{M}(FX, FY) \cong \mathcal{M}(X, Y)$.*

(iv) *$F : \mathcal{M} \to Ho\,\mathcal{M}$ identifies left/right homotopic maps.*

(v) *If $f : A \to B$ is in $\mathcal{M}$ such that Ff is an isomorphism then f is a weak equivalence.*

Proof. (i) This follows directly from proposition 3.5 and corollary 3.15.

(ii) We first invert the equivalence $\text{Ho}\,\mathcal{M}_{cf} \to \text{Ho}\,\mathcal{M}$ by $\text{Ho}\,Q\text{Ho}\,R$ (equivalently $\text{Ho}\,R\text{Ho}\,Q$). Using corollary 3.15 we obtain the desired natural isomorphisms.

(iii) Recall that we have natural weak equivalences $QX \to X \to RX$. Using parts (iii) and (iv) of proposition 3.12 we obtain the desired isomorphisms.

(iv) Suppose that $F : \mathcal{M}_{cf} \to \mathcal{D}$ is a functor which sends weak equivalences to isomorphisms. Let $\text{Cyl}(B)$ be a cylinder object for B, then we have the factorisation $B \coprod B \xrightarrow{(i_0,i_1)} \text{Cyl}(B) \underset{r}{\xrightarrow{\sim}} B$. Then $ri_0 = ri_1 = \text{id}_B$. Suppose that $f, g : B \to X$ with $f \overset{\ell}{\sim} g$. Let the left homotopy be $H : \text{Cyl}(B) \to X$. Then we have $Ff = F(H)F(i_0) = F(H)F(i_1) = Fg$ and hence F identifies left homotopic maps. A dual argument applies for right homotopic maps. Then we can apply part (i) to get this identification for $F : \mathcal{M} \to \text{Ho}\,\mathcal{M}$.

(v) If Ff is an isomorphism then QRf is an isomorphism in $\mathcal{M}_{cf}/\sim$ and hence a homotopy equivalence. Then QRf is a weak equivalence and since $QX \to X$ and $X \to RX$ are weak equivalences we must have that f is a weak equivalence.

□

4. Conclusion

Model categories have given us a new lens through which we can approach homotopy theory. What we have presented here, however, only scratches the surface on model categories. A next step would be to examine special functors between model categories called Quillen functors. These give rise to derived functors between the homotopy categories of model categories. We can use these as a starting point to begin examining a notion of localisation in model categories along with other types of model categories (such as monoidal and simplicial). We can even continue to build our notion of homotopy theory into stable, equivariant, and G-equivariant stable homotopy theory (for some group G). A particularly nice journey through model categories is given in Riehl (2019), which has provided much motivation throughout this project.

Appendix I: Necessary Category Theory

All of the following definitions and results can be found in Riehl (2017) (or any other introductory category theory text).

Definition 4.1. A *category* $\mathcal{C}$ is a collection of objects, $\operatorname{ob}\mathcal{C}$, and a collection of morphisms, $\operatorname{mor}\mathcal{C}$, between objects such that:

- Each morphism has specified objects for its domain and codomain, $f : X \to Y$ for $X, Y \in \operatorname{ob}\mathcal{C}$.
- Every object has an identity morphism $\mathrm{id}_X : X \to X$ for $X \in \operatorname{ob}\mathcal{C}$.
- For any two morphisms $f : X \to Y$ and $g : Y \to Z$ with $X, Y, Z \in \operatorname{ob}\mathcal{C}$, the composite morphism $gf : X \to Z$ exists.

These objects and morphisms must also adhere to the following axioms:

(i) For $f : X \to Y$ with $X, Y \in \operatorname{ob}\mathcal{C}$, $f\mathrm{id}_X = \mathrm{id}_Y f = f$.

(ii) For any three morphisms f, g, h which are composable $h(gf) = (hg)f$ which we denote hgf.

We will denote the collection of all morphisms between two objects X and Y in a category $\mathcal{C}$ by $\mathcal{C}(X, Y)$. We will often drop the notation $\operatorname{ob}\mathcal{C}$ and just write $X \in \mathcal{C}$. We will sometimes refer to morphisms as "arrows".

Example 4.2.
- **Top** - topological spaces are objects with continuous maps as morphisms.
- **Group** - groups are objects with group homomorphisms as morphisms.
- **AbGrp** - abelian groups are objects with group homomorphisms as morphisms.
- **Set** - sets are objects with functions between sets as morphisms.

For any category we can define another category by simply flipping the domain and codomain on each morphism. This is the first notion of duality that we will encounter.

Definition 4.3. For a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$ and a morphism $f^{op} : Y \to X$ for each morphism $f : X \to Y$ in $\mathcal{C}$.

Definition 4.4. A morphism $f : X \to Y$ is called an *isomorphism* if there is another morphism $g : Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Definition 4.5. A *subcategory* $\mathcal{D}$ of a category $\mathcal{C}$ is a category with objects a subcollection of the objects of $\mathcal{C}$ and morphisms a subcollection of the morphisms of $\mathcal{C}$.

Definition 4.6. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a *full subcategory* of $\mathcal{C}$ if, for all objects $X, Y \in D$, $f \in \mathcal{C}(X, Y)$ implies $f \in \mathcal{D}(X, Y)$.

Often we will deal with categories which have too many objects or morphisms such that we run into problems with the the Zermelo-Fraenkl axiomatisation of set theory. We thus use this axiomatisation to distinguish between "small" collections (sets) and "large" collections (classes).

Definition 4.7. A category is called *small* if it has a set's worth of morphisms. A category is *locally* small if $\mathrm{Hom}(X, Y)$ is a set for all objects X and Y.

Definition 4.8. A functor $F : \mathcal{C} \to \mathcal{D}$ between categories has an object $FX \in \mathcal{D}$ for each $X \in \mathcal{C}$ and a morphism $Ff \in \mathcal{D}(FX, FY)$ for each morphism $f \in \mathcal{C}(X, Y)$ and is subject to the following functoriality axioms:

(i) For $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$, $F(gf) = FgFf$.

(ii) For each $X \in \mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.

Lemma 4.9. *Functors preserve isomorphisms.*

The proof of the above lemma can be found in Riehl (2017) (Lemma 1.3.8).

Example 4.10. A particularly simple example of a functor is a *forgetful functor* - a functor which forgets some structure of its domain. An example of this is F : **AbGrp** $\to$ **Group**, which just 'forgets' that the commutativity of the group operation. Another example is F : **Group** $\to$ **Set** which forgets the group operation and returns the set of elements of the group.

Definition 4.11. Let $\mathcal{C}$ be a category. A *functorial factorisation* on $\mathcal{C}$ assigns to any morphism $f : X \to Y$ in $\mathcal{C}$ a factorisation $f = f_R f_L$ together with a functor E which acts on commutative squares of the form

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & Z \end{array}$$

sending them to commutative diagrams of the following form.

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ {\scriptstyle f_L}\downarrow & & \downarrow{\scriptstyle g_L} \\ X' & \xrightarrow{E(j,k)} & W' \\ {\scriptstyle f_R}\downarrow & & \downarrow{\scriptstyle g_R} \\ Y & \xrightarrow[k]{} & Z \end{array} \qquad (f : X \to Y,\ g : W \to Z)$$

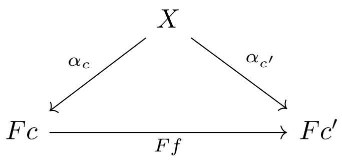
The functorial part of the above definition comes from E satisfying the functoriality axioms in Definition 4.8. In this case, referring to the commutative squares by the horizontal morphisms, it acts as $E(j_1 j_2, k_1 k_2) = E(j_1, k_1)E(j_2, k_2)$.

Definition 4.12. An object 0 in a category $\mathcal{C}$ is called an *initial object* if for every $X \in \mathcal{C}$ there is a unique morphism $0 \to X$. Dually, an object 1 is called a *terminal object* if for every $X \in \mathcal{C}$ there is a unique morphism $X \to 1$.

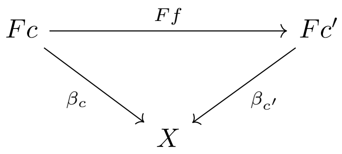
Definition 4.13. Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \implies G$ has an arrow $\alpha_X : FX \to GX$ for each $X \in \mathcal{C}$ such that for each morphism $f \in \mathcal{C}(X,Y)$ the following square commutes.

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\alpha_Y} & GY \end{array}$$

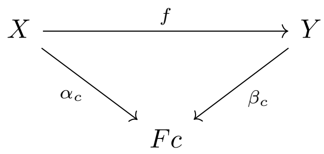
Definition 4.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *cone* over F is an object $X \in \mathcal{D}$ and a natural transformation $\alpha : \Delta X \implies F$ where ΔX is the constant functor at X sending all objects to X and all morphisms to id_X. The components of this natural transformation $\{\alpha_c : X \to Fc\}_{c \in \mathcal{C}}$ are such that for any morphism $f : c \to c'$ in $\mathcal{C}$ the following diagram commutes.



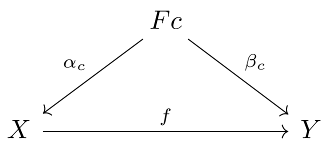
Dually, a *cocone* under F is an object $X \in \mathcal{D}$ and a natural transformation $\beta : F \implies \Delta X$. The components of this natural transformation $\{\beta_c : Fc \to X\}_{c \in \mathcal{C}}$ are such that for any morphism $f : c \to c'$ in $\mathcal{C}$ the following diagram commutes.



We will often identify a (co)cone over a functor by (X, α) where X is the fixed object and α is the natural transformation. For any functor $F : \mathcal{C} \to \mathcal{D}$ we can consider the category of cones over F and the category of cocones under F. We can define a morphism between two (co)cones $(X, \alpha) \to (Y, \beta)$ as a morphism $f : X \to Y$ such that for all $c \in \mathcal{C}$ the following diagram commutes for cones,



and the following diagram commutes for cocones,



Definition 4.15. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If the category of cones over F has a terminal object (Definition 4.12) (X, α) then we call X the *limit* of F and write $\lim F = X$. Dually, if the category of cocones under F has an initial object (X, α) then we call X the *colimit* of F and write $\operatorname{colim} F = X$.
We call a (co)limit small if $\mathcal{C}$ is a small category (Definition 4.7).

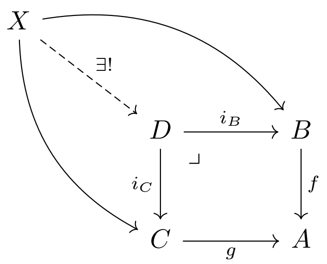
We are particularly interested in categories which contain all of their small limits and colimits. That is, for every functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ is small, $\lim F$ and $\operatorname{colim} F$ exist, respectively.

Definition 4.16. A category $\mathcal{D}$ is *complete* if contains all small limits. Similarly, a category $\mathcal{D}$ is *cocomplete* if it contains all small colimits. A category $\mathcal{D}$ is *bicomplete* if it is both complete and cocomplete.

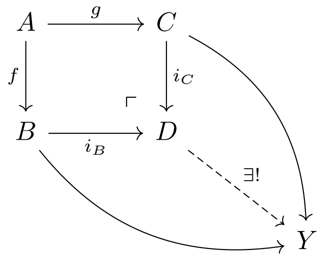
Definition 4.17. Let $\mathcal{C}$ be a discrete category with only identity morphisms. A functor $F : \mathcal{C} \to \mathcal{D}$ picks out a collection of objects in $\mathcal{D}$ indexed over $c \in \mathcal{C}$. The limit of F is called the *product* of $\{Fc\}_{c \in \mathcal{C}}$ and is written as $\prod_{c \in \mathcal{C}} Fc$. The colimit of F is called the *coproduct* of $\{Fc\}_{c \in \mathcal{C}}$ and is written as $\coprod_{c \in \mathcal{C}} Fc$.

If $\mathcal{C}$ only has two objects then $F : \mathcal{C} \to \mathcal{D}$ picks out two objects $X, Y \in \mathcal{D}$ giving $\lim F = X \times Y$ and $\operatorname{colim} F = X \coprod Y$.

Definition 4.18. Let $\mathcal{C} = \{\bullet \to \bullet \leftarrow \bullet\}$. The image of the functor $F : \mathcal{C} \to \mathcal{D}$ is $B \xrightarrow{f} A \xleftarrow{g} C$. A *pullback* is the limit of F, if it exists, consisting of an object D with morphisms $C \xleftarrow{i_C} D \xrightarrow{i_B} B$ such that for any $X \in \mathcal{C}$ with morphisms $C \leftarrow X \to B$ which make the outside of the following diagram commute there is a unique morphism $X \to D$ such that the entire diagram commutes.

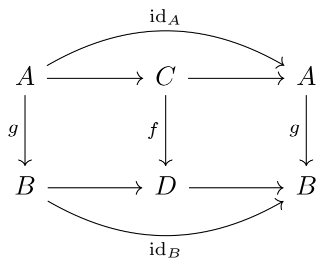


Definition 4.19. Let $\mathcal{C} = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$. The image of the functor $F : \mathcal{C} \to \mathcal{D}$ is $B \xleftarrow{f} A \xrightarrow{g} C$. A *pushout* is the colimit of F, if it exists, consisting of an object D with morphisms $C \xrightarrow{i_C} D \xleftarrow{i_B} B$ such that for any $Y \in \mathcal{C}$ with morphisms $B \to Y \leftarrow C$ which make the outside of the following diagram commute there is a unique morphism $D \to Y$ such that the entire diagram commutes.



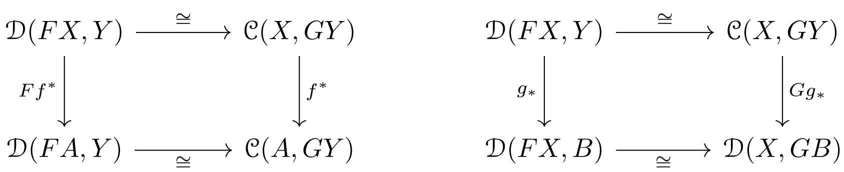
Note in the above two definitions the usage of ⌟ and ⌜. Typically when we have a pullback or pushout we will only require the square and use these symbols to denote whether it is a pullback or pushout square respectively.

Definition 4.20. Let $f : C \to D$ be a morphism in a category $\mathcal{C}$. A *retract* of f is a morphism $g : A \to B$ in $\mathcal{C}$ such that there exists a commutative diagram of the following form.



Definition 4.21. An *adjunction* is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, which we will write $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$, together with an isomorphism $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$ for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ which is natural in X and Y. We call F the *left adjoint functor* and G the *right adjoint functor.*

The naturality condition in the above means that for any morphisms $f : A \to X$ and $g : Y \to B$ the following diagrams commute.



Appendix II: Essential Algebraic Topology

Whilst we assume familiarity with basic homotopy theory, we put some essential definitions here to save ourselves defining them in the main text. There is the following standard notation that we will use:

- S^n is the n-sphere, that is $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$.
- D^n is the n-disc, that is $D^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$.
- $*$ is the single point space.

Definition 5.1. Let $f, g : (X, x_0) \to (Y, y_0)$ be basepoint preserving maps. A *homotopy* between f and g is a family of continuous maps $h_t : X \to Y$ with $t \in [0, 1]$ such that $h_0 = f$ and $h_1 = g$ and the associated map $H : X \times [0, 1] \to Y$ defined by $(x, t) \mapsto h_t(x)$ is continuous.

If there exists a homotopy between two maps f and g then we say they are homotopic and write $f \simeq g$. It is fairly simple to check that begin homotopic defines an equivalence relation (Hatcher, 2002) with the equivalences classes of homotopic maps being called *homotopy classes* which we will denote $[f]$. The set of all homotopy classes of basepoint preserving maps between two spaces X and Y is denoted as $[X, Y]$.

Definition 5.2. The *wedge* sum of two pointed spaces (X, x_0) and (Y, y_0), denoted $X \vee Y$, is defined as the quotient

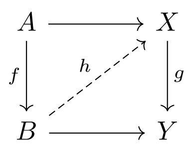
$$X \vee Y = (X \coprod Y)/(x_0 \sim y_0).$$

The wedge product can effectively be visualised as glueing spaces at their basepoints. As an example, $S^1 \vee S^1$ can be viewed as a figure-eight.

Definition 5.3. Let (X, x_0) be a topological space with a chosen basepoint. The *n-th homotopy group* of X with respect to x_0 $\pi_n(X, x_0)$ is the set of homotopy classes of maps $f : (S^n, s_0) \to (X, x_0)$. For $n \geq 1$ the group operation is defined so that $[f] + [g]$ is the homotopy class of the composition $S^n \to S^n \vee S^n \to X$. The first map in the composition collapses the equator of S^n to a point and we choose the basepoint s_0 to lie on this equator. The second map in the composition is $f \vee g$ which is well-defined as f and g are basepoint preserving. If $n = 0$ we define $\pi_0(X, x_0)$ to be the set of path-components of X.

Appendix III: Weak Factorisation Systems

Definition 6.1. Let $f : A \to B$ and $g : X \to Y$ be morphisms in a category $\mathcal{C}$. We say that f has the *left lifting property* with respect to g, or g has the *right lifting property* with respect to f if there is a morphism $h : B \to X$ such that the following diagram commutes.



We denote this as $f \boxslash g$.

Given a collection of morphisms J in a category $\mathcal{C}$ we denote the collection of morphisms which have the right lifting property with respect to each $j \in J$ by $J^{\boxslash}$ and the collection of morphisms which have the left lifting property with respect to each $j \in J$ by ${}^{\boxslash}J$. That is, $f \in J^{\boxslash}$ if and only if $j \boxslash f$ for each $j \in J$ and similarly $g \in {}^{\boxslash}J$ if and only if $g \boxslash j$ for each $j \in J$.

Definition 6.2. A pair of distinguished classes of morphisms $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{C}$ is a *weak factorisation system* if

(i) Each morphism f in $\mathcal{C}$ can be factored as $f = lr$ where $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

(ii) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

(iii) $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts (see Definition 4.20).

Many results about weak factorisation systems can be found in Riehl (2008) although we will not necessarily be requiring them in this report.

Appendix IV: The Small Object Argument

Ordinals, Cardinals, and Transfinite Composition

We assume some basic knowledge of orderings on the reader.

Definition 7.1. An *ordinal* is a well ordered set of all lesser ordinals equipped with a successor operation which returns the next ordinal. Every well ordered set is isomorphic to a unique ordinal.

Definition 7.2. A *limit ordinal* is an ordinal which cannot be obtained from a lesser ordinal by repeated (possibly infinite) applications of the successor operation.

Definition 7.3. A *cardinal* is an ordinal which has greater cardinality than all lesser ordinals.

The above definition is set so that it coincides with the regular definition of cardinality encountered in set theory. Both ordinals and cardinals can also be defined in terms of isomorphism classes of sets.

Definition 7.4. A cardinal κ is called *regular* if for any set S with cardinal less than κ, for every $s \in S$ there is a set U_s with cardinal less than κ such that the set $\cup_{s \in S} U_s$ has cardinal less than κ.

Definition 7.5. Let $\mathcal{C}$ be a cocomplete category (Definition 4.16) and λ and ordinal. A λ*-sequence* in $\mathcal{C}$ is a functor $X : \lambda \to \mathcal{C}$ which preserves colimits and is often written as the following diagram.

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots \qquad (\beta < \lambda)$$

For all limit ordinals $\gamma < \lambda$ the induced map

$$\mathrm{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$

is an isomorphism.

Definition 7.6. Let $\mathcal{W}$ be a class of maps in a cocomplete category $\mathcal{C}$. The composition $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ for a λ-sequence X is called a *transfinite composition* if every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{W}$.

Small Objects and Relative Cell Complexes

Definition 7.7. Let $\mathcal{C}$ be a category, $\mathcal{D}$ a subcategory, and κ a cardinal. An object $A \in \mathcal{C}$ is κ*-small relative to* $\mathcal{D}$ if for every regular cardinal $\lambda \geq \kappa$ and every λ-sequence X in $\mathcal{C}$ such that $X_\beta \to X_{\beta+1}$ $(\beta + 1 < \lambda)$ is in $\mathcal{D}$ the morphism

$$\mathrm{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \mathrm{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism.

Definition 7.8. Let $\mathcal{C}$ be a category and $\mathcal{D}$ a subcategory. An object $A \in \mathcal{C}$ is called *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some cardinal κ and *small* if it is small relative to $\mathcal{C}$.

Note. It is very important that they be sets as we want to be able to use them in the small object argument. We will then let $\mathcal{R}$ be the class of maps with the right lifting property with respect to all maps in I and $\mathcal{L}$ to be the class of maps with the left lifting property with respect to all maps in $\mathcal{R}$. If necessary we will use subscripts to denote which set of maps is generating the class.

The following results are fairly immediate from the definition.

Lemma 7.9. *Let $\mathcal{C}$ be a category. If I and J are sets of maps in $\mathcal{C}$ and every map in $\mathcal{R}_I$ is in $\mathcal{R}_J$ then every map in $\mathcal{L}_I$ is in $\mathcal{L}_J$.*

Lemma 7.10. *For a cocomplete category $\mathcal{C}$ with subcategory $\mathcal{D}$ and set K of all objects in $\mathcal{C}$ which are small relative to $\mathcal{D}$ there is a cardinal κ such that every element of I is κ-small relative to $\mathcal{D}$.*

Definition 7.11. Let $\mathcal{C}$ a cocomplete category and I a set of maps in $\mathcal{C}$. A *relative I-cell complex* is a map which can be constructed as a transfinite composition of pushouts (Definition 7.6 and Definition 4.19) of elements of I. An object is an *I-cell complex* if every map from the initial object (Definition 4.12) to it is a relative cell complex.

Remark 7.12. For a category $\mathcal{C}$ and set of maps I the relative I-cell complexes form a subcategory of $\mathcal{C}$.

Definition 7.13. Let $\mathcal{C}$ be a category, I a set of maps in $\mathcal{C}$, and $\mathcal{D}$ the subcategory of relative I-cell complexes. An object in $\mathcal{C}$ is called *small relative to I* if it is κ-small relative to $\mathcal{D}$ for some cardinal κ.

We need two results about the subcategory of relative I-cell complexes. We give them here without proof (see Hovey (1999) Lemmas 2.1.11,12,13 on pages 31-32 for details).

Lemma 7.14. *Let $\mathcal{C}$ be a cocomplete category and I a set of maps in $\mathcal{C}$. Then the subcategory of relative I-cell complexes is closed under transfinite composition.*

Lemma 7.15. *Let $\mathcal{C}$ be a cocomplete category and I a set of maps in $\mathcal{C}$. Then the subcategory of relative I-cell complexes is closed under pushouts.*

Remark 7.16. The importance of these two lemmas is as follows: if I is a set of maps in a cocomplete category $\mathcal{C}$ and $\mathbb{R}$ is as usual. Then the subcategory of relative I-cell complexes is contained in $\mathcal{L}$. If we have a relative I-cell complex, then it is in $\mathcal{L}$.

The Small Object Argument

Before we present the small argument we need to briefly discuss why we care. One of the modern developments in model categories is a definition, which is equivalent to Quillen (1967), in terms of weak factorisation systems. The rough idea, which is explored in Section 2.2, is to have a set of generating cofibrations and a set of generating acyclic fibrations which will (hopefully) give us two weak factorisation systems. Of course we need to know when this is possible and hence we need the small object argument. As a warning, we will not be presenting this as a proposition and then a proof, but rather a running proof - constantly indicating what we want to prove. This can be found in Hirschhorn (2003) and Hovey (1999).

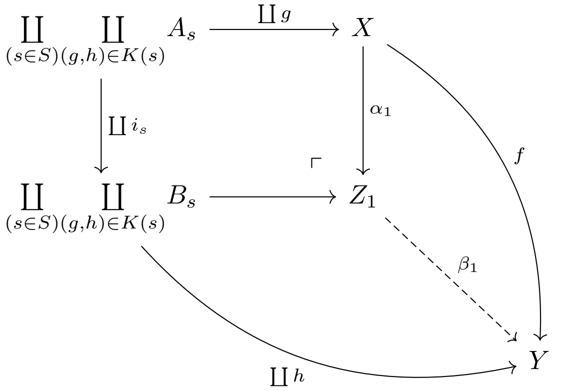
Let $\mathcal{M}$ be a cocomplete category and let $I = \{i_s : A_s \to B_s\}_{s \in S}$ be a set of maps in $\mathcal{M}$. Let I-cell denote the subcategory of relative I-cell complexes. We will build a factorisation for every morphism in $\mathcal{M}$ using I. First, let $f : X \to Y$ be a morphism in $\mathcal{M}$. Consider the following set of commutative squares.

$$\begin{array}{ccc} A_s & \xrightarrow{g} & X \\ {\scriptstyle i_s}\downarrow & & \downarrow{\scriptstyle f} \\ B_s & \xrightarrow[h]{} & Y \end{array}$$

Since the morphisms g and h which make the above squares commute are not necessarily the same for various $s \in S$ we let $K(s)$ denote the collection of all pairs of maps (g, h) such that the the above square commutes. We can put all of the above squares into the following single commutative square.

$$\begin{array}{ccc} \coprod_{(s\in S)} \coprod_{(g,h)\in K(s)} A_s & \xrightarrow{\coprod g} & X \\ \Big\downarrow{\scriptstyle \coprod i_s} & & \Big\downarrow{\scriptstyle f} \\ \coprod_{(s\in S)} \coprod_{(g,h)\in K(s)} B_s & \xrightarrow[\coprod h]{} & Y \end{array}$$

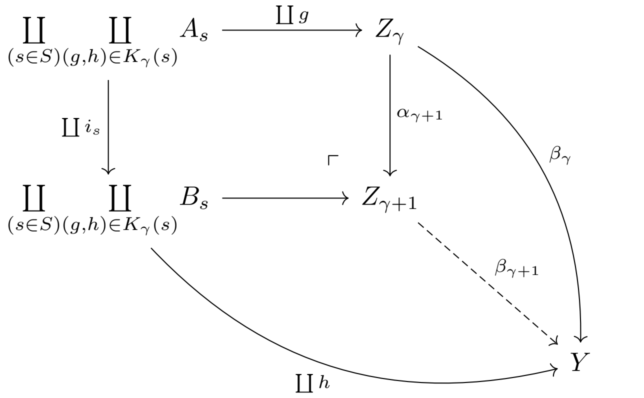
Let λ be a regular cardinal. Using Lemma 7.10 we can choose λ such that $\{A_s\}_{s\in S}$ is λ-small relative to I-cell. Now want to construct a λ-sequence $Z : \lambda \to \mathcal{C}$. We start by letting $Z_0 = X$. Then, we let Z_1 be the pushout of the above commuting square giving us the following pushout diagram.



This first step in our factorisation of f is good, but we still have some properties we want. First, we would like β_1 to have the right lifting property with respect to I. That is, we would like it to be in the class $\mathcal{R} = I^{\boxslash}$. Let γ be a limit ordinal and suppose that we have defined Z_δ along with β_δ for all $\delta < \gamma$. We can then define $Z_\gamma = \text{colim}_{\delta<\gamma} Z_\delta$ and β_γ to be the the map induced by β_δ. We now need to define $Z_{\gamma+1}$. To do this, we use the same pushout method as above. Let $K_\gamma(s)$ denote all pairs of morphisms (g, h) such that the following square commutes.

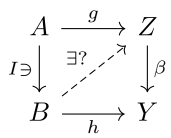
$$\begin{array}{ccc} \coprod_{(s\in S)} \coprod_{(g,h)\in K_\gamma(s)} A_s & \xrightarrow{g} & Z_\gamma \\ {\scriptstyle i_s}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_\gamma} \\ \coprod_{(s\in S)} \coprod_{(g,h)\in K_\gamma(s)} B_s & \xrightarrow[h]{} & Y \end{array}$$

We then define $Z_{\gamma+1}$ to be the pushout of the above commuting square. This gives us the following pushout diagram.

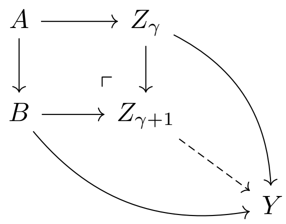


This gives us a factorisation of f as $X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y$ where $Z = \text{colim} Z_\gamma$, $\alpha = \text{colim}\alpha_\gamma$, and $\beta = \text{colim}\beta_\gamma$. By our construction this factorisation is functorial.

The next step is to show that $\beta \in \mathcal{R}^{\boxslash}$. That is, given the commutative solid arrow square

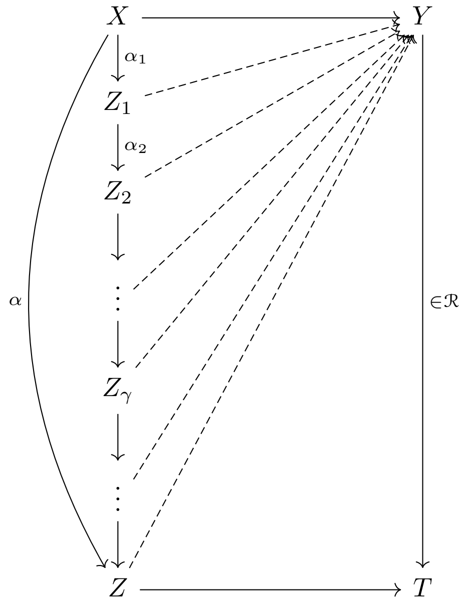


we want to find a lift $B \to Z$. However, since A is λ-small relative to I-cell we get the following pushout diagram.



But this gives us a factorisation $B \to Z_{\gamma+1} \hookrightarrow Z \to Y$. Taking the composition of the first two arrows we get the desired lift and so $\beta \in \mathcal{R} = I^{\boxslash}$.

Finally, we want to show that $\alpha \in \mathcal{L} = {}^{\boxslash}\mathcal{R}$. To do this, we first need two results. The first we have given in Lemma 2.9. The result stated is for (acyclic) (co)fibrations but it holds in general. In particular for any map $r : X \to Y$ in a category the class of maps with the left lifting property with respect to r is closed under pushout. This tells us that $\mathcal{L}$ is closed under pushout. The second result is less trivial. We claim that $\mathcal{L}$ is closed under transfinite composition. We claim that this can be done using a transfinite induction on a λ-sequence of maps with the left lifting property. Details can be found in Hirschhorn (2003) (Lemma 10.3.1, page 193). Let $f : X \to Y$ as we have been doing. Using the construction of the factorisation for f, using the λ-sequence, we can make the following commutative diagram.



However, by the way we have formed α we know that it is a relative I-cell complex and so the lift $Z \to Y$ exists and $\alpha \in \mathcal{L}$.

This completes the small object argument. It's a lot to take in but it is fundamental in putting a weak factorisation system on a model category.

References

W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995. doi: 10.1016/B978-044481779-2/50003-1. URL https://doi-org.virtual.anu.edu.au/10.1016/B978-044481779-2/50003-1.

P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. ISBN 0-521-79160-X; 0-521-79540-0.

P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003. ISBN 0-8218-3279-4.

P. S. Hirschhorn. The Quillen model category of topological spaces. *Expo. Math.*, 37(1):2–24, 2019. ISSN 0723-0869. doi: 10.1016/j.exmath.2017.10.004. URL https://doi-org.virtual.anu.edu.au/10.1016/j.exmath.2017.10.004.

M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. ISBN 0-8218-1359-5.

D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

E. Riehl. Factorization systems, 2008. URL http://www.math.jhu.edu/~eriehl/factorization.pdf.

E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017. ISBN 9780486820804. URL http://www.math.jhu.edu/~eriehl/context.pdf.

E. Riehl. Homotopical categories: from model categories to $(\infty, 1)$-categories, 2019.